Controlling representations of frame matroids.

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joint work in progress
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CanaDAM

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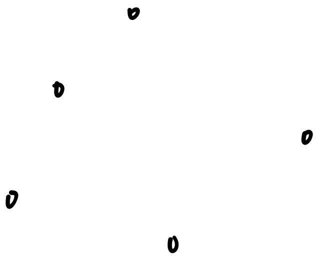
Frame matroids (Zaslavsky, 1989)

- every element is in the span of ≤ 2 points in a distinguished basis



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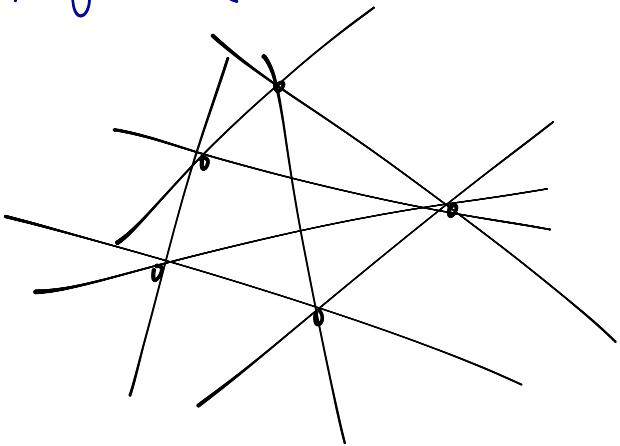

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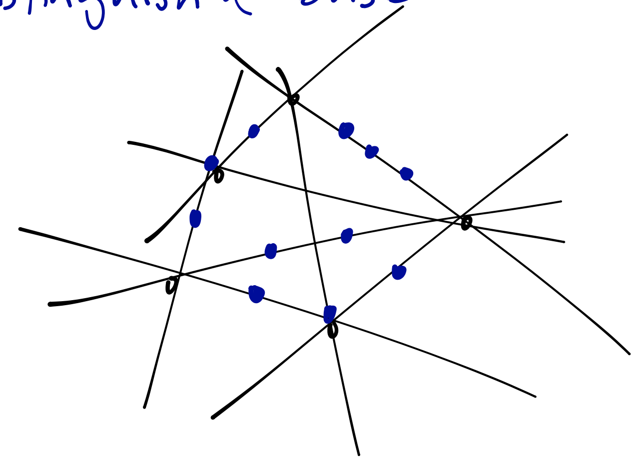

M

- restrict to points not in the special basis

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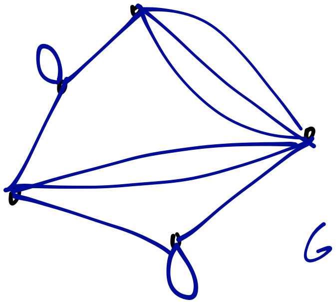
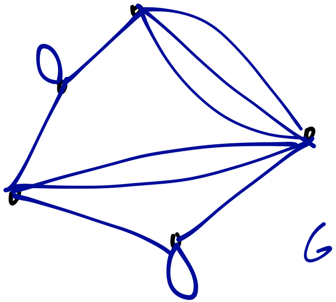


- distinguished basis elements $\leftrightarrow$ vertices
- $e \in \text{span}(u,v) \leftrightarrow$ edge with endpoints u,v

Frame matroids (Zaslavsky, 1989)

- put $\mathcal{B} = \{ C \mid C$ is a cycle of G and $E(C)$ is a circuit of $M \}$
- $(G, \mathcal{B})$ is a biased graph representing M



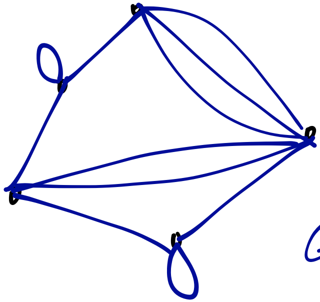
$M = F(G, \mathcal{B})$

or

$M = F(G)$

where G is a graph each cycle of which is either balanced or unbalanced according to M.

