Douglas College Physics 1107

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OpenStax and Department of Physics and Astronomy at Douglas College

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Introduction to Open Textbooks at Douglas College

Territorial Acknowledgment

We recognize and acknowledge the QayQayt (Ki-Kite) and the Kwikwetlem First Nation, as well as all Coast Salish Peoples, on whose traditional and unceded territories we live, we learn, we play, and we do our work.

The welcome Pole at Douglas College. She faces up river and her arms are raised in welcome. Photo credit: Jennifer Kirkey CC0

Open Textbooks

As stated on the BCcampus website, https://open.bccampus.ca/open-textbook-101/, An open textbook is a textbook licensed under an open copyright license and made available online to be freely used by students, teachers, and members of the public. They are available for free as online versions and in a variety of file formats (e.g., for eReaders, editable files like XML and HTML), and as low-cost printed versions, should students or faculty opt for these. Open textbooks are a way to significantly reduce student textbook costs while giving instructors the flexibility to reformat and customize their course material. They are an affordable, flexible alternative to traditionally-published textbooks.

Jennifer Kirkey and Open Textbooks

Looking up at the sky as a child, I was fascinated by the stars. The local library supplied a book on the constellations, and that led to a book explaining why the stars were different colours, and that led to a book on physics. I kept asking questions and that led me to a Bachelor of Science degree in Physics from Trent University in Peterborough, Ontario, Canada and then to a Master's degree in Physics from Simon Fraser University in Burnaby, British Columbia, Canada.

I have been working at Douglas College in New Westminster, British Columbia, Canada for more than twenty-five years. I teach a first year course in astronomy to liberal arts majors and physics to people whose background is no physics in high school to those who want to be engineers. I also teach in a post-graduate program for Elementary School Teachers.

I volunteer with our local science centre, Science World at Telus World of Science in Vancouver, B.C., doing outreach visits to elementary and secondary schools, as well as being active with the local branch of the Royal Astronomical Society of Canada.

Jennifer Kirkey and the 8 inch Celestron Telescope at Douglas College Photo credit: Jennifer Kirkey June 2015 (CC-BY license)

Open textbooks became a cause of mine as I found more and more of my students could not afford the "required" textbook. BCcampus organized an Open Textbook Summit in 2015 and the keynote speaker, Rajiv Jhangiani from Kwantlen Polytechnic University was so inspiring that I became a BCcampus Faculty Fellow as an Advocate for Open Textbooks in 2016. I do wish to thank BCcampus for their work with open educational resources such as this textbook, and for the opportunities they have given me. You can learn more about them by visiting https://bccampus.ca/

Being able to modify a textbook so that it better matched our courses was the next logical step. This book is the start of the process.

Here is a picture I took at the start of the semester in September 2016. This bulletin board is located outside our college's bookstore. Students who clearly did not need their old textbooks, or who needed the money to buy new ones, put up posters in an attempt to sell their old books. I do hope that you find this textbook useful and worth keeping. Please tell us what worked for you in this book, and what did not. We have the legal ability to change the book and your feedback will help us.

Cork board outside Douglas College Bookstore. Most of those posters are students offering their textbooks for sale. Photo credit: Jennifer Kirkey September 2016 (CC-BY license)

Preface to Open Stax College Physics

This textbook is based on the Open Stax College Physics textbook, which can can be downloaded for free at https://openstax.org/details/college-physics

About OpenStax

OpenStax is a non-profit organization committed to improving student access to quality learning materials. Our free textbooks are developed and peer-reviewed by educators to ensure they are readable, accurate, and meet the scope and sequence requirements of modern college courses. Unlike traditional textbooks, OpenStax resources live online and are owned by the community of educators using them. Through our partnerships with companies and foundations committed to reducing costs for students, OpenStax is working to improve access to higher education for all. OpenStax is an initiative of Rice University and is made possible through the generous support of several philanthropic foundations.

About This Book

Welcome to *College Physics*, an OpenStax resource created with several goals in mind: accessibility, affordability, customization, and student engagement—all while encouraging learners toward high levels of learning. Instructors and students alike will find that this textbook offers a strong foundation in introductory physics, with algebra as a prerequisite. It is available for free online and in low-cost print and e-book editions.

To broaden access and encourage community curation, College Physics is “open source” licensed under a Creative Commons Attribution (CC-BY) license. Everyone is invited to submit examples, emerging research, and other feedback to enhance and strengthen the material and keep it current and relevant for today’s students. You can make suggestions by contacting us at info@openstaxcollege.org.

To the Student

This book is written for you. It is based on the teaching and research experience of numerous physicists and influenced by a strong recollection of their own struggles as students. After reading this book, we hope you see that physics is visible everywhere. Applications range from driving a car to launching a rocket, from a skater whirling on ice to a neutron star spinning in space, and from taking your temperature to taking a chest X-ray.

General Approach

College Physics is organized such that topics are introduced conceptually with a steady progression to precise definitions and analytical applications. The analytical aspect (problem solving) is tied back to the conceptual before moving on to another topic. Each introductory chapter, for example, opens with an engaging photograph relevant to the subject of the chapter and interesting applications that are easy for most students to visualize.

Concepts and Calculations

The ability to calculate does not guarantee conceptual understanding. In order to unify conceptual, analytical, and calculation skills within the learning process, we have integrated Strategies and Discussions throughout the text.

Supplements

Accompanying the main text are a Student Solutions Manual and an Instructor Solutions Manual. The Student Solutions Manual provides worked-out solutions to select end-of-module Problems and Exercises. The Instructor Solutions Manual provides worked-out solutions to all Exercises.

Features of OpenStax *College Physics*

The following briefly describes the special features of this text.

Learning Objectives

Every module begins with a set of learning objectives. These objectives are designed to guide the instructor in deciding what content to include or assign, and to guide the student with respect to what he or she can expect to learn. After completing the module and end-of-module exercises, students should be able to demonstrate mastery of the learning objectives.

Call-Outs

Key definitions, concepts, and equations are called out with a special design treatment. Call-outs are designed to catch readers' attention, to make it clear that a specific term, concept, or equation is particularly important, and to provide easy reference for a student reviewing content.

Key Terms

Key terms are in bold and are followed by a definition in context. Definitions of key terms are also listed in the Glossary, which appears at the end of the module.

Worked Examples

Worked examples have four distinct parts to promote both analytical and conceptual skills. Worked examples are introduced in words, always using some application that should be of interest. This is followed by a Strategy section that emphasizes the concepts involved and how solving the problem relates to those concepts. This is followed by the mathematical Solution and Discussion.

Many worked examples contain multiple-part problems to help the students learn how to approach normal situations, in which problems tend to have multiple parts. Finally, worked examples employ the techniques of the problem-solving strategies so that students can see how those strategies succeed in practice as well as in theory.

Problem-Solving Strategies

Problem-solving strategies are first presented in a special section and subsequently appear at crucial points in the text where students can benefit most from them. Problem-solving strategies have a logical structure that is reinforced in the worked examples and supported in certain places by line drawings that illustrate various steps.

Misconception Alerts

Students come to physics with preconceptions from everyday experiences and from previous courses. Some of these preconceptions are misconceptions, and many are very common among students and the general public. Some are inadvertently picked up through misunderstandings of lectures and texts. The Misconception Alerts feature is designed to point these out and correct them explicitly.

Take-Home Investigations

Take Home Investigations provide the opportunity for students to apply or explore what they have learned with a hands-on activity.

Things Great and Small

In these special topic essays, macroscopic phenomena (such as air pressure) are explained with submicroscopic phenomena (such as atoms bouncing off walls). These essays support the modern perspective by describing aspects of modern physics before they are formally treated in later chapters. Connections are also made between apparently disparate phenomena.

Simulations

Where applicable, students are directed to the interactive PHeT physics simulations developed by the University of Colorado (http://phet.colorado.edu). There they can further explore the physics concepts they have learned about in the module.

Summary

Module summaries are thorough and functional and present all important definitions and equations. Students are able to find the definitions of all terms and symbols as well as their physical relationships. The structure of the summary makes plain the fundamental principles of the module or collection and serves as a useful study guide.

Glossary

At the end of every module or chapter is a glossary containing definitions of all of the key terms in the module or chapter.

End-of-Module Problems

At the end of every chapter is a set of Conceptual Questions and/or skills-based Problems & Exercises. Conceptual Questions challenge students' ability to explain what they have learned conceptually, independent of the mathematical details. Problems & Exercises challenge students to apply both concepts and skills to solve mathematical physics problems. Online, every other problem includes an answer that students can reveal immediately by clicking on a "Show Solution" button. Fully worked solutions to select problems are available in the Student Solutions Manual and the Teacher Solutions Manual.

In addition to traditional skills-based problems, there are three special types of end-of-module problems: Integrated Concept Problems, Unreasonable Results Problems, and Construct Your Own Problems. All of these problems are indicated with a subtitle preceding the problem.

Integrated Concept Problems

In Integrated Concept Problems, students are asked to apply what they have learned about two or more concepts to arrive at a solution to a problem. These problems require a higher level of thinking because, before solving a problem, students have to recognize the combination of strategies required to solve it.

Unreasonable Results

In Unreasonable Results Problems, students are challenged to not only apply concepts and skills to solve a problem, but also to analyze the answer with respect to how likely or realistic it really is. These problems contain a premise that produces an unreasonable answer and are designed to further emphasize that properly applied physics must describe nature accurately and is not simply the process of solving equations.

Construct Your Own Problem

These problems require students to construct the details of a problem, justify their starting assumptions, show specific steps in the problem's solution, and finally discuss the meaning of the result. These types

of problems relate well to both conceptual and analytical aspects of physics, emphasizing that physics must describe nature. Often they involve an integration of topics from more than one chapter. Unlike other problems, solutions are not provided since there is no single correct answer. Instructors should feel free to direct students regarding the level and scope of their considerations. Whether the problem is solved and described correctly will depend on initial assumptions.

Acknowledgements

This text is based on the work completed by Dr. Paul Peter Urone in collaboration with Roger Hinrichs, Kim Dirks, and Manjula Sharma. We would like to thank the authors as well as the numerous professors (a partial list follows) who have contributed their time and energy to review and provide feedback on the manuscript. Their input has been critical in maintaining the pedagogical integrity and accuracy of the text.

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Chapter 1 The Nature of Science and Physics

Galaxies are as immense as atoms are small. Yet the same laws of physics describe both, and all the rest of nature—an indication of the underlying unity in the universe. The laws of physics are surprisingly few in number, implying an underlying simplicity to nature's apparent complexity. (credit: NASA, JPL-Caltech, P. Barmby, Harvard-Smithsonian Center for Astrophysics)

What is your first reaction when you hear the word "physics"? Did you imagine working through difficult equations or memorizing formulas that seem to have no real use in life outside the physics classroom? Many people come to the subject of physics with a bit of fear. But as you begin your exploration of this broad-ranging subject, you may soon come to realize that physics plays a much larger role in your life than you first thought, no matter your life goals or career choice.

For example, take a look at the image above. This image is of the Andromeda Galaxy, which contains billions of individual stars, huge clouds of gas, and dust. Two smaller galaxies are also visible as bright blue spots in the background. At a staggering 2.5 million light years from the Earth, this galaxy is the nearest one to our own galaxy (which is called the Milky Way). The stars and planets that make up Andromeda might seem to be the furthest thing from most people's regular, everyday lives. But Andromeda is a great starting point to think about the forces that hold together the universe. The forces that cause Andromeda to act as it does are the same forces we contend with here on Earth, whether we are planning to send a rocket into space or simply raise the walls for a new home. The same gravity that causes the stars of Andromeda to rotate and revolve also causes water to flow over hydroelectric dams here on Earth. Tonight, take a moment to look up at the stars. The forces out there are the same as the ones here on Earth. Through a study of physics, you may gain a greater understanding of the interconnectedness of everything we can see and know in this universe.

Think now about all of the technological devices that you use on a regular basis. Computers, smart phones, GPS systems, MP3 players, and satellite radio might come to mind. Next, think about the most exciting modern technologies that you have heard about in the news, such as trains that levitate above tracks, "invisibility cloaks" that bend light around them, and microscopic robots that fight cancer cells in our bodies. All of these groundbreaking advancements, commonplace or unbelievable, rely on

the principles of physics. Aside from playing a significant role in technology, professionals such as engineers, pilots, physicians, physical therapists, electricians, and computer programmers apply physics concepts in their daily work. For example, a pilot must understand how wind forces affect a flight path and a physical therapist must understand how the muscles in the body experience forces as they move and bend. As you will learn in this text, physics principles are propelling new, exciting technologies, and these principles are applied in a wide range of careers.

In this text, you will begin to explore the history of the formal study of physics, beginning with natural philosophy and the ancient Greeks, and leading up through a review of Sir Isaac Newton and the laws of physics that bear his name. You will also be introduced to the standards scientists use when they study physical quantities and the interrelated system of measurements most of the scientific community uses to communicate in a single mathematical language. Finally, you will study the limits of our ability to be accurate and precise, and the reasons scientists go to painstaking lengths to be as clear as possible regarding their own limitations.

1.1 Physics: An Introduction

Summary

- Explain the difference between a principle and a law.
- Explain the difference between a model and theory.

Figure 1. *The flight formations of migratory birds such as Canada geese are governed by the laws of physics. (credit: David Merrett).*

The physical universe is enormously complex in its detail. Every day, each of us observes a great variety of objects and phenomena. Over the centuries, the curiosity of the human race has led us collectively to explore and catalog a tremendous wealth of information. From the flight of birds to the colors of flowers, from lightning to gravity, from quarks to clusters of galaxies, from the flow of time to the mystery of the creation of the universe, we have asked questions and assembled huge arrays of facts. In the face of all these details, we have discovered that a surprisingly small and unified set of physical laws can explain what we observe. As humans, we make generalizations and seek order. We have found that nature is remarkably cooperative—it exhibits the *underlying order and simplicity* we so value.