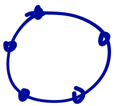
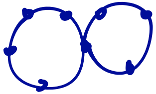
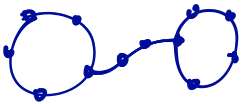
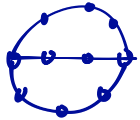
Frame matroids (Zaslavsky, 1989)



G

$M = F(G)$

- C is circuit of M $\Longleftrightarrow$ C induces a "circuit subgraph":

 $\in \mathcal{B}$, , , 

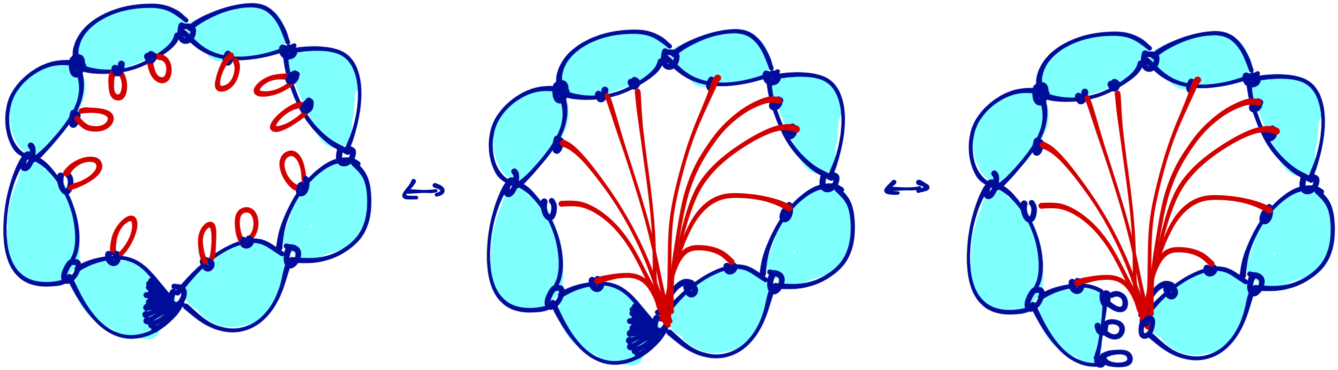
Committed vertices

Let M be a frame matroid represented by biased graph $(G, \mathcal{B})$.

Def'n. A vertex $v \in V(G)$ is committed if in every biased graph representing M there is a vertex whose set of incident edges is exactly that of v.

Otherwise v is uncommitted.

Families of 3-biconnected biased graphs with abitarily many uncommitted vertices



looped bangle ↔ bonnet ↔ broach

3-biconnected: no 2-separation with one side balanced

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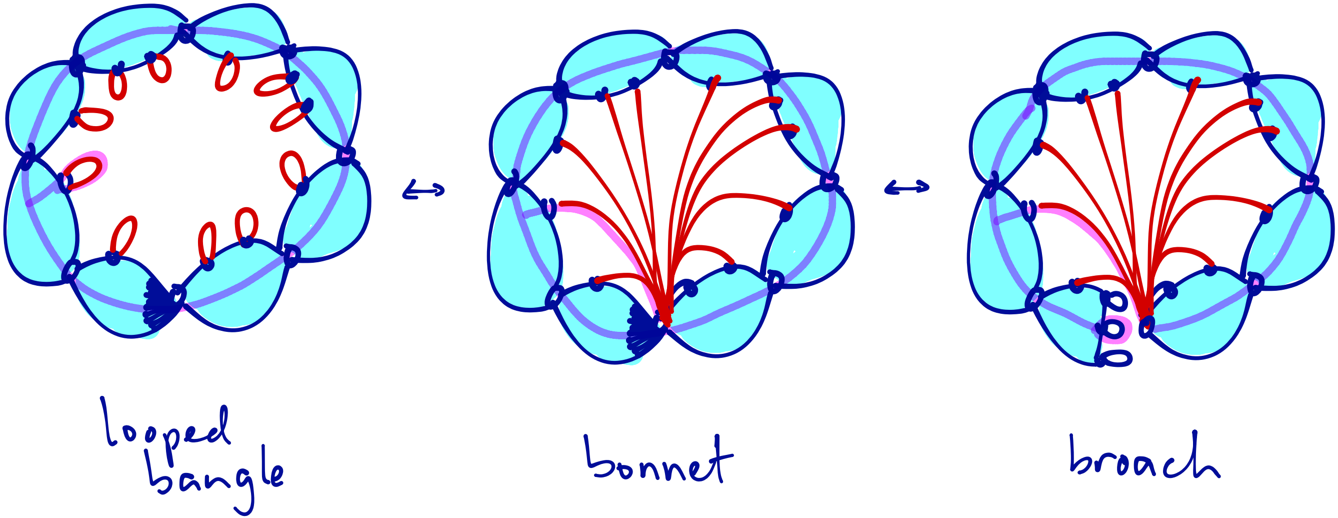

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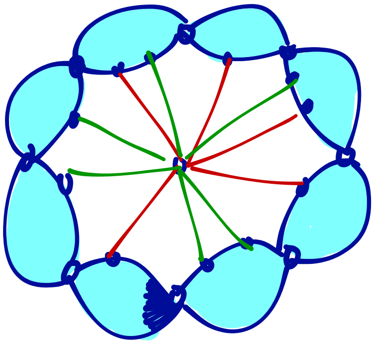
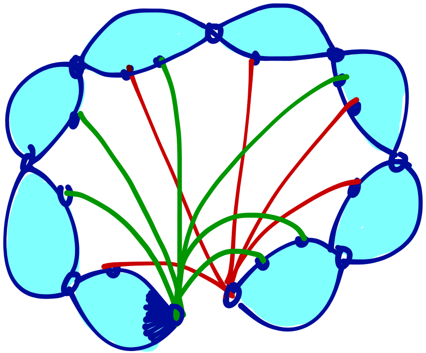


buckle



open buckle

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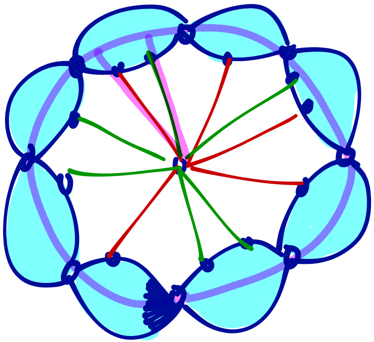
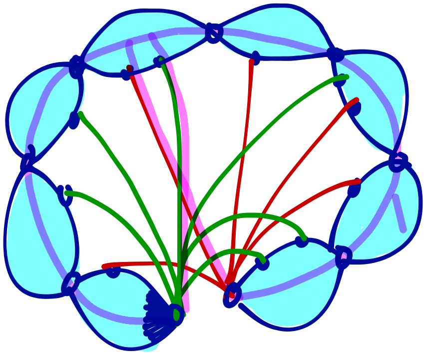


buckle

$\longleftrightarrow$



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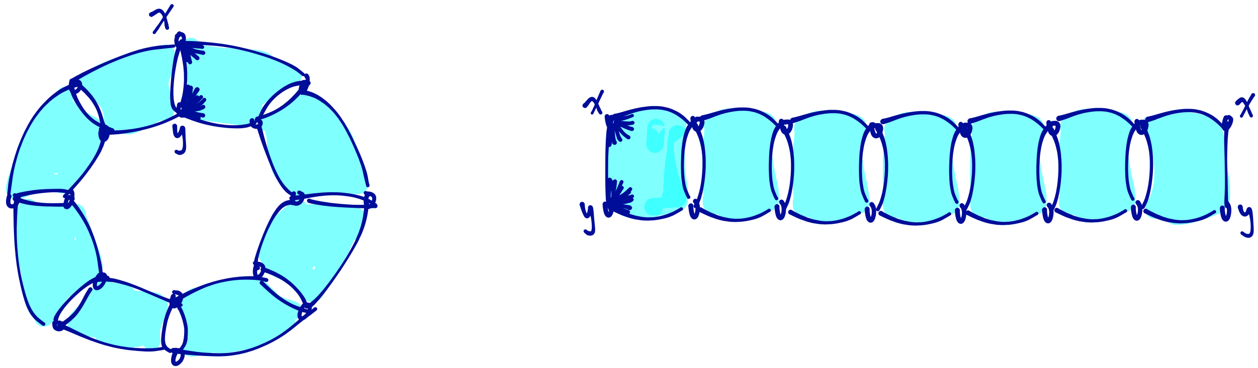

Belts

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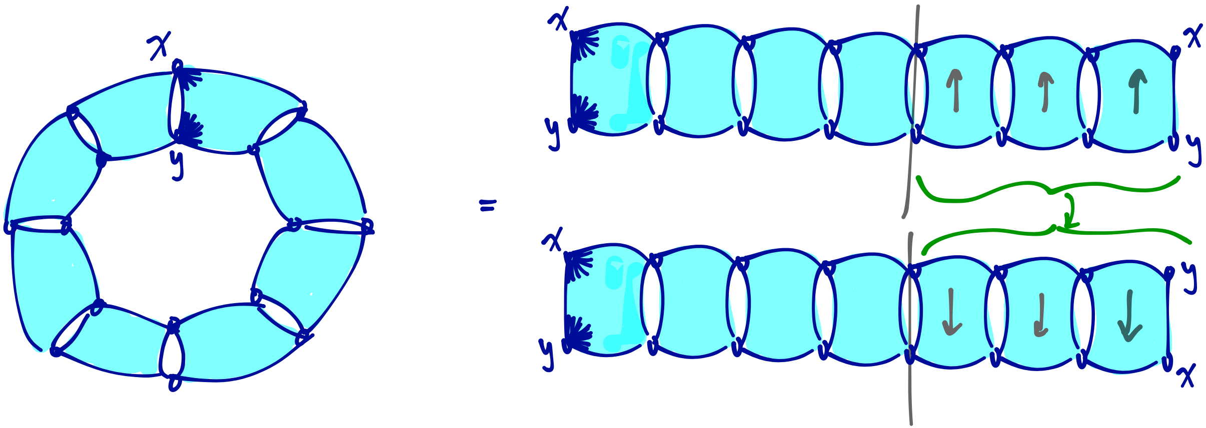