It is the underlying order of nature that makes science in general, and physics in particular, so enjoyable to study. For example, what do a bag of chips and a car battery have in common? Both contain energy that can be converted to other forms. The law of conservation of energy (which says that energy can change form but is never lost) ties together such topics as food calories, batteries, heat, light, and watch springs. Understanding this law makes it easier to learn about the various forms energy takes and how they relate to one another. Apparently unrelated topics are connected through broadly applicable physical laws, permitting an understanding beyond just the memorization of lists of facts.

The unifying aspect of physical laws and the basic simplicity of nature form the underlying themes of this text. In learning to apply these laws, you will, of course, study the most important topics in physics. More importantly, you will gain analytical abilities that will enable you to apply these laws

far beyond the scope of what can be included in a single book. These analytical skills will help you to excel academically, and they will also help you to think critically in any professional career you choose to pursue. This module discusses the realm of physics (to define what physics is), some applications of physics (to illustrate its relevance to other disciplines), and more precisely what constitutes a physical law (to illuminate the importance of experimentation to theory).

Science and the Realm of Physics

Science consists of the theories and laws that are the general truths of nature as well as the body of knowledge they encompass. Scientists are continually trying to expand this body of knowledge and to perfect the expression of the laws that describe it. **Physics** is concerned with describing the interactions of energy, matter, space, and time, and it is especially interested in what fundamental mechanisms underlie every phenomenon. The concern for describing the basic phenomena in nature essentially defines the *realm of physics*.

Physics aims to describe the function of everything around us, from the movement of tiny charged particles to the motion of people, cars, and spaceships. In fact, almost everything around you can be described quite accurately by the laws of physics. Consider a smart phone (Figure 2). Physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and circuit layout when building the smart phone. Next, consider a GPS system. Physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS device in a vehicle, it utilizes these physics equations to determine the travel time from one location to another.

Figure 2. *The Apple "iPhone" is a common smart phone with a GPS function. Physics describes the way that electricity flows through the circuits of this device. Engineers use their knowledge of physics to construct an iPhone with features that consumers will enjoy. One specific feature of an iPhone is the GPS function. GPS uses physics equations to determine the driving time between two locations on a map. (credit: @gletham GIS, Social, Mobile Tech Images).*

Applications of Physics

You need not be a scientist to use physics. On the contrary, knowledge of physics is useful in everyday situations as well as in nonscientific professions. It can help you understand how microwave ovens work, why metals should not be put into them, and why they might affect pacemakers. (See Figure 3 and Figure 4.) Physics allows you to understand the hazards of radiation and rationally evaluate these hazards more easily. Physics also explains the reason why a black car radiator helps remove heat in a car engine, and it explains why a white roof helps keep the inside of a house cool. Similarly, the operation of a car's ignition system as well as the transmission of electrical signals through our body's nervous system are much easier to understand when you think about them in terms of basic physics.

Physics is the foundation of many important disciplines and contributes directly to others. Chemistry, for example—since it deals with the interactions of atoms and molecules—is rooted in atomic and molecular

physics. Most branches of engineering are applied physics. In architecture, physics is at the heart of structural stability, and is involved in the acoustics, heating, lighting, and cooling of buildings. Parts of geology rely heavily on physics, such as radioactive dating of rocks, earthquake analysis, and heat transfer in the Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.

Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cell walls and cell membranes (Figure 5 and Figure 6). On the macroscopic level, it can explain the heat, work, and power associated with the human body. Physics is involved in medical diagnostics, such as x-rays, magnetic resonance imaging (MRI), and ultrasonic blood flow measurements. Medical therapy sometimes directly involves physics; for example, cancer radiotherapy uses ionizing radiation. Physics can also explain sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers can transmit information.

It is not necessary to formally study all applications of physics. What is most useful is knowledge of the basic laws of physics and a skill in the analytical methods for applying them. The study of physics also can improve your problem-solving skills. Furthermore, physics has retained the most basic aspects of science, so it is used by all of the sciences, and the study of physics makes other sciences easier to understand.

Figure 3. *The laws of physics help us understand how common appliances work. For example, the laws of physics can help explain how microwave ovens heat up food, and they also help us understand why it is dangerous to place metal objects in a microwave oven. (credit: MoneyBlogNewz).*

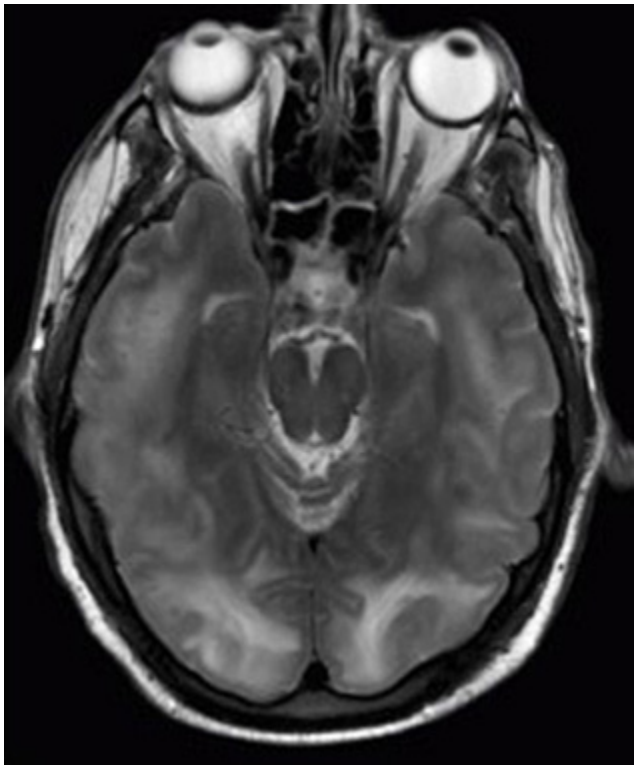


Figure 4. *These two applications of physics have more in common than meets the eye. Microwave ovens use electromagnetic waves to heat food. Magnetic resonance imaging (MRI) also uses electromagnetic waves to yield an image of the brain, from which the exact location of tumors can be determined. (credit: Rashmi Chawla, Daniel Smith, and Paul E. Marik).*

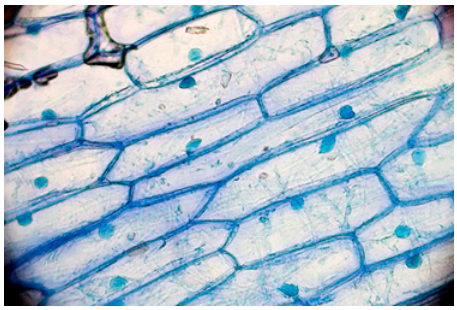


Figure 5. *Physics, chemistry, and biology help describe the properties of cell walls in plant cells, such as the onion cells seen here. (credit: Umberto Salvagnin).*

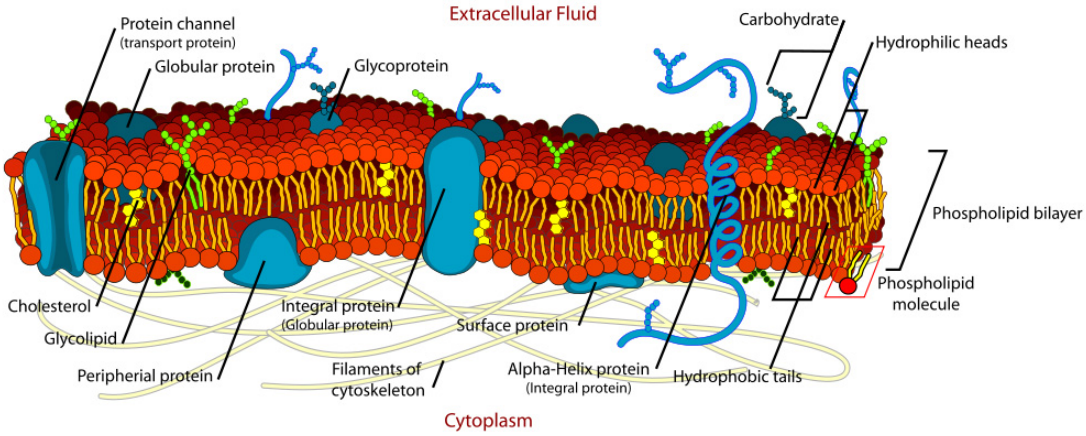


Figure 6. *An artist's rendition of the the structure of a cell membrane. Membranes form the boundaries of animal cells and are complex in structure and function. Many of the most fundamental properties of life, such as the firing of nerve cells, are related to membranes. The disciplines of biology, chemistry, and physics all help us understand the membranes of animal cells. (credit: Mariana Ruiz).*

Models, Theories, and Laws; The Role of Experimentation

The laws of nature are concise descriptions of the universe around us; they are human statements of the underlying laws or rules that all natural processes follow. Such laws are intrinsic to the universe; humans did not create them and so cannot change them. We can only discover and understand them. Their discovery is a very human endeavour, with all the elements of mystery, imagination, struggle, triumph, and disappointment inherent in any creative effort. (See Figure 7 and Figure 8.) The cornerstone of discovering natural laws is observation; science must describe the universe as it is, not as we may imagine it to be.

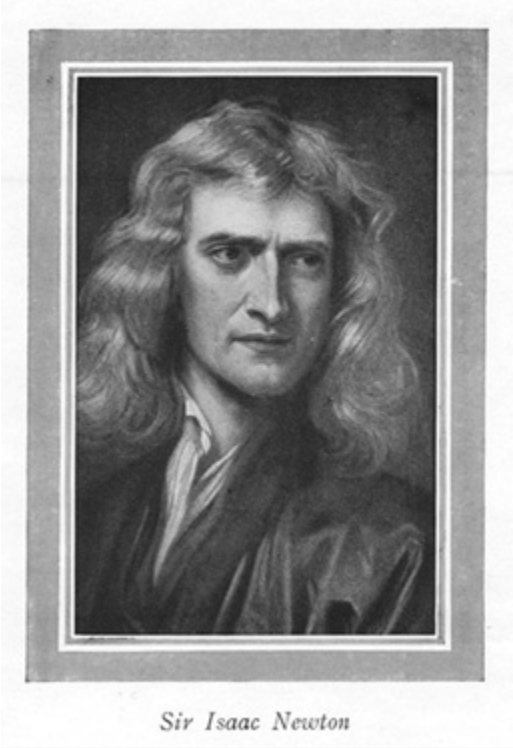


Figure 7. *Isaac Newton (1642–1727) was very reluctant to publish his revolutionary work and had to be convinced to do so. In his later years, he stepped down from his academic post and became exchequer of the Royal Mint. He took this post seriously, inventing reeding (or creating ridges) on the edge of coins to prevent unscrupulous people from trimming the silver off of them before using them as currency. (credit: Arthur Shuster and Arthur E. Shipley: Britain's Heritage of Science. London, 1917.).*

Figure 8. *Marie Curie (1867–1934) sacrificed monetary assets to help finance her early research and damaged her physical well-being with radiation exposure. She is the only person to win Nobel prizes in both physics and chemistry. One of her daughters also won a Nobel Prize. (credit: Wikimedia Commons).*

We all are curious to some extent. We look around, make generalizations, and try to understand what we see—for example, we look up and wonder whether one type of cloud signals an oncoming storm. As we become serious about exploring nature, we become more organized and formal in collecting and analyzing data. We attempt greater precision, perform controlled experiments (if we can), and write down ideas about how the data may be organized and unified. We then formulate models, theories, and laws based on the data we have collected and analyzed to generalize and communicate the results of these experiments.

A **model** is a representation of something that is often too difficult (or impossible) to display directly. While a model is justified with experimental proof, it is only accurate under limited situations. An example is the planetary model of the atom in which electrons are pictured as orbiting the nucleus, analogous to the way planets orbit the Sun. (See Figure 9.) We cannot observe electron orbits directly, but the mental image helps explain the observations we can make, such as the emission of light from hot gases (atomic spectra). Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation, or they can be used to represent a situation in the form of a computer simulation. A **theory** is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers. Some theories include models to help visualize phenomena, whereas others do not. Newton’s theory of gravity, for example,

does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, is a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what our instruments tell us about the behaviour of gases.

A **law** uses concise language to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the designation *law* is reserved for a concise and very general statement that describes phenomena in nature, such as the law that energy is conserved during any process, or Newton's second law of motion, which relates force, mass, and acceleration by the simple equation $F = ma$. A theory, in contrast, is a less concise statement of observed phenomena. For example, the Theory of Evolution and the Theory of Relativity cannot be expressed concisely enough to be considered a law. The biggest difference between a law and a theory is that a theory is much more complex and dynamic. A law describes a single action, whereas a theory explains an entire group of related phenomena. And, whereas a law is a postulate that forms the foundation of the scientific method, a theory is the end result of that process.

Less broadly applicable statements are usually called principles (such as Pascal's principle, which is applicable only in fluids), but the distinction between laws and principles often is not carefully made.

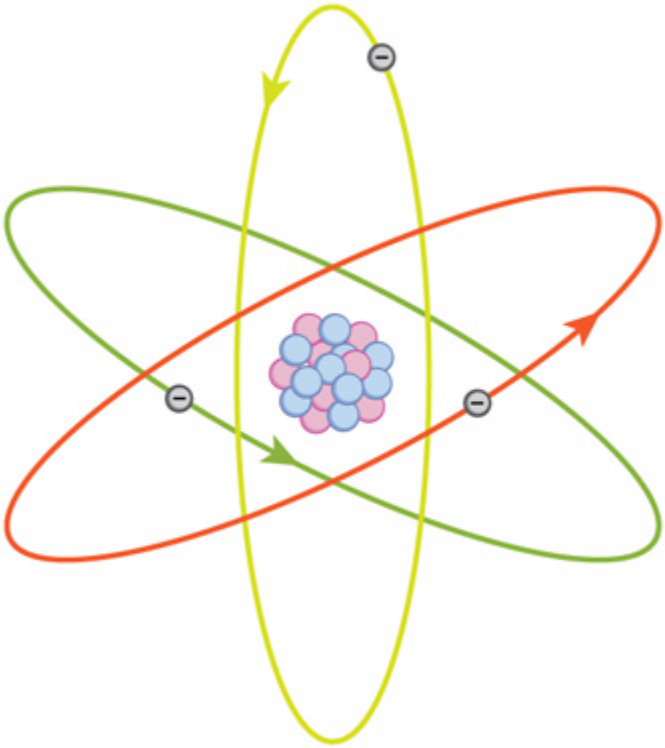


Figure 9. *What is a model? This planetary model of the atom shows electrons orbiting the nucleus. It is a drawing that we use to form a mental image of the atom that we cannot see directly with our eyes because it is too small.*

MODELS, THEORIES, AND LAWS

Models, theories, and laws are used to help scientists analyze the data they have already collected. However, often after a model, theory, or law has been developed, it points scientists toward new discoveries they would not otherwise have made.

The models, theories, and laws we devise sometimes *imply the existence of objects or phenomena as yet unobserved.* These predictions are remarkable triumphs and tributes to the power of science. It is the underlying order in the universe that enables scientists to make such spectacular predictions. However, if *experiment* does not verify our predictions, then the theory or law is wrong, no matter how elegant or convenient it is. Laws can never be known with absolute certainty because it is impossible to perform every imaginable experiment in order to confirm a law in every possible scenario. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable experiment contradicts a well-established law, then the law must be modified or overthrown completely.

The study of science in general and physics in particular is an adventure much like the exploration of uncharted ocean. Discoveries are made; models, theories, and laws are formulated; and the beauty of the physical universe is made more sublime for the insights gained.

THE SCIENTIFIC METHOD

As scientists inquire and gather information about the world, they follow a process called the **scientific method**. This process typically begins with an observation and question that the scientist will research. Next, the scientist typically performs some research about the topic and then devises a hypothesis. Then, the scientist will test the hypothesis by performing an experiment. Finally, the scientist analyzes the results of the experiment and draws a conclusion. Note that the scientific method can be applied to many situations that are not limited to science, and this method can be modified to suit the situation.

Consider an example. Let us say that you try to turn on your car, but it will not start. You undoubtedly wonder: Why will the car not start? You can follow a scientific method to answer this question. First off, you may perform some research to determine a variety of reasons why the car will not start. Next, you will state a hypothesis. For example, you may believe that the car is not starting because it has no engine oil. To test this, you open the hood of the car and examine the oil level. You observe that the oil is at an acceptable level, and you thus conclude that the oil level is not contributing to your car issue. To troubleshoot the issue further, you may devise a new hypothesis to test and then repeat the process again.

The Evolution of Natural Philosophy into Modern Physics

Physics was not always a separate and distinct discipline. It remains connected to other sciences to this day. The word *physics* comes from Greek, meaning nature. The study of nature came to be called "natural philosophy." From ancient times through the Renaissance, natural philosophy encompassed many fields, including astronomy, biology, chemistry, physics, mathematics, and medicine. Over the last few centuries, the growth of knowledge has resulted in ever-increasing specialization and branching of natural philosophy into separate fields, with physics retaining the most basic facets. (See Figure 10, Figure 11, and Figure 12.) Physics as it developed from the Renaissance to the end of the 19th century is called **classical physics**. It was transformed into modern physics by revolutionary discoveries made starting at the beginning of the 20th century.

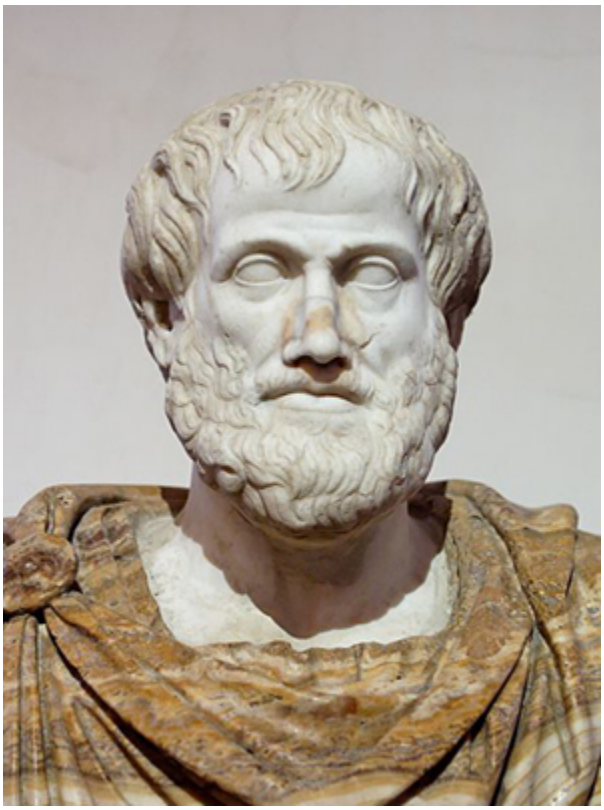


Figure 10. *Over the centuries, natural philosophy has evolved into more specialized disciplines, as illustrated by the contributions of some of the greatest minds in history. The Greek philosopher Aristotle (384–322 B.C.) wrote on a broad range of topics including physics, animals, the soul, politics, and poetry. (credit: Jastrow (2006)/Ludovisi Collection).*

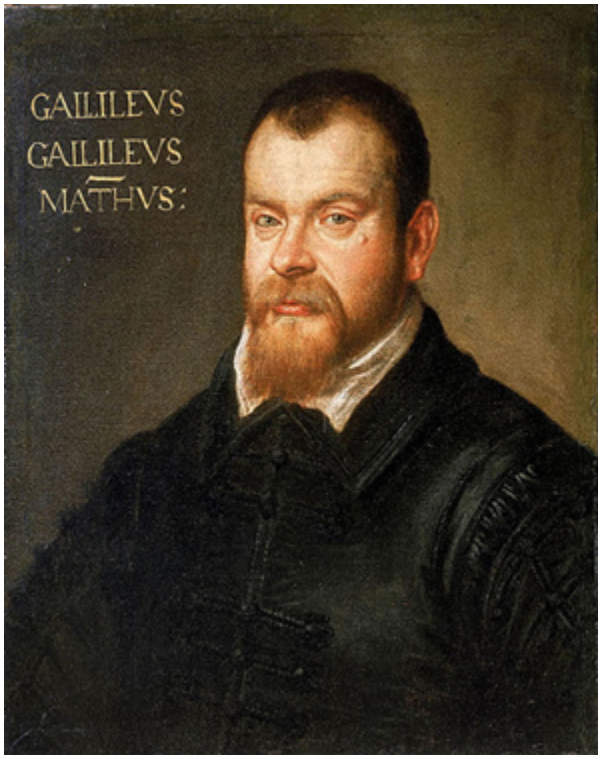


Figure 11. *Galileo Galilei (1564–1642) laid the foundation of modern experimentation and made contributions in mathematics, physics, and astronomy. (credit: Domenico Tintoretto).*

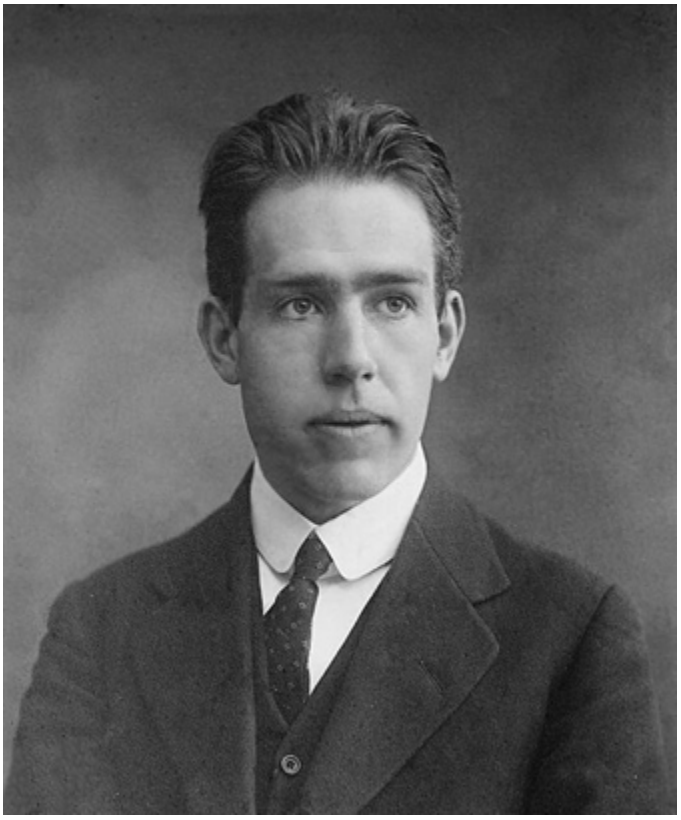


Figure 12. *Niels Bohr (1885–1962) made fundamental contributions to the development of quantum mechanics, one part of modern physics. (credit: United States Library of Congress Prints and Photographs Division).*

Classical physics is not an exact description of the universe, but it is an excellent approximation under the following conditions: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields, such as the field generated by the Earth, can be involved. Because humans live under such circumstances, classical physics seems intuitively reasonable, while many aspects of modern physics seem bizarre. This is why models are so useful in modern physics—they let us conceptualize phenomena we do not ordinarily experience. We can relate to models in human terms and visualize what happens when objects move at high speeds or imagine what objects too small to observe with our senses might be like. For example, we can understand an atom's properties because we can picture it in our minds, although we have never seen an atom with our eyes. New tools, of course, allow us to better picture phenomena we cannot see. In fact, new instrumentation has allowed us in recent years to actually "picture" the atom.

LIMITS ON THE LAWS OF CLASSICAL PHYSICS

For the laws of classical physics to apply, the following criteria must be met: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be

seen with a microscope, and only weak gravitational fields (such as the field generated by the Earth) can be involved.

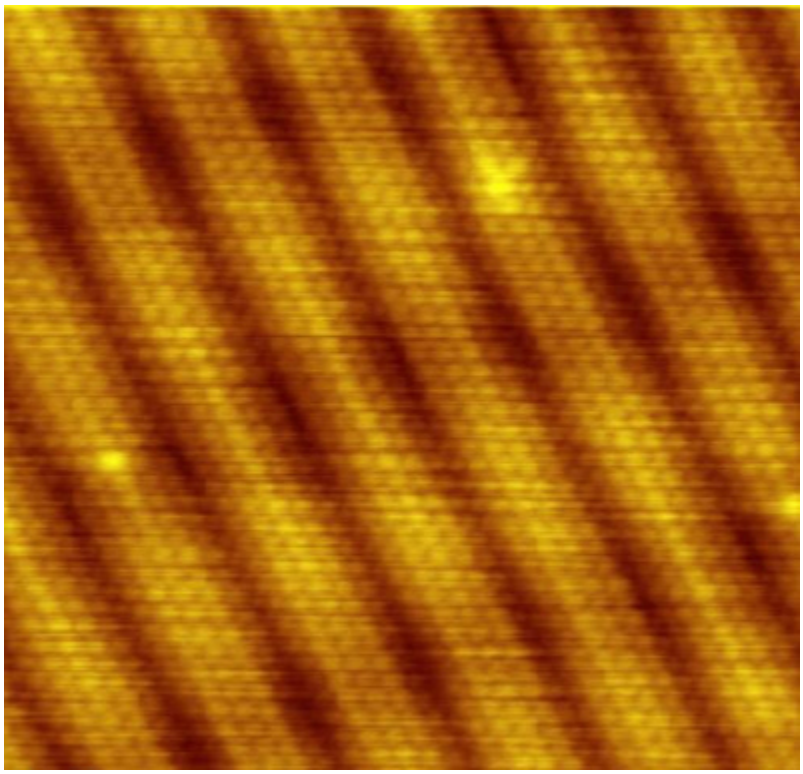


Figure 13. *Using a scanning tunneling microscope (STM), scientists can see the individual atoms that compose this sheet of gold. (credit: Erwinrossen).*

Some of the most spectacular advances in science have been made in modern physics. Many of the laws of classical physics have been modified or rejected, and revolutionary changes in technology, society, and our view of the universe have resulted. Like science fiction, modern physics is filled with fascinating objects beyond our normal experiences, but it has the advantage over science fiction of being very real. Why, then, is the majority of this text devoted to topics of classical physics? There are two main reasons: Classical physics gives an extremely accurate description of the universe under a wide range of everyday circumstances, and knowledge of classical physics is necessary to understand modern physics.

Modern physics itself consists of the two revolutionary theories, relativity and quantum mechanics. These theories deal with the very fast and the very small, respectively. **Relativity** must be used whenever an object is traveling at greater than about 1% of the speed of light or experiences a strong gravitational field such as that near the Sun. **Quantum mechanics** must be used for objects smaller than can be seen with a microscope. The combination of these two theories is *relativistic quantum mechanics*, and it describes the behavior of small objects traveling at high speeds or experiencing a strong gravitational field. Relativistic quantum mechanics is the best universally applicable theory we have. Because of its mathematical complexity, it is used only when necessary, and the other theories are used whenever they

will produce sufficiently accurate results. We will find, however, that we can do a great deal of modern physics with the algebra and trigonometry used in this text.

Check Your Understanding

1: A friend tells you he has learned about a new law of nature. What can you know about the information even before your friend describes the law? How would the information be different if your friend told you he had learned about a scientific theory rather than a law?

PHET EXPLORATIONS: EQUATION GRAPHER

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. $y=bx$) to see how they add to generate the polynomial curve.

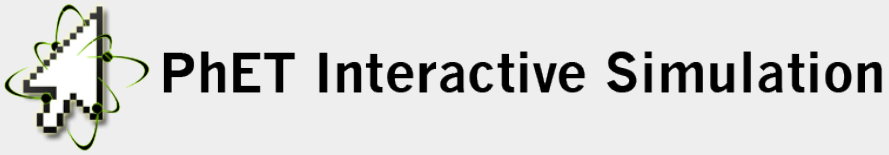


Figure 14. *Equation Grapher*.