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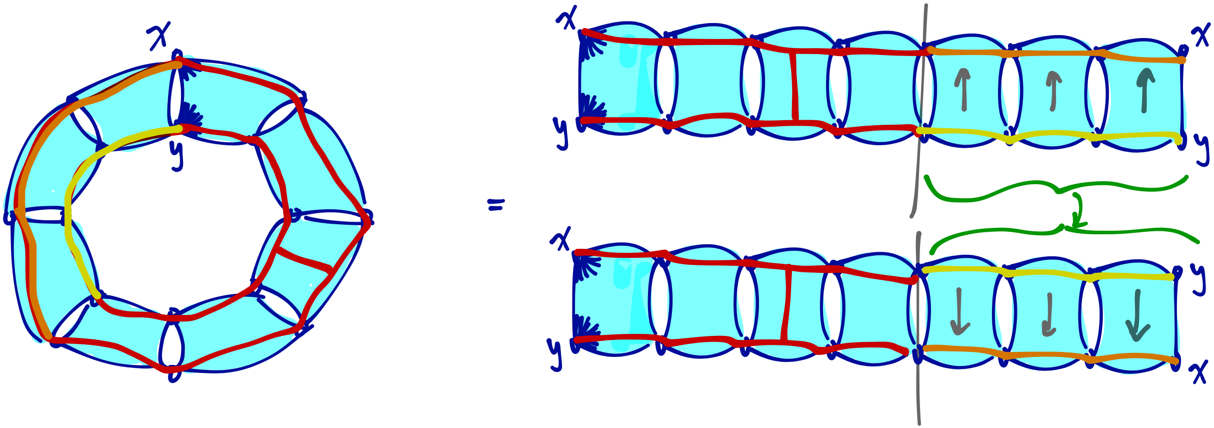

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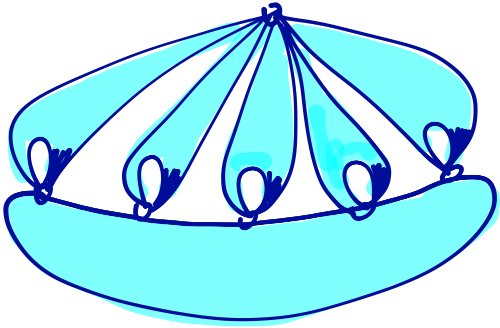

Twisted flips

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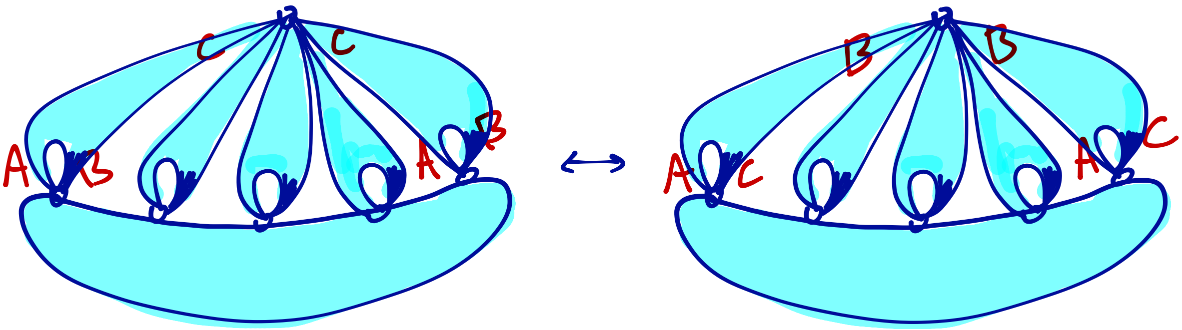