Summary

- Science seeks to discover and describe the underlying order and simplicity in nature.
- Physics is the most basic of the sciences, concerning itself with energy, matter, space and time, and their interactions.
- Scientific laws and theories express the general truths of nature and the body of knowledge they encompass. These laws of nature are rules that all natural processes appear to follow.

Conceptual Questions

1: Models are particularly useful in relativity and quantum mechanics, where conditions are outside those normally encountered by humans. What is a model?

2: How does a model differ from a theory?

3: If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?

4: What determines the validity of a theory?

5: Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?

6: Can the validity of a model be limited, or must it be universally valid? How does this compare to the required validity of a theory or a law?

7: Classical physics is a good approximation to modern physics under certain circumstances. What are they?

8: When is it *necessary* to use relativistic quantum mechanics?

9: Can classical physics be used to accurately describe a satellite moving at a speed of 7500 m/s? Explain why or why not.

Glossary

classical physics
: physics that was developed from the Renaissance to the end of the 19th century

physics
: the science concerned with describing the interactions of energy, matter, space, and time; it is especially interested in what fundamental mechanisms underlie every phenomenon

model
: representation of something that is often to difficult (or impossible) to display directly

theory
: an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers

law
: a description, using concise language or a mathematical formula, a generalized pattern in nature that is supported by scientific evidence and repeated examples

scientific method
: a method that typically begins with an observation and question that the scientist will research; next, the scientist typically performs some research about the topic and then devises a hypothesis; then, the scientist will test the hypothesis by performing an experiment; finally, the scientist analyzes the results of the experiment and draws a conclusion

modern physics
: the study of relativity, quantum mechanics, or both

relativity
: the study of objects moving at speeds greater than about 1% of the speed of light, or of objects being affected by a strong gravitational field

quantum mechanics
: the study of objects smaller than can be seen with a microscope

Solutions

Check Your Understanding

1: Without knowing the details of the law, you can still infer that the information your friend has learned conforms to the requirements of all laws of nature: it will be a concise description of the universe around us; a statement of the underlying rules that all natural processes follow. If the information had been a theory, you would be able to infer that the information will be a large-scale, broadly applicable generalization.

1.2 Physical Quantities and Units

Summary

- Perform unit conversions both in the SI and English units
- Explain the most common prefixes in the SI units and be able to write them in scientific notation

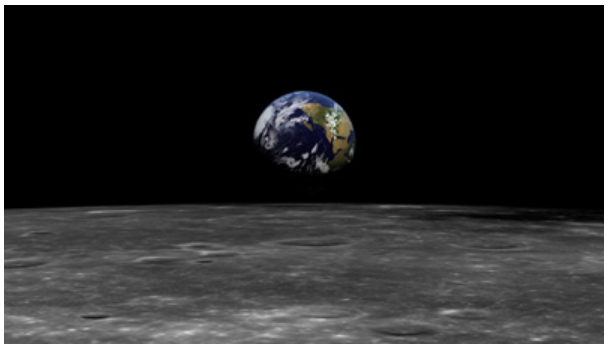


Figure 1. *The distance from Earth to the Moon may seem immense, but it is just a tiny fraction of the distances from Earth to other celestial bodies. (credit: NASA).*

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of the Earth, from the tiny sizes of sub-nuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than does qualitative description alone. To comprehend these vast ranges, we must also have accepted units in which to express them. And we shall find that (even in the potentially mundane discussion of meters, kilograms, and seconds) a profound simplicity of nature appears—all physical quantities can be expressed as combinations of only four fundamental physical quantities: length, mass, time, and electric current.

We define a **physical quantity** either by *specifying how it is measured* or by *stating how it is calculated* from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define *average speed* by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way. (See Figure 2.)

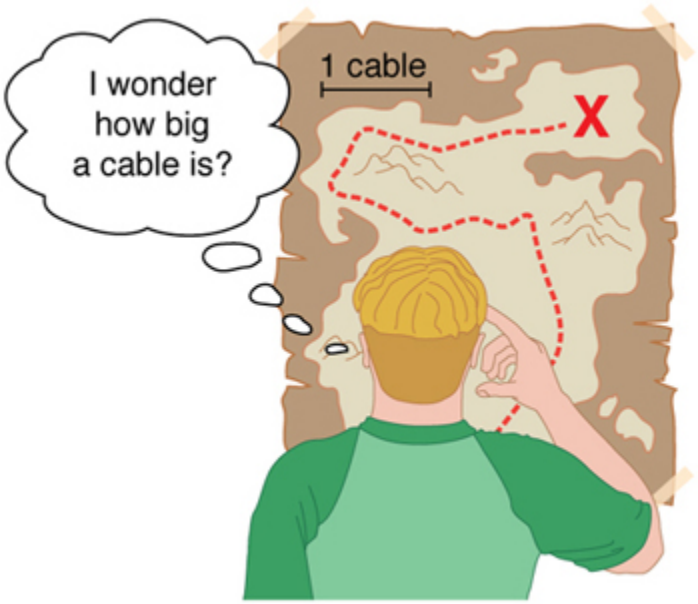


Figure 2. *Distances given in unknown units are maddeningly useless.*

There are two major systems of units used in the world: **SI units** (also known as the metric system) and **English units** (also known as the customary or imperial system). **English units** were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym "SI" is derived from the French *Système International*.

SI Units: Fundamental and Derived Units

Table 1 gives the fundamental SI units that are used throughout this textbook. This text uses non-SI units in a few applications where they are in very common use, such as the measurement of blood pressure in millimeters of mercury (mm Hg). Whenever non-SI units are discussed, they will be tied to SI units through conversions.

Length	Mass	Time	Electric Current
meter (m)	kilogram (kg)	second (s)	ampere (A)

Table 1. Fundamental SI Units.

It is an intriguing fact that some physical quantities are more fundamental than others and that the most fundamental physical quantities can be defined *only* in terms of the procedure used to measure them. The units in which they are measured are thus called **fundamental units**. In this textbook, the fundamental physical quantities are taken to be length, mass, time, and electric current. (Note that electric current will not be introduced until much later in this text.) All other physical quantities, such as force and electric

charge, can be expressed as algebraic combinations of length, mass, time, and current (for example, speed is length divided by time); these units are called **derived units**.

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The Second

The SI unit for time, the second (abbreviated s), has a long history. For many years it was defined as 1/86,400 of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a non-varying, or constant, physical phenomenon (because the solar day is getting longer due to very gradual slowing of the Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967 the second was redefined as the time required for 9,192,631,770 of these vibrations. (See Figure 3.) Accuracy in the fundamental units is essential, because all measurements are ultimately expressed in terms of fundamental units and can be no more accurate than are the fundamental units themselves.

Figure 3. *An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image is looking down from the top of an atomic fountain nearly 30 feet tall! (credit: Steve Jurvetson/Flickr).*

The Meter

The SI unit for length is the meter (abbreviated m); its definition has also changed over time to become more accurate and precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum-iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its present definition (partly for greater accuracy) as the distance light travels in a vacuum in 1/299,792,458 of a second. (See Figure 4.) This change defines the speed of light to be exactly 299,792,458 meters per second. The length of the meter will change if the speed of light is someday measured with greater accuracy.

The Kilogram

The SI unit for mass is the kilogram (abbreviated kg); it is defined to be the mass of a platinum-iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. Exact replicas of the standard kilogram are also kept at the United States' National Institute of Standards and Technology, or NIST, located in Gaithersburg, Maryland outside of Washington D.C., and at other locations around the world. The determination of all other masses can be ultimately traced to a comparison with the standard mass.

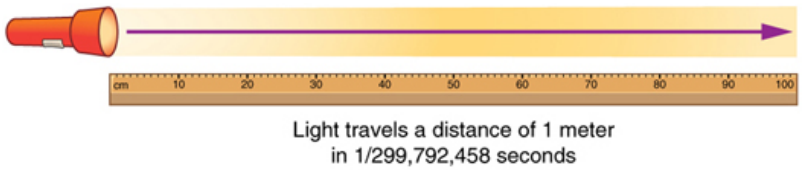


Figure 4. *The meter is defined to be the distance light travels in 1/299,792,458 of a second in a vacuum. Distance traveled is speed multiplied by time.*

Metric Prefixes

SI units are part of the **metric system**. The metric system is convenient for scientific and engineering calculations because the units are categorized by factors of 10. Table 2 gives metric prefixes and symbols used to denote various factors of 10.

Metric systems have the advantage that conversions of units involve only powers of 10. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on. Another advantage of the metric system is that the same unit can be used over extremely large ranges of values simply by using an appropriate metric prefix. For example, distances in meters are suitable in construction, while distances in kilometers are appropriate for air travel, and the tiny measure of nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications.

The term **order of magnitude** refers to the scale of a value expressed in the metric system. Each power of 10 in the metric system represents a different order of magnitude. For example,10^1, 10^2, 10^3, and so forth are all different orders of magnitude. All quantities that can be expressed as a product of a specific power of 10 are said to be of the *same* order of magnitude. For example, the number 800 can be written as 8×10^2 , and the number 450 can be written as 4.5×10^2. Thus, the numbers 800 and 450 are of the same order of magnitude: 10^2. Order of magnitude can be thought of as a ballpark estimate for the scale of a value. The diameter of an atom is on the order of 10^{-9} m while the diameter of the Sun is on the order of 10^9 m.

THE QUEST FOR MICROSCOPIC STANDARDS FOR BASIC UNITS

The fundamental units described in this chapter are those that produce the greatest accuracy and precision in measurement. There is a sense among physicists that, because there is an underlying microscopic substructure to matter, it would be most satisfying to base our standards of measurement on microscopic objects and fundamental physical phenomena such as the speed of light. A microscopic standard has been accomplished for the standard of time, which is based on the oscillations of the cesium atom.

The standard for length was once based on the wavelength of light (a small-scale length) emitted by a certain type of atom, but it has been supplanted by the more precise measurement of the speed of light. If it becomes possible to measure the mass of atoms or a particular arrangement of atoms such as a silicon sphere to greater precision than the kilogram standard, it may become possible to base mass measurements on the small scale. There are also possibilities that electrical phenomena on the small scale may someday allow us to base a unit of charge on the charge of electrons and protons, but at present current and charge are related to large-scale currents and forces between wires.

Prefix	Symbol	Value	Example (some are approximate)			
exa	E	10^{18}	exameter	Em	10^{18} m	distance light travels in a century
peta	P	10^{15}	petasecond	Ps	10^{15} s	30 million years
tera	T	10^{12}	terawatt	TW	10^{12} W	powerful laser output
giga	G	10^{9}	gigahertz	GHz	10^{9} Hz	a microwave frequency
mega	M	10^{6}	megacurie	MCi	10^{6} Ci	high radioactivity
kilo	k	10^{3}	kilometer	km	10^{3} m	about 6/10 mile
hecto	h	10^{2}	hectoliter	hL	10^{2} L	26 gallons
deka	da	10^{1}	dekagram	dag	10^{1} g	teaspoon of butter
—	—	10^{0} (=1)				
deci	d	10^{-1}	deciliter	dL	10^{-1} L	less than half a soda
centi	c	10^{-2}	centimeter	cm	10^{-2} m	fingertip thickness
milli	m	10^{-3}	millimeter	mm	10^{-3} m	flea at its shoulders
micro	μ	10^{-6}	micrometer	μm	10^{-6} m	detail in microscope
nano	n	10^{-9}	nanogram	ng	10^{-9} g	small speck of dust
pico	p	10^{-12}	picofarad	pF	10^{-12} F	small capacitor in radio
femto	f	10^{-15}	femtometer	fm	10^{-15} m	size of a proton
atto	a	10^{-18}	attosecond	as	10^{-18} s	time light crosses an atom

Table 2. Metric Prefixes for Powers of 10 and their Symbols.

Known Ranges of Length, Mass, and Time

The vastness of the universe and the breadth over which physics applies are illustrated by the wide range of examples of known lengths, masses, and times in Table 3. Examination of this table will give you some feeling for the range of possible topics and numerical values. (See Figure 5 and Figure 6.)

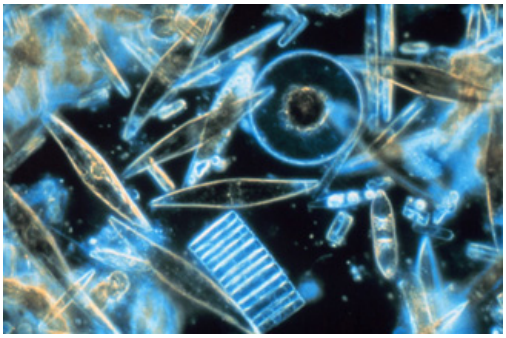


Figure 5. *Tiny phytoplankton swims among crystals of ice in the Antarctic Sea. They range from a few micrometers to as much as 2 millimeters in length. (credit: Prof. Gordon T. Taylor, Stony Brook University; NOAA Corps Collections).*

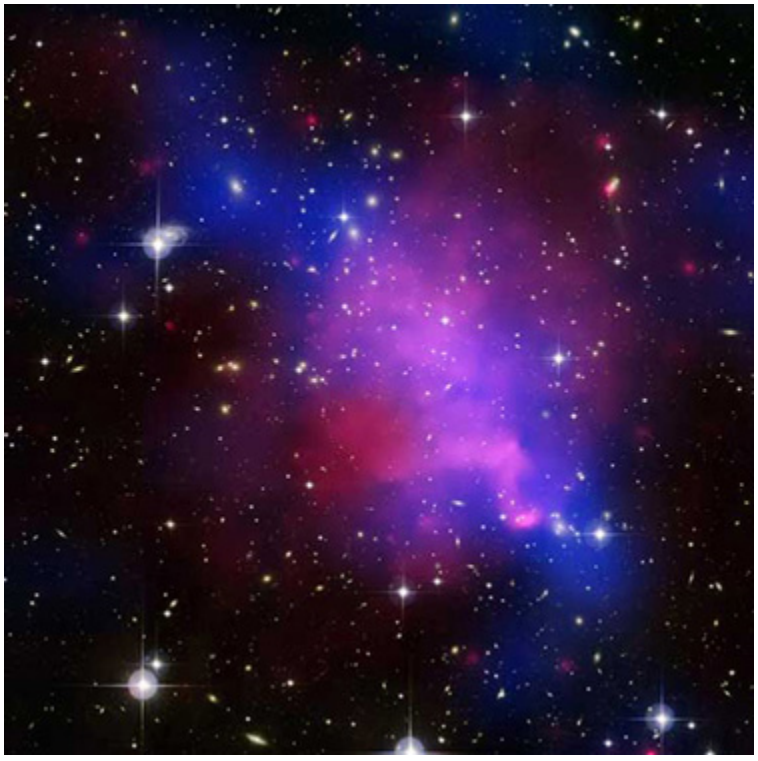


Figure 6. *Galaxies collide 2.4 billion light years away from Earth. The tremendous range of observable phenomena in nature challenges the imagination. (credit: NASA/CXC/UVic./A. Mahdavi et al. Optical/lensing: CFHT/UVic./H. Hoekstra et al.).*

Unit Conversion and Dimensional Analysis

It is often necessary to convert from one type of unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or, perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you will need to convert units of feet to miles.

Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km).

The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in *meters* and we want to convert to *kilometers*.

Next, we need to determine a **conversion factor** relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

$$8~\rule[0.5ex]{1.0em}{0.1ex}\hspace{-1.0em}\text{m} \times \dfrac{1~\text{km}}{1000~\rule[0.5ex]{1.0em}{0.1ex}\hspace{-1.0em}\text{m}} = 1000~\text{km}$$

Note that the unwanted m unit cancels, leaving only the desired km unit. You can use this method to convert between any types of unit.

Click Appendix A Useful Information for a more complete list of conversion factors.

Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
10^{-18}	Present experimental limit to smallest observable detail	10^{-30}	Mass of an electron (9.11×10^{-31} kg)	10^{-23}	Time for light to cross a proton
10^{-15}	Diameter of a proton	10^{-27}	Mass of a hydrogen atom (1.67×10^{-27} kg)	10^{-22}	Mean life of an extremely unstable nucleus
10^{-14}	Diameter of a uranium nucleus	10^{-15}	Mass of a bacterium	10^{-15}	Time for one oscillation of visible light
10^{-10}	Diameter of a hydrogen atom	10^{-5}	Mass of a mosquito	10^{-13}	Time for one vibration of an atom in a solid
10^{-8}	Thickness of membranes in cells of living organisms	10^{-2}	Mass of a hummingbird	10^{-8}	Time for one oscillation of an FM radio wave
10^{-6}	Wavelength of visible light	1	Mass of a liter of water (about a quart)	10^{-3}	Duration of a nerve impulse
10^{-3}	Size of a grain of sand	10^{2}	Mass of a person	1	Time for one heartbeat
1	Height of a 4-year-old child	10^{3}	Mass of a car	10^{5}	One day (8.64×10^{4} s)
10^{2}	Length of a football field	10^{8}	Mass of a large ship	10^{7}	One year (y) (3.16×10^{7} s)
10^{4}	Greatest ocean depth	10^{12}	Mass of a large iceberg	10^{9}	About half the life expectancy of a human
10^{7}	Diameter of the Earth	10^{15}	Mass of the nucleus of a comet	10^{11}	Recorded history
10^{11}	Distance from the Earth to the Sun	10^{23}	Mass of the Moon (7.35×10^{22} kg)	10^{17}	Age of the Earth
10^{16}	Distance traveled by light in 1 year (a light year)	10^{25}	Mass of the Earth (5.97×10^{24} kg)	10^{18}	Age of the universe
10^{21}	Diameter of the Milky Way galaxy	10^{30}	Mass of the Sun (1.99×10^{30} kg)		
10^{22}	Distance from the Earth to the nearest large galaxy (Andromeda)	10^{42}	Mass of the Milky Way galaxy (current upper limit)		
10^{26}	Distance from the Earth to the edges of the known universe	10^{53}	Mass of the known universe (current upper limit)		

Table 3. Approximate Values of Length, Mass, and Time.

Example 1: Unit Conversions: A Short Drive Home

Suppose that you drive the 10.0 km from your university to home in 20.0 min. Calculate your average speed (a) in kilometers per hour (km/h) and (b) in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

Strategy

First we calculate the average speed using the given units. Then we can get the average speed into the desired units by picking the correct conversion factor and multiplying by it. The correct conversion factor is the one that cancels the unwanted unit and leaves the desired unit in its place.

Solution for (a)

(1) Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now—average speed and other motion concepts will be covered in a later module.) In equation form,

$$\text{Average speed} = \frac{Distance}{Time}.$$

(2) Substitute the given values for distance and time.

$$\text{Average speed} = \frac{10.0\ \text{km}}{20.0\ \text{min}} = 0.500\ \frac{\text{km}}{\text{min}}.$$

(3) Convert km/min to km/h: multiply by the conversion factor that will cancel minutes and leave hours. That conversion factor is 60 min/hr. Thus,

$$\text{Average speed} = 0.500\ \frac{\text{km}}{\text{min}} \times \frac{60\ \text{min}}{1\ \text{h}} = 30.0\ \frac{\text{km}}{\text{h}}.$$

Discussion for (a)

To check your answer, consider the following:

(1) Be sure that you have properly cancelled the units in the unit conversion. If you have written the unit conversion factor upside down, the units will not cancel properly in the equation. If you accidentally get the ratio upside down, then the units will not cancel; rather, they will give you the wrong units as follows:

$$\frac{\text{km}}{\text{min}} \times \frac{1\ \text{h}}{60\ \text{min}} = \frac{1\ \text{km} \cdot \text{h}}{60\ \text{min}^2},$$

which are obviously not the desired units of km/h.

(2) Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of km/h and we have indeed obtained these units.

(3) Check the significant figures. Because each of the values given in the problem has three significant figures, the answer should also have three significant figures. The answer 30.0 km/h does indeed have three significant figures, so this is appropriate. Note that the significant figures in the conversion factor are not relevant because an hour is *defined* to be 60 minutes, so the precision of the conversion factor is perfect.

(4) Next, check whether the answer is reasonable. Let us consider some information from the problem—if you travel 10 km in a third of an hour (20 min), you would travel three times that far in an hour. The answer does seem reasonable.

Solution for (b)

There are several ways to convert the average speed into meters per second.

(1) Start with the answer to (a) and convert km/h to m/s. Two conversion factors are needed—one to convert hours to seconds, and another to convert kilometers to meters.

(2) Multiplying by these yields

$$\begin{aligned}\text{Average speed} &= 30.0\ \frac{\text{km}}{\text{h}} \times \frac{1\ \text{h}}{3,600\ \text{s}} \times \frac{1,000\ \text{m}}{1\ \text{km}} \\ &= 8.33\ \text{m/s}\end{aligned}$$

Discussion for (b)

If we had started with 0.500 km/min, we would have needed different conversion factors, but the answer would have been the same: 8.33 m/s.

You may have noted that the answers in the worked example just covered were given to three digits. Why? When do you need to be concerned about the number of digits in something you calculate? Why not write down all the digits your calculator produces? The module Chapter 1.3 Accuracy, Precision, and Significant Figures will help you answer these questions.

NONSTANDARD UNITS

While there are numerous types of units that we are all familiar with, there are others that are much more obscure. For example, a **firkin** is a unit of volume that was once used to measure beer. One firkin equals about 34 liters. To learn more about nonstandard units, use a dictionary or encyclopedia to research different “weights and measures.” Take note of any unusual units, such as a barleycorn, that are not listed in the text. Think about how the unit is defined and state its relationship to SI units.

Check Your Understanding 1

1: Some hummingbirds beat their wings more than 50 times per second. A scientist is measuring the time it takes for a hummingbird to beat its wings once. Which fundamental unit should the scientist use to describe the measurement? Which factor of 10 is the scientist likely to use to describe the motion precisely? Identify the metric prefix that corresponds to this factor of 10.

Check Your Understanding 2

1: One cubic centimeter is equal to one milliliter. What does this tell you about the different units in the SI metric system?

Summary

- Physical quantities are a characteristic or property of an object that can be measured or calculated from other measurements.
- Units are standards for expressing and comparing the measurement of physical quantities. All units can be expressed as combinations of four fundamental units.
- The four fundamental units we will use in this text are the meter (for length), the kilogram (for mass), the second (for time), and the ampere (for electric current). These units are part of the metric system, which uses powers of 10 to relate quantities over the vast ranges encountered in nature.
- The four fundamental units are abbreviated as follows: meter, m; kilogram, kg; second, s; and ampere, A. The metric system also uses a standard set of prefixes to denote each order of magnitude greater than or lesser than the fundamental unit itself.
- Unit conversions involve changing a value expressed in one type of unit to another type of unit. This is done by using conversion factors, which are ratios relating equal quantities of different units.

Glossary

physical quantity
: a characteristic or property of an object that can be measure or calculated from other measurements

units
: a standard used for expressing and comparing measurements

SI units
: the international system of units that scientist in most countries have agreed to use; includes units such as meters, liters, and grams

English units
: system of measurement used in the United States; includes units of measurement such as feet, gallons, and pounds

fundamental units
: units that can only be expressed relative to the procedure used to measure them

derived units
: units that can be calculated using algebraic combinations of the fundamental units

second
: the SI unit for time, abbreviated (s)

meter
: the SI unit for length, abbreviated (m)

kilogram
: the SI unit for mass, abbreviated (kg)

metric system
: a system in which values can be calculated in factors of 10

order of magnitude
: refers to the size of a quantity as it related to a power of 10

conversion factor
: a ratio expression how many of one unit are equal to another unit

Conceptual Questions

1: Identify some advantages of metric units.

Problems & Exercises

1: The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?

2: A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/h speed limit?

3: Show that 1.0 m/s = 3.6 km/h. Hint: Show the explicit steps involved in converting 1.0 m/s = 3.6 km/h.

4: American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 meter equals 3.281 feet.)

5: Soccer fields vary in size. A large soccer field is 115 m long and 85 m wide. What are its dimensions in feet and inches? (Assume that 1 meter equals 3.281 feet.)

6: What is the height in meters of a person who is 6 ft 1.0 in. tall? (Assume that 1 meter equals 39.37 in.)

7: Mount Everest, at 29,031.7 feet, is the tallest mountain on the Earth. What is its height in kilometers? (Assume that 1 kilometer equals 3,281 feet.)

8: The speed of sound is measured to be 342 m/s on a certain day. What is this in km/h?

9: Tectonic plates are large segments of the Earth's crust that move slowly. Suppose that one such plate has an average speed of 4.0 cm/year. (a) What distance does it move in 1 s at this speed? (b) What is its speed in kilometers per million years?

10: (a) Refer to Table 3 to determine the average distance between the Earth and the Sun. Then calculate the average speed of the Earth in its orbit in kilometers per second. (b) What is this in meters per second?

Solutions

Check Your Understanding 1

1: The scientist will measure the time between each movement using the fundamental unit of seconds. Because the wings beat so fast, the scientist will probably need to measure in milliseconds, or 10^{-3} seconds. (50 beats per second corresponds to 20 milliseconds per beat.)

Check Your Understanding 2

1: The fundamental unit of length (meter) is probably used to create the derived unit of volume (liter). The measure of a milliliter is dependent on the measure of a centimeter.

Problems & Exercises

1: (a) 27.8 m/s b) 62.1 miles per hour or mph

3: 118.8 km/h so yes it is speeding)

3: $$\frac{1.0\text{ m}}{\text{s}} = \frac{1.0\text{ m}}{\text{s}} \times \frac{3600\text{ s}}{1\text{ h}} \times \frac{1\text{ km}}{1000\text{ m}} = 3.6\text{ km/h}\cdot$$

4: 91.44 m

5: length: 377 ft; 4.53 x 10^3 in width: 280 ft ; 3.3 x 10^3 inches

6: 1.85 m

7: 8.84886 km

8: 1120 km/h

9: (a) 1.3 x 10^{-9} m (b) 40 km/My or mega year.

1.3 Accuracy, Precision, and Significant Figures

Summary

- Determine the appropriate number of significant figures in both addition and subtraction, as well as multiplication and division calculations.
- Calculate the percent uncertainty of a measurement.

Figure 1. *A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The "known masses" are typically metal cylinders of standard mass such as 1 gram, 10 grams, and 100 grams. (credit: Serge Melki).*

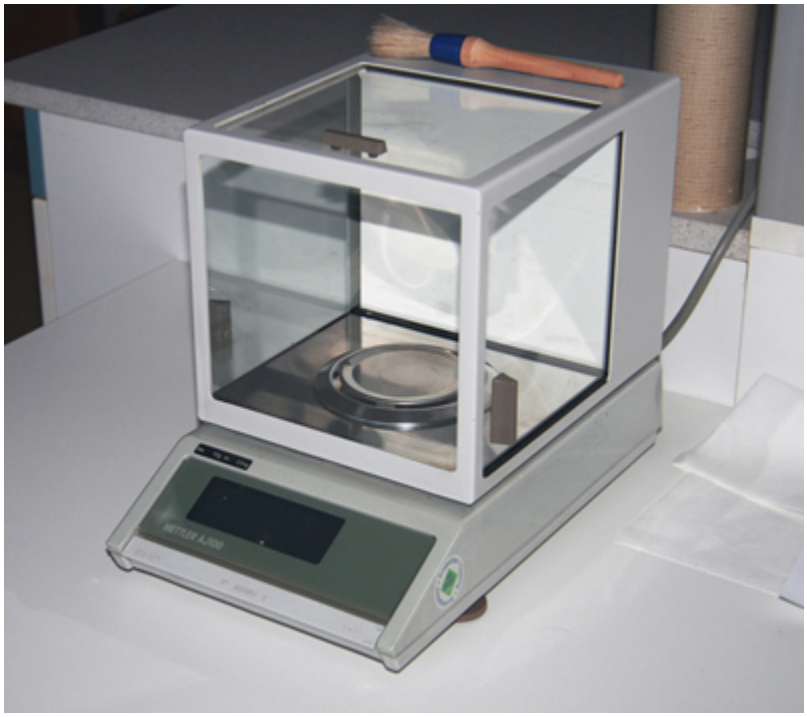


Figure 2. *Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. Whereas a mechanical balance may only read the mass of an object to the nearest tenth of a gram, many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit: Karel Jakubec).*

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let us say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system is refers to how close the agreement is between repeated measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements would be to determine the range, or difference, between the lowest and the highest measured values. In that case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by at most 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9, 11.1, and 11.9, then the measurements would not be very precise because there would be significant variation from one measurement to another.