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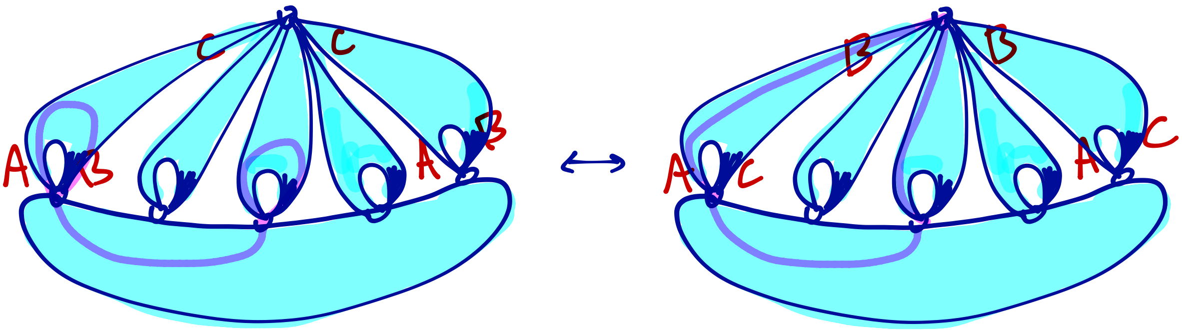
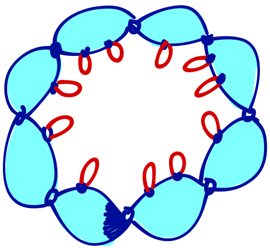


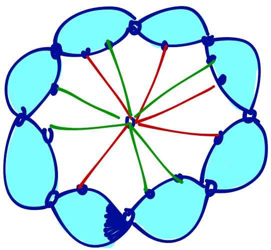
Twisted flips

3-biconnected: no 2-separation with one side balanced

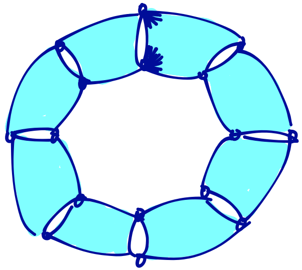
<u>Thm</u> (F. and Mayhew, '23+). Let G be a biased graph with $F(G)$ 3-connected and non-graphic. There is an integer k such that if G has more than k uncommitted vertices then G is in one of these infinite families.



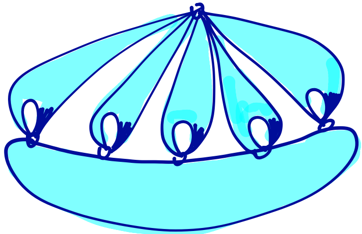
bangles



buckles



belts



twisted flips

Excluded Minors

<u>Thm</u> (F. and Mayhew, '18). For each positive integer r, there are only a finite number of excluded minors of rank r for the class of frame matroids.

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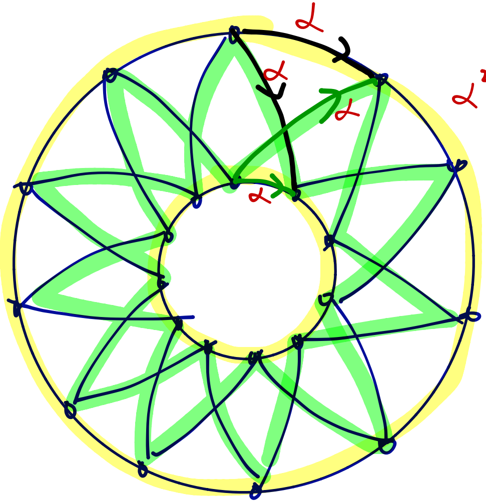
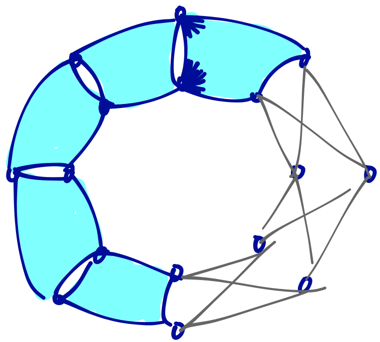
<u>Thm</u> (Chen and Geelen, '18). There is an infinite set of excluded minors for the class of frame matroids.