The measurements in the paper example are both accurate and precise, but in some cases, measurements

are accurate but not precise, or they are precise but not accurate. Let us consider an example of a GPS system that is attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull's-eye target, and think of each GPS attempt to locate the restaurant as a black dot. In Figure 3, you can see that the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low precision, high accuracy measuring system. However, in Figure 4, the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high precision, low accuracy measuring system.

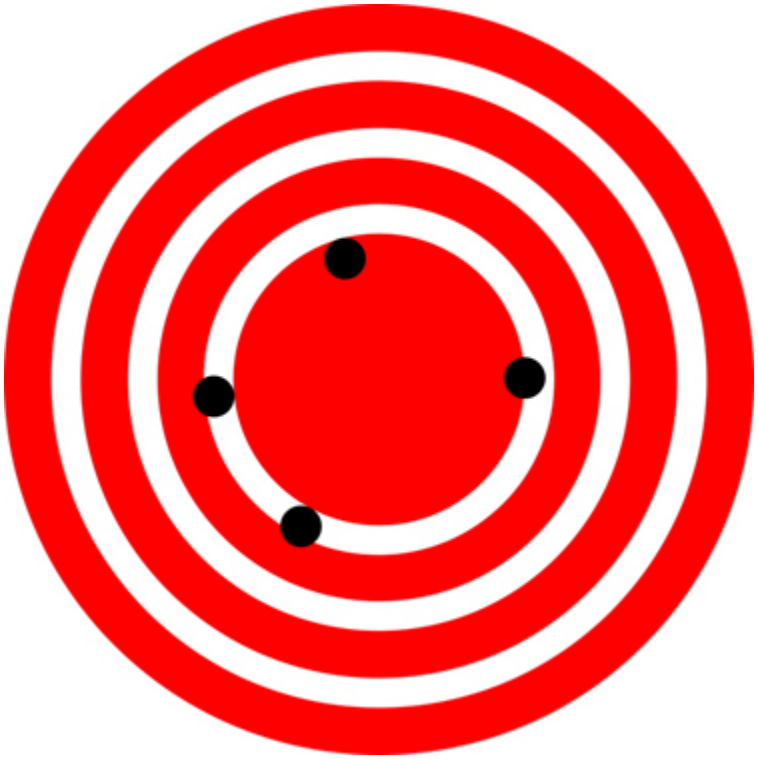


Figure 3. *A GPS system attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy. (credit: Dark Evil).*

Figure 4. *In this figure, the dots are concentrated rather closely to one another, indicating high precision, but they are rather far away from the actual location of the restaurant, indicating low accuracy. (credit: Dark Evil).*

Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the **uncertainty** in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement, A, is often denoted as δA ("delta A"), so the measurement result would be recorded as $A \pm \delta A$. In our paper example, the length of the paper could be expressed as 11 in. ± 0.2.

The factors contributing to uncertainty in a measurement include:

1. Limitations of the measuring device,
2. The skill of the person making the measurement,
3. Irregularities in the object being measured,

4. Any other factors that affect the outcome (highly dependent on the situation).

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.

MAKING CONNECTIONS: REAL-WORLD CONNECTIONS – FEVER OR CHILLS?

Uncertainty is a critical piece of information, both in physics and in many other real-world applications. Imagine you are caring for a sick child. You suspect the child has a fever, so you check his or her temperature with a thermometer. What if the uncertainty of the thermometer were 3.0 °C? If the child's temperature reading was 37.0 °C (which is normal body temperature), the "true" temperature could be anywhere from a hypothermic 34.0 °C to a dangerously high 40.0 °C. A thermometer with an uncertainty of 3.0 °C would be useless.

Percent Uncertainty

One method of expressing uncertainty is as a percent of the measured value. If a measurement A is expressed with uncertainty, δA, the **percent uncertainty** (%unc) is defined to be:

$$\%\text{unc} = \frac{\delta A}{A} \times 100\%$$

Example 1: Calculating Percent Uncertainty: A Bag of Apples

A grocery store sells 5-lb bags of apples. You purchase four bags over the course of a month and weigh the apples each time. You obtain the following measurements:

- Week 1 weight: 4.8 lb
- Week 2 weight: 5.3 lb
- Week 3 weight: 4.9 lb
- Week 4 weight: 5.4 lb

You determine that the weight of the 5-lb bag has an uncertainty of ±0.4 lb. What is the percent uncertainty of the bag's weight?

Strategy

First, observe that the expected value of the bag's weight, A, is 5 lb. The uncertainty in this value, δA, is 0.4 lb. We can use the following equation to determine the percent uncertainty of the weight:

$$\%\text{unc} = \frac{\delta A}{A} \times 100\%$$

Solution

Plug the known values into the equation:

$$\%\text{unc} = \frac{0.4 \text{ lb}}{5 \text{ lb}} \times 100\% = 8\%$$

Discussion

We can conclude that the weight of the apple bag is 5 lb ±8%. Consider how this percent uncertainty would change if the bag of apples were half as heavy, but the uncertainty in the weight remained the same. Hint for future calculations: when calculating percent uncertainty, always remember that you must multiply the fraction by 100%. If you do not do this, you will have a decimal quantity, not a percent value.

Uncertainties in Calculations

There is an uncertainty in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the **method of adding percents** can be used for multiplication or division. This method says that *the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation*. For example, if a floor has a length of 4.00 m and a width of 3.00 m, with uncertainties of 2% and 1%, respectively, then the area of the floor is 12.0 m and has an uncertainty of 3%. (Expressed as an area this is 0.36 m^2, which we round to 0.4 m^2 since the area of the floor is given to a tenth of a square meter.)

Check Your Understanding 1

1: A high school track coach has just purchased a new stopwatch. The stopwatch manual states that the stopwatch has an uncertainty of ±0.05 s. Runners on the track coach's team regularly clock 100-m sprints of 11.49 s to 15.01 s. At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s. Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

Precision of Measuring Tools and Significant Figures

An important factor in the accuracy and precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter, while a caliper can measure length to the nearest 0.01 millimeter. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise and accurate the measurements can be.

When we express measured values, we can only list as many digits as we initially measured with our measuring tool. For example, if you use a standard ruler to measure the length of a stick, you may measure it to be 36.7 cm. You could not express this value as 36.71 cm because your measuring tool was not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices that the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the method of **significant figures**, the rule is that *the last digit written down in a measurement is the first digit with some uncertainty*. In order to determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or significant figures. Significant figures indicate the precision of a measuring tool that was used to measure a value.

Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant, because they are only placekeepers that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placekeepers but are significant—this number has five significant figures. The zeros in 1300 may or may not be significant depending on the style of writing numbers. They could mean the number is known to the last digit, or they could be placekeepers. So 1300 could have two, three, or four significant figures. (To avoid this ambiguity, write 1300 in scientific notation.) *Zeros are significant except when they serve only as placekeepers*.

Check Your Understanding 2

2: Determine the number of significant figures in the following measurements:

a: 0.0009

b: 15,450.0

c: 6×10^{3}

d: 87.990

e: 30.42

Significant Figures in Calculations

When combining measurements with different degrees of accuracy and precision, *the number of significant digits in the final answer can be no greater than the number of significant digits in the least precise measured value*. There are two different rules, one for multiplication and division and the other for addition and subtraction, as discussed below.

1. For multiplication and division: *The result should have the same number of significant figures as the quantity having the least significant figures entering into the calculation*. For example, the area of a circle can be calculated from its radius using $A = \pi r^2$. Let us see how many significant figures the area has if the radius has only two—say, r = 1.2 m. Then,

$$A = \pi r^2 = (3.1415927\ldots) \times (1.2\ \text{m})^2 = 4.5238934\ \text{m}^2$$

is what you would get using a calculator that has an eight-digit output. But because the radius has only two significant figures, it limits the calculated quantity to two significant figures or

$$\text{A} = 4.5\ \text{m}^2,$$

even though π is good to at least eight digits.

2. For addition and subtraction: *The answer can contain no more decimal places than the least precise measurement*. Suppose that you buy 7.56-kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg. Then you drop off 6.052-kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Finally, you go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do you now have, and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:

$$\begin{aligned} &7.56\ \text{kg} \\ &-6.052\ \text{kg} \\ &+\ \ 13.7\ \text{kg} \\ \hline &15.208\ \text{kg} \quad = 15.2\ \text{kg} \end{aligned}$$

Next, we identify the least precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

Significant Figures in this Text

In this text, most numbers are assumed to have three significant figures. Furthermore, consistent numbers of significant figures are used in all worked examples. You will note that an answer given to three digits is based on input good to at least three digits, for example. If the input has fewer significant figures, the answer will also have fewer significant figures. Care is also taken that the number of significant figures is reasonable for the situation posed. In some topics, particularly in optics, more accurate numbers are needed and more than three significant figures will be used. Finally, if a number is *exact*, such as the two in the formula for the circumference of a circle, $c = 2\pi r$, it does not affect the number of significant figures in a calculation.

Check Your Understanding 3

1: Perform the following calculations and express your answer using the correct number of significant digits.

(a) A woman has two bags weighing 13.5 pounds and one bag with a weight of 10.2 pounds. What is the total weight of the bags?

(b) The force $\vec{F}$ on an object is equal to its mass *m* multiplied by its acceleration $\vec{a}$. If a wagon with mass 55 kg accelerates at a rate of 0.0255 m/s^2, what is the force on the wagon? (The unit of force is called the newton, and it is expressed with the symbol N.)

PHET EXPLORATION: ESTIMATION

Explore size estimation in one, two, and three dimensions! Multiple levels of difficulty allow for progressive skill improvement.

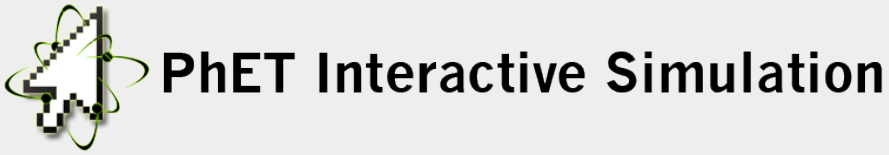


Figure 5. *Estimation.*

Summary

- Accuracy of a measured value refers to how close a measurement is to the correct value. The uncertainty in a measurement is an estimate of the amount by which the measurement result may differ from this value.
- Precision of measured values refers to how close the agreement is between repeated measurements.
- The precision of a *measuring tool* is related to the size of its measurement increments. The smaller the measurement increment, the more precise the tool.
- Significant figures express the precision of a measuring tool.
- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least precise value.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least precise value.

Conceptual Questions

1: What is the relationship between the accuracy and uncertainty of a measurement?

2: Prescriptions for vision correction are given in units called *diopters* (D). Determine the meaning of that unit. Obtain information (perhaps by calling an optometrist or performing an internet search) on the minimum uncertainty with which corrections in diopters are determined and the accuracy with which corrective lenses can be produced. Discuss the sources of uncertainties in both the prescription and accuracy in the manufacture of lenses.

Problems & Exercises

Express your answer to problems in this section to the correct number of significant figures and proper units.

1: Suppose that your bathroom scale reads your mass as 65 kg with a 3% uncertainty. What is the uncertainty in your mass (in kilograms)?

2: A good-quality measuring tape can be off by 0.50 cm over a distance of 20 m. What is its percent uncertainty?

3: (a) A car speedometer has a 5.0% uncertainty. What is the range of possible speeds when it reads 90 km/h? (b) Convert this range to miles per hour. (1 km = 0.6214 mi)

4: An infant's pulse rate is measured to be 130 ± 5 beats/min. What is the percent uncertainty in this measurement?

5: (a) Suppose that a person has an average heart rate of 72.0 beats/min. How many beats does he or she have in 2.0 y? (b) In 2.00 y? (c) In 2.000 y?

6: A can contains 375 mL of soda. How much is left after 308 mL is removed?

7: State how many significant figures are proper in the results of the following calculations: (a) (106.7)(98.2)\(46.210)(1.01) (b) $(18.7)^2$ (c) $(1.60 \times 10^{-19})(3712)$.

8: (a) How many significant figures are in the numbers 99 and 100? (b) If the uncertainty in each number is 1, what is the percent uncertainty in each? (c) Which is a more meaningful way to express the accuracy of these two numbers, significant figures or percent uncertainties?

9: (a) If your speedometer has an uncertainty of 2.0 km/h at a speed of 90 km/h, what is the percent uncertainty? (b) If it has the same percent uncertainty when it reads 60 km/h, what is the range of speeds you could be going?

10: (a) A person's blood pressure is measured to be 120 ± 2 mm Hg. What is its percent uncertainty? (b) Assuming the same percent uncertainty, what is the uncertainty in a blood pressure measurement of 80 mm Hg?

11: A person measures his or her heart rate by counting the number of beats in 30 s. If 40 ± 1 beats are counted in 30.0 ± 0.5 s, what is the heart rate and its uncertainty in beats per minute?

12: What is the area of a circle 3.102 cm in diameter?

13: If a marathon runner averages 9.5 mi/h, how long does it take him or her to run a 26.22-mi marathon?

14: A marathon runner completes a 42.188-km course in 2 h, 30 min, and 12 s. There is an uncertainty of 25 m in the distance traveled and an uncertainty of 1 s in the elapsed time. (a) Calculate the percent uncertainty in the distance. (b) Calculate the uncertainty in the elapsed time. (c) What is the average speed in meters per second? (d) What is the uncertainty in the average speed?

15: The sides of a small rectangular box are measured to be 1.80 ± 0.01 cm, 2.05 ± 0.02 cm, and 3.1 ± 0.1 cm long. Calculate its volume and uncertainty in cubic centimeters.

16: When non-metric units were used in the United Kingdom, a unit of mass called the *pound-mass* (lbm) was employed, where 1 lbm = 0.4539 kg. (a) If there is an uncertainty of 0.0001 kg in the pound-mass unit, what is its percent uncertainty? (b) Based on that percent uncertainty, what mass in pound-mass has an uncertainty of 1 kg when converted to kilograms?

17: The length and width of a rectangular room are measured to be 3.955 ± 0.005 m and 3.050 ± 0.005 m. Calculate the area of the room and its uncertainty in square meters.

18: A car engine moves a piston with a circular cross section of 7.500 ± 0.002 cm diameter a distance of 3.250 ± 0.001 cm to compress the gas in the cylinder. (a) By what amount is the gas decreased in volume in cubic centimeters? (b) Find the uncertainty in this volume.

Glossary

accuracy
: the degree to which a measure value agrees with the correct value for that measurement

method of adding percents
: the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation

percent uncertainty
: the ratio of the uncertainty of a measurement to the measure value, express as a percentage

precision
: the degree to which repeated measurements agree with each other

significant figures
: express the precision of a measuring tool used to measure a value

uncertainty
: a quantitative measure of how much your measured values deviate from a standard or expected value

Solutions

Check Your Understanding 1

1: No, the uncertainty in the stopwatch is too great to effectively differentiate between the sprint times.

Check Your Understanding 2

1: (a) 1; the zeros in this number are placekeepers that indicate the decimal point

(b) 6; here, the zeros indicate that a measurement was made to the 0.1 decimal point, so the zeros are significant

(c) 1; the value 10^3 signifies the decimal place, not the number of measured values

(d) 5; the final zero indicates that a measurement was made to the 0.001 decimal point, so it is significant

(e) 4; any zeros located in between significant figures in a number are also significant

Check Your Understanding 3

1: (a) 37.2 pounds; Because the number of bags is an exact value, it is not considered in the significant figures.

(b) 1.4 N; Because the value 55 kg has only two significant figures, the final value must also contain two significant figures.

Problems & Exercises

1: 2 kg

3: (a) 90.0 ± 4.50 km/h (b) 55.9 ± 2.8 mi/h

5: (a) 7.6×10^7 beats (b) 7.57×10^7 beats (c) 7.57^7 beats

7. (a) 3 (b) 3 (c) 3

9: (a) 2.2% (b) 60 ± 1 km/h

11: 80 ± 3 beats/min

13: 2.8h

15: 11.1 ± 1 m^3

17: 12.06 ± 0.04 m^2

1.4 Approximation

Summary

- Make reasonable approximations based on given data.

On many occasions, physicists, other scientists, and engineers need to make **approximations** or "guesstimates" for a particular quantity. What is the distance to a certain destination? What is the approximate density of a given item? About how large a current will there be in a circuit? Many approximate numbers are based on formulae in which the input quantities are known only to a limited accuracy. As you develop problem-solving skills (that can be applied to a variety of fields through a study of physics), you will also develop skills at approximating. You will develop these skills through thinking more quantitatively, and by being willing to take risks. As with any endeavor, experience helps, as well as familiarity with units. These approximations allow us to rule out certain scenarios or unrealistic numbers. Approximations also allow us to challenge others and guide us in our approaches to our scientific world. Let us do two examples to illustrate this concept.

Example 1: Approximate the Height of a Building

Can you approximate the height of one of the buildings on your campus, or in your neighborhood? Let us make an approximation based upon the height of a person. In this example, we will calculate the height of a 39-story building.

Strategy

Think about the average height of an adult male. We can approximate the height of the building by scaling up from the height of a person.

Solution

Based on information in the example, we know there are 39 stories in the building. If we use the fact that the height of one story is approximately equal to about the length of two adult humans (each human is about 2-m tall), then we can estimate the total height of the building to be

$$\frac{2\text{ m}}{1\text{ person}} \times \frac{2\text{ person}}{1\text{ story}} \times 39\text{ stories} = 156\text{ m}$$

Discussion

You can use known quantities to determine an approximate measurement of unknown quantities. If your hand measures 10 cm across, how many hand lengths equal the width of your desk? What other measurements can you approximate besides length?

Example 2: Approximating Vast Numbers: a Trillion Dollars

Figure 1. *A bank stack contains one-hundred $100 bills, and is worth $10,000. How many bank stacks make up a trillion dollars? (credit: Andrew Magill).*

The U.S. federal deficit in the 2008 fiscal year was a little greater than $10 trillion. Most of us do not have any concept of how much even one trillion actually is. Suppose that you were given a trillion dollars in $100 bills. If you made 100-bill stacks and used them to evenly cover a football field (between the end zones), make an approximation of how high the money pile would become. (We will use feet/inches rather than meters here because football fields are measured in yards.) One of your friends says 3 in., while another says 10 ft. What do you think?

Strategy

When you imagine the situation, you probably envision thousands of small stacks of 100 wrapped $100 bills, such as you might see in movies or at a bank. Since this is an easy-to-approximate quantity, let us start there. We can find the volume of a stack of 100 bills, find out how many stacks make up one trillion dollars, and then set this volume equal to the area of the football field multiplied by the unknown height.

Solution

(1) Calculate the volume of a stack of 100 bills. The dimensions of a single bill are approximately 3 in. by 6 in. A stack of 100 of these is about 0.5 in. thick. So the total volume of a stack of 100 bills is:

$$\begin{aligned}\text{Volume of stack} &= \text{length} \times \text{width} \times \text{height} \\ &= 6\text{ in} \times 3\text{ in} \times 0.5\text{ in} \\ &= 9\text{ in}^3\end{aligned}$$

(2) Calculate the number of stacks. Note that a trillion dollars is equal to $\$1 \times 10^{12}$, and a stack of one-hundred $100 bills is equal to $10,000, or $\$1 \times 10^4$. The number of stacks you will have is:

$$\frac{\$1 \cdot 10^{12}\ (\text{a trillion dollars})}{1} \cdot \frac{1\text{ stack}}{\$1 \cdot 10^4} = 1 \cdot 10^8\text{ stacks}.$$

(3) Calculate the area of a football field in square inches. The area of a football field is 100 yd × 50 yd, which gives 5000 yd^2 Because we are working in inches, we need to convert square yards to square inches:

$$\begin{aligned}\text{Area} &= 5000\ \text{yd}^2 \times \frac{3\ \text{ft}}{1\ \text{yd}} \times \frac{3\ \text{ft}}{1\ \text{yd}} \times \frac{12\ \text{in}}{1\ \text{ft}} \times \frac{12\ \text{in}}{1\ \text{ft}} \\ &= 6,480,000\ \text{in}^2\end{aligned}$$

$$\text{Area} \approx 6 \cdot 10^6\ \ \text{in}^2$$

This conversion gives us 6 × 10^6 $in.^2$ for the area of the field. (Note that we are using only one significant figure in these calculations.)

(4) Calculate the total volume of the bills. The volume of all the $100-bill stacks is

$$\frac{9\ \text{in}^3}{1\ \text{stack}} \times 10^8\, stacks = 9 \cdot 10^8\ \text{in}^3.$$

(5) Calculate the height. To determine the height of the bills, use the equation:

$$\text{Volume of bills} = \text{Area of field} \times \text{Height of bills}$$

$$\begin{aligned}\text{Height of bills} &= \frac{\text{Volume of bills}}{\text{Area of field}} \\ &= \frac{9 \cdot 10^8\ \text{in}^3}{6 \cdot 10^6\ \text{in}^2} = 1.33 \cdot 10^2\ \text{in}\end{aligned}$$

$$\text{Height of bills} \approx 1 \cdot 10^2\ \text{in} = 100\ \text{in}$$

The height of the money will be about 100 inches high. Converting this value to feet gives

$$100\ \text{in} \times \frac{1\ \text{ft}}{12\ \text{in}} = 8.33\ \text{ft} \approx 8\ \text{ft}$$

Discussion

The final approximate value is much higher than the early estimate of 3 in., but the other early estimate of 10 ft (120 in.) was roughly correct. How did the approximation measure up to your first guess? What can this exercise tell you in terms of rough “guesstimates” versus carefully calculated approximations?

Check Your Understanding

1: Using mental math and your understanding of fundamental units, approximate the area of a regulation basketball court. Describe the process you used to arrive at your final approximation.

Summary

Scientists often approximate the values of quantities to perform calculations and analyze system

Glossary

approximation
an estimated value based on prior experience and reasoning

Problems & Exercises

1: How many heartbeats are there in a lifetime?

2: A generation is about one-third of a lifetime. Approximately how many generations have passed since the year 0 AD?

3: How many times longer than the mean life of an extremely unstable atomic nucleus is the lifetime of a human? (Hint: The lifetime of an unstable atomic nucleus is on the order of 10^{-22} seconds.)

4: Calculate the approximate number of atoms in a bacterium. Assume that the average mass of an atom in the bacterium is ten times the mass of a hydrogen atom. (Hint: The mass of a hydrogen atom is on the order of 10^{-27} kg and the mass of a bacterium is on the order of 10^{-15} kg.)

Figure 2. *This color-enhanced photo shows Salmonella typhimurium (red) attacking human cells. These bacteria are commonly known for causing foodborne illness. Can you estimate the number of atoms in each bacterium? (credit: Rocky Mountain Laboratories, NIAID, NIH).*

5: Approximately how many atoms thick is a cell membrane, assuming all atoms there average about twice the size of a hydrogen atom?

6: (a) What fraction of Earth's diameter is the greatest ocean depth? (b) The greatest mountain height?

7: (a) Calculate the number of cells in a hummingbird assuming the mass of an average cell is ten times the mass of a bacterium. (b) Making the same assumption, how many cells are there in a human?

8: Assuming one nerve impulse must end before another can begin, what is the maximum firing rate of a nerve in impulses per second?

Solutions

Check Your Understanding

1: An average male is about two meters tall. It would take approximately 15 men laid out end to end to cover the length, and about 7 to cover the width. That gives an approximate area of 420 m^2.

Problems & Exercises

1: Sample answer: 2×10^9 heartbeats if an average human lifetime is taken to be about 70 years.

3: Sample answer: 2×10^{31} if an average human lifetime is taken to be about 70 years.

5: Sample answer: 50 atoms

7: Sample answers: (a) 10^{12} cells/hummingbird (b) 10^{16} cells/human

Chapter 2 One-Dimensional Kinematics

Figure 1. *The motion of an American kestrel through the air can be described by the bird's displacement, speed, velocity, and acceleration. When it flies in a straight line without any change in direction, its motion is said to be one dimensional. (credit: Vince Maidens, Wikimedia Commons).*

Objects are in motion everywhere we look. Everything from a tennis game to a space-probe flyby of the planet Neptune involves motion. When you are resting, your heart moves blood through your veins. And even in inanimate objects, there is continuous motion in the vibrations of atoms and molecules. Questions about motion are interesting in and of themselves: *How long will it take for a space probe to get to Mars? Where will a football land if it is thrown at a certain angle?* But an understanding of motion is also key to understanding other concepts in physics. An understanding of acceleration, for example, is crucial to the study of force.

Our formal study of physics begins with **kinematics** which is defined as the *study of motion without considering its causes*. The word "kinematics" comes from a Greek term meaning motion and is related to other English words such as "cinema" (movies) and "kinesiology" (the study of human motion). In one-dimensional kinematics and Two-Dimensional Kinematics we will study only the *motion* of a football, for example, without worrying about what forces cause or change its motion. Such considerations come in other chapters. In this chapter, we examine the simplest type of motion—namely, motion along a straight line, or one-dimensional motion. In Two-Dimensional Kinematics, we apply concepts developed here to study motion along curved paths (two- and three-dimensional motion); for example, that of a car rounding a curve.

2.1 Displacement

Summary

- Define position, displacement, distance, and distance traveled.
- Explain the relationship between position and displacement.
- Distinguish between displacement and distance traveled.
- Calculate displacement and distance given initial position, final position and the path between the two.

Figure 1. *These cyclists in Vietnam can be described by their position relative to buildings and a canal. Their motion can be described by their change in position, or displacement, in the frame of reference. (credit: Suzan Black, Fotopedia).*

Position

In order to describe the motion of an object, you must first be able to describe its **position**—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor's position could be described in terms of where she is in relation to the nearby white board. (See Figure 2.) In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See Figure 3.)

Displacement

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object's position changes. This change in position is known as **displacement**. The word "displacement" implies that an object has moved, or has been displaced.

DISPLACEMENT

Displacement is the *change in position* of an object:

$$\Delta x = x_f - x_0$$

where Δx is displacement, x_f is the final position, and x_0 is the initial position.

In this text the upper case Greek letter Δ (delta) always means "change in" whatever quantity follows it; thus, Δx means *change in position*. Always solve for displacement by subtracting initial position x_0 from final x_f.

Note that the SI unit for displacement is the meter (m) (see Chapter 1.2 Physical Quantities and Units), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are used in a problem, you may need to convert them into meters to complete the calculation.