- Chen and Geelen exhibit an infinite family of excluded minors

The Chen-Geelen excluded minors are belts

- having no committed vertices





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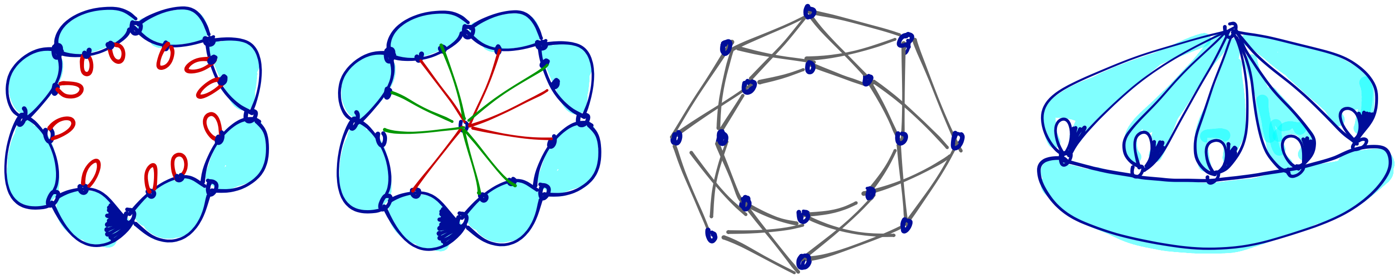


<u>Conjecture</u>. There is an integer k such that every excluded minor of rank > k for the class of frame matroids is a bangle, buckle, or belt.

Let N be an excluded minor for the class of frame matroids, of rank $> k^k$!

- N is connected, simple, cosimple, and if not 3-connected then $N = N' \oplus_2 U_{2,4}$ where N' is 3-connected.

  (F., DeVos, Pivotto, '16)

- N has an element e such that $N \backslash e$ is 3-connected up to series classes, represented by a 2-connected graph (or N' does).

Let G be a 2-connected biased graph representing $N \backslash e$.

A twin for N w.r.t. $X \subseteq E$

- Let M, N be matroids on common ground set E
- M and N are twins w.r.t. $X \subseteq E$ if

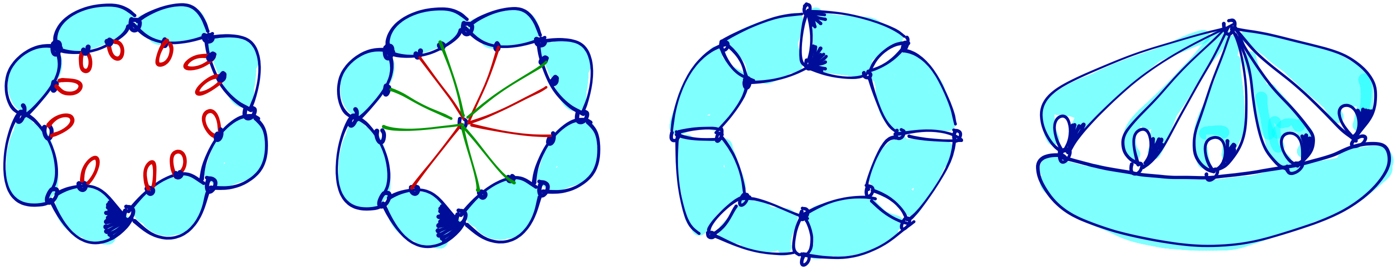
$$M/x = N/x \quad \forall x \in X$$

A twin for N w.r.t. $X \subseteq E$

- G is a frame representation for $N \backslash e$.
- build a twin for N by "putting e back":
  - find a set $X \subseteq E$ for which $\forall Z \subseteq X$ representations of $N \backslash e / Z = F(G/Z)$ are understood
  - $\forall Z \subseteq X$ let H_Z be representation for N/Z
  - G/Z and $H_Z \backslash e$ both represent $N \backslash e / Z$
  - Let H be biased graph obtained from G by adding e according to H_Z's

$M = F(H)$ is our twin

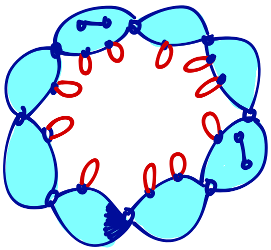
N\e



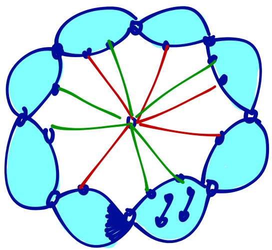
bangles   buckles   belts   twisted flips

- If G has ==many uncommitted vertices==, then G is a bangle, buckle, belt or twisted flip.

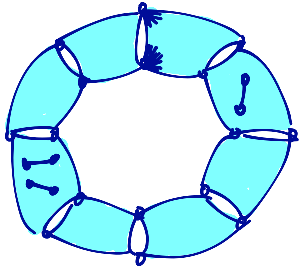
Either
- find a bag or two containing many committed vertices, or
- find that N contains an excluded minor, a contradiction.



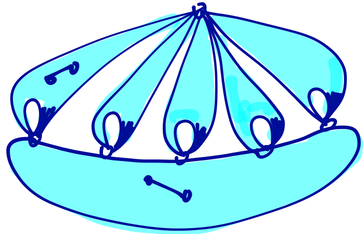
bangles



buckles



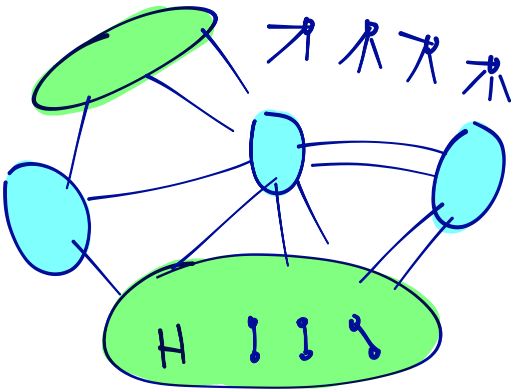
belts



twisted flips

- If G has many uncommitted vertices, then G is a bangle, buckle, belt or twisted flip.
    - find a bag or two containing many committed vertices

- Find a set of edges X for which $N \backslash e / Z$ have only understood, controlled representations $\forall\, Z \subseteq X$
    - build a frame twin for N



- If G has few uncommitted vertices, find a large sufficiently connected biased subgraph H representing a restriction of N whose representation is fixed in every representation of $N \setminus e$.

  → contract edges while controlling representations

  → build a frame twin for N

Thm (F. + Mayhew, '23+).

Let N, M be matroids on common ground set E.

Let $X \subseteq E$ such that

(i) N and M are twins r.w.t. X,

(ii) $\forall Z \subseteq X$, $N/Z = M/Z$ is vertically 3-connected,

(iii) $M = F(G)$ and $|V_G(X)| \geq 5$.

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Then $\exists Y$ s.t. $r_N(Y) \neq r_M(Y)$, Y is a circuit of N that forms a pair of disjoint cycles in G.

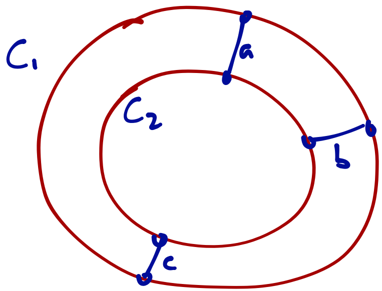
Moreover, in G each element in X has one end in each of these cycles.

A twin for N w.r.t. $X \subseteq E$

Thm (F. + Mayhew, '23+).

Twin $F(H)$ for excluded minor N

$X = \{a, b, c\}$

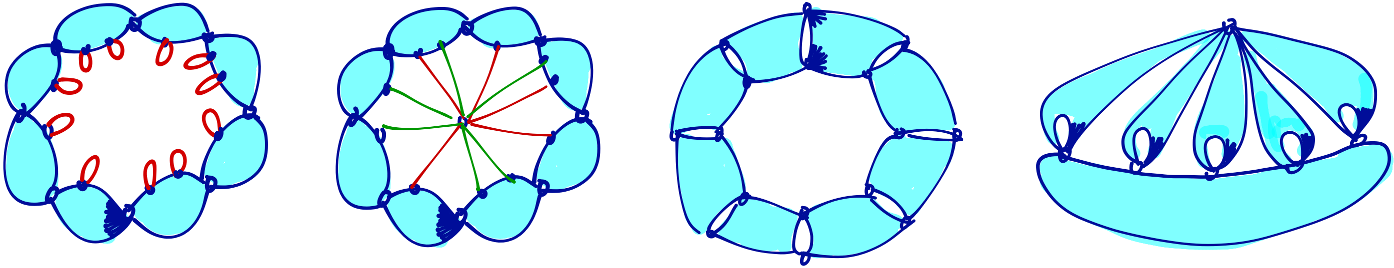


$C_1 \cup C_2 \in \mathcal{C}(N)$

$C_1 \cup C_2 \in \mathcal{I}(M)$

The Chen-Geelen excluded minors are belts

• having no committed vertices



bangles buckles belts twisted flips

<u>Conjecture</u>. There is an integer k such that every excluded minor of rank > k for the class of frame matroids is a bangle, buckle, or belt.

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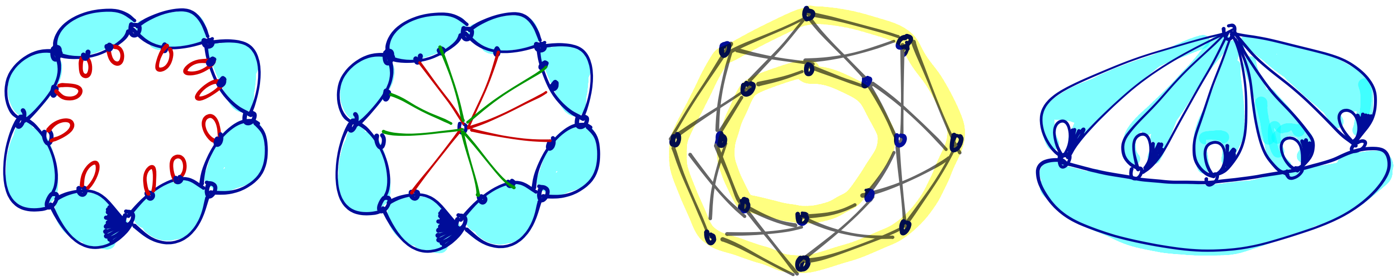

bangles buckles belts twisted flips

<u>Conjecture</u>. There is an integer k such that every excluded minor of rank > k for the class of frame matroids is in the Chen-Geelen family.

Rota's Conjecture for gain-graphs over a fixed finite group

<u>Conjecture</u> (Rota for fixed finite groups).

Let Γ be a finite group. The set of excluded minors for Γ-gain frame matroids is finite.

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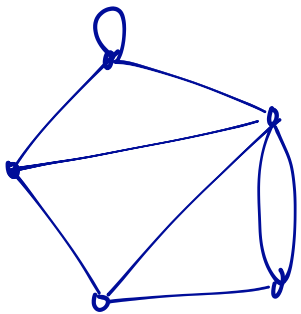

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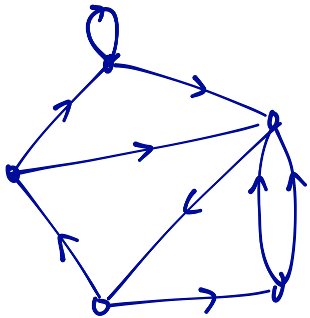

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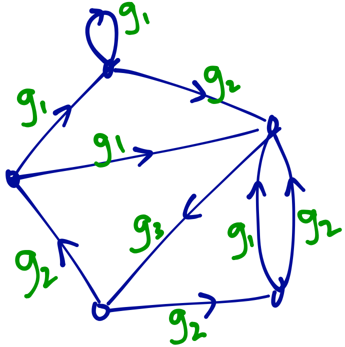

- Let $\gamma : \vec{E}(G) \to \Gamma$ be a gain function.

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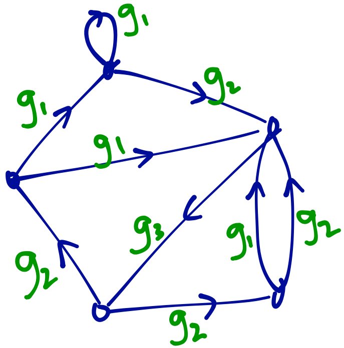
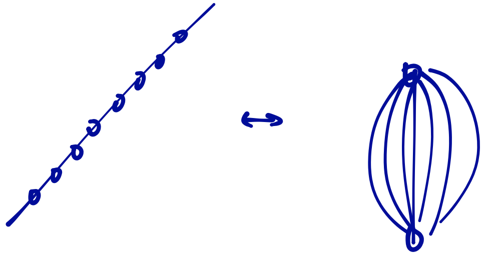


- Let $\gamma : \vec{E}(G) \to \Gamma$ be a gain function.
- Set $\mathcal{B}_\gamma = \{ C : \gamma(C) = \mathrm{id.} \}$
- $(G, \mathcal{B}_\gamma)$ is a biased graph
- $M = F(G, \mathcal{B}_\gamma)$ is a frame matroid.

Thm (F. and Mayhew, '18). For each positive integer r, there are only a finite number of excluded minors of rank r for the class of frame matroids.

Proof.



- because an excluded minor does not have arbitrarily long lines

Observation. For each positive integer r, there are only a finite number of excluded minors of rank r for the class of gain-graphic matroids over a fixed finite group.

$\rightarrow$ an excluded minor does not have arbitrarily long lines