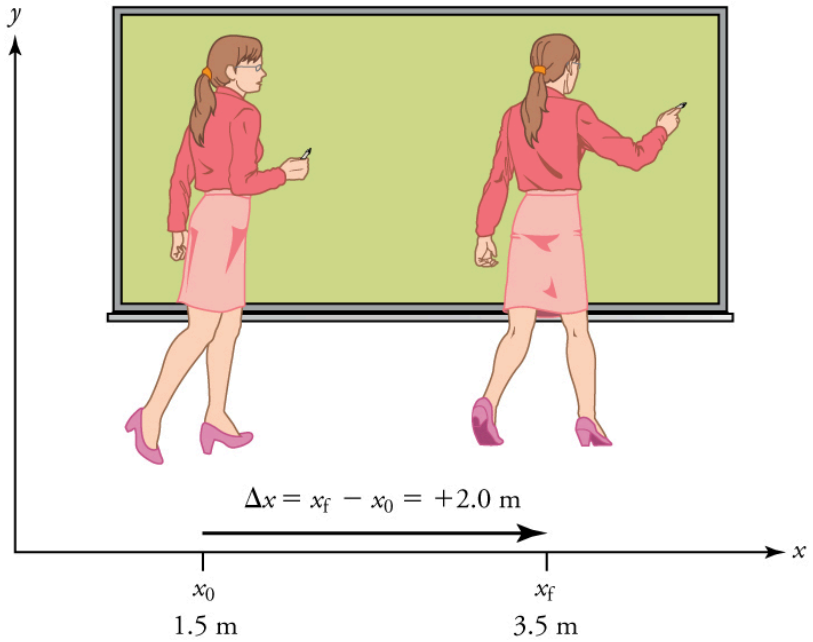


Figure 2. *A professor paces left and right while lecturing. Her position relative to Earth is given by x. The +2.0 m displacement of the professor relative to Earth is represented by an arrow pointing to the right.*

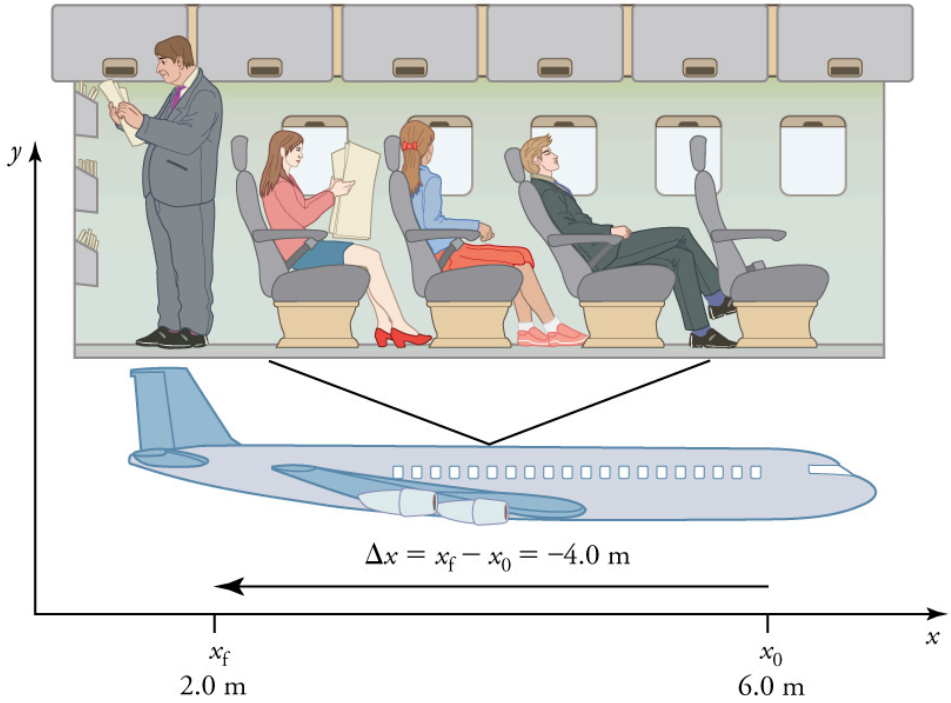


Figure 3. *A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by* ***x****. The* ***-4.0-m*** *displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in Figure 2.*

Note that displacement has a direction as well as a magnitude. The professor's displacement is 2.0 m to the right, and the airline passenger's displacement is 4.0 m toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor's initial position is $x_0 = 1.5\ \text{m}$ and her final position is $x_f = 3.5\ \text{m}$. Thus her displacement is

$$\Delta x = x_f - x_0 = 3.5\ \text{m} - 1.5\ \text{m} = 2\ \text{m}$$

In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger's initial position is $x_0 = 6.0\ \text{m}$ and his final position is $x_f = 2.0\ \text{m}$, so his displacement is

$$\Delta x = x_f - x_0 = 2.0\ \text{m} - 6.0\ \text{m} = -4.0\ \text{m}.$$

His displacement is negative because his motion is toward the rear of the plane, or in the negative x direction in our coordinate system.

Distance

See Figure 1. Although displacement is described in terms of direction, distance is not. **Distance** is defined to be *the magnitude or size of displacement between two positions*. Note that the distance between two positions is not the same as the distance traveled between them. **Distance traveled** is *the total length of the path traveled between two positions*. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

MISCONCEPTION ALERT: DISTANCE TRAVELED VS. MAGNITUDE OF DISPLACEMENT

It is important to note that the *distance traveled*, however, can be greater than the magnitude of the displacement (by magnitude, we mean just the size of the displacement without regard to its direction; that is, just a number with a unit). For example, the professor could pace back and forth many times, perhaps walking a distance of 150 m during a lecture, yet still end up only 2.0 m to the right of her starting point. In this case her displacement would be +2.0 m, the magnitude of her displacement would be 2.0 m, but the distance she traveled would be 150 m. In kinematics we nearly always deal with displacement and magnitude of displacement, and almost never with distance traveled. One way to think about this is to assume you marked the start of the motion and the end of the motion. The displacement is simply the difference in the position of the two marks and is independent of the path taken in traveling between the two marks. The distance traveled, however, is the total length of the path taken between the two marks.

Check Your Understanding 1

1: A cyclist rides 3 km west and then turns around and rides 2 km east. (a) What is her displacement? (b) What distance does she ride? (c) What is the magnitude of her displacement?

Section Summary

- Kinematics is the study of motion without considering its causes. In this chapter, it is limited to motion along a straight line, called one-dimensional motion.
- Displacement is the change in position of an object.
- In symbols, displacement Δx is defined to be

$$\Delta x = x_f - x_0$$

where x_0 is the initial position and x_f is the final position. In this text, the Greek letter Δ (delta) always means “change in” what ever quantity follows it. The SI unit for displacement is the meter (m). Displacement has a direction as well as a magnitude.

- When you start a problem, assign which direction will be positive.
- Distance is the magnitude of displacement between two positions.
- Distance traveled is the total length of the path traveled between two positions.

Glossary

kinematics
: the study of motion without considering its causes

position
: the location of an object at a particular time

displacement
: the change in position of an object

distance
: the magnitude of displacement between two positions

distance traveled
: the total length of the path traveled between two positions

Conceptual Questions

1: Give an example in which there are clear distinctions among distance traveled, displacement, and magnitude of displacement. Specifically identify each quantity in your example.

2: Under what circumstances does distance traveled equal magnitude of displacement? What is the only case in which magnitude of displacement and displacement are exactly the same?

3: Bacteria move back and forth by using their flagella (structures that look like little tails). Speeds of up to 50 um/s (50×10^{-6} m/s) have been observed. The total distance traveled by a bacterium is large for its size, while its displacement is small. Why is this?

Problems & Exercises

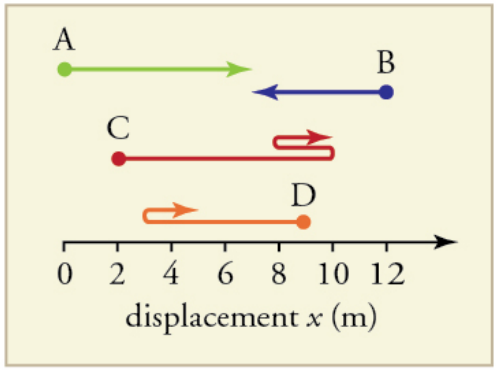


Figure 4. Note: The x axis should be labelled position not displacement.

1: Find the following for path A in Figure 4: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

2: Find the following for path B in Figure 4: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

3: Find the following for path C in Figure 4: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

4: Find the following for path D in Figure 4: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

Solutions

Check Your Understanding 1

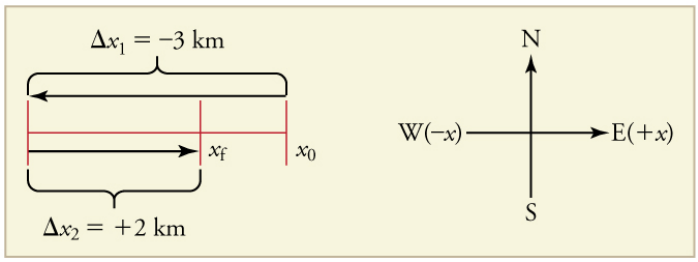


Figure 5.

1: (a) The rider's displacement is $\Delta x = x_f - x_0 = -1$ km. The displacement is negative because we take east to be positive and west to be negative. Or you could just say "1 km to the West". Note that the drawing clearly showed that West was chosen to be negative. (b) The distance traveled is 3 km + 2 km = 5 km. (c) The magnitude of the displacement is 1 km.

Problems & Exercises

1: (a) 7 m (b) 7 m (c) + 7 m

2: (a) 5 m (b) 5 m (c) – 5 m

3: This is badly drawn so the answers are debatable. Assuming it went from a position of 2 m to 10 then back to 8 and then back again to 10 m that gives a) distance of 12 m b) magnitude of the displacement as 8 m and c) a displacement of +8 m or 8 metres ot the right.

4: (a) 8 m (b) 4 m (c) – 4 m

2.2 Vectors, Scalars, and Coordinate Systems

Summary

- Define and distinguish between scalar and vector quantities.
- Assign a coordinate system for a scenario involving one-dimensional motion.

Figure 1. *The motion of this Eclipse Concept jet can be described in terms of the distance it has traveled (a scalar quantity) or its displacement in a specific direction (a vector quantity). In order to specify the direction of motion, its displacement must be described based on a coordinate system. In this case, it may be convenient to choose motion toward the left as positive motion (it is the forward direction for the plane), although in many cases, the* ***x****-coordinate runs from left to right, with motion to the right as positive and motion to the left as negative. (credit: Armchair Aviator, Flickr).*

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A **vector** is any quantity with both *magnitude and direction*. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.

The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (−) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length

proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A **scalar** is any quantity that has a magnitude, but no direction. For example, a 20 °C temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars—quantities with no specified direction. Note, however, that a scalar can be negative, such as a -20 °C temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

Coordinate Systems for One-Dimensional Motion

In order to describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive and motion down is negative. In some cases, however, as with the jet in Figure 1, it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.

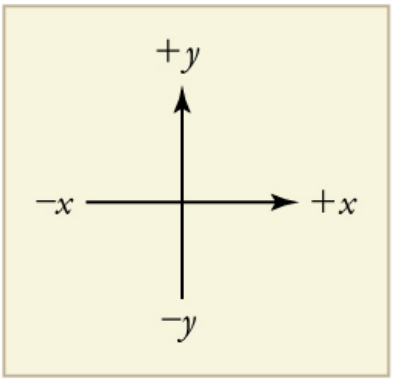


Figure 2. *It is usually convenient to consider motion upward or to the right as positive* ***(+)*** *and motion downward or to the left as negative* ***(−)****.*

Check Your Understanding

1: A person's speed can stay the same as he or she rounds a corner and changes direction. Given this information, is speed a scalar or a vector quantity? Explain.

Section Summary

- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.
- In one-dimensional motion, direction is specified by a plus or minus sign to signify left or right, up or down, and the like.

Glossary

scalar
a quantity that is described by magnitude, but not direction

vector
a quantity that is described by both magnitude and direction

Conceptual Questions

1: A student writes, "*A bird that is diving for prey has a speed of* -10 m/s." What is wrong with the student's statement? What has the student actually described? Explain.

2: What is the speed of the bird in Question 1?

3: Acceleration is the change in velocity over time. Given this information, is acceleration a vector or a scalar quantity? Explain.

4: A weather forecast states that the temperature is predicted to be -5 °C the following day. Is this temperature a vector or a scalar quantity? Explain.

Solutions

Check Your Understanding

1: Speed is a scalar quantity. It does not change at all with direction changes; therefore, it has magnitude only. If it were a vector quantity, it would change as direction changes (even if its magnitude remained constant).

2.3 Time, Velocity, and Speed

Summary

- Explain the relationship between instantaneous velocity, average velocity, instantaneous speed, average speed, displacement, and time.
- Calculate velocity and speed given initial position, initial position, final position, and final time.
- Derive a graph of velocity vs. time given a graph of position vs. time.
- Interpret a graph of velocity vs. time.

There is more to motion than distance and displacement. Questions such as, “How long does a foot race take?” and “What was the runner’s speed?” cannot be answered without an understanding of other concepts. In this section we add definitions of time, velocity, and speed to expand our description of motion.

Time

As discussed in Chapter 1.2 Physical Quantities and Units, the most fundamental physical quantities are defined by how they are measured. This is the case with time. Every measurement of time involves measuring a change in some physical quantity. It may be a number on a digital clock, a heartbeat, or the position of the Sun in the sky. In physics, the definition of time is simple—**time** is *change*, or the interval over which change occurs. It is impossible to know that time has passed unless something changes.

The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min.**Elapsed time** Δt is the difference between the ending time and beginning time,

$$\Delta t = t_f - t_0,$$

where Δt is the change in time or elapsed time, t_f is the time at the end of the motion, and t_0 is the time at the beginning of the motion. (As usual, the delta symbol, Δ, means the change in the quantity that follows it.)

Life is simpler if the beginning time t_0 is taken to be zero, as when we use a stopwatch. If we were using a stopwatch, it would simply read zero at the start of the lecture and 50 min at the end. If $t_0 = 0$, then $\Delta t = t_f \equiv t$.

In this text, for simplicity's sake,

- motion starts at time equal to zero $t_0 = 0$
- the symbol ***t*** is used for elapsed time unless otherwise specified ($\Delta t = t_f \equiv t$)

Velocity

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

AVERAGE VELOCITY

Average velocity is *displacement (change in position) divided by the time of travel*,

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0},$$

where $\bar{v}$ is the *average* (indicated by the bar over the v) velocity, Δx is the change in position (or displacement), and x_f and x_0 are the final and beginning positions at times t_f and t_0, respectively. If the starting time t_0 is taken to be zero, then the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}.$$

Notice that this definition indicates that *velocity is a vector because displacement is a vector*. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move −4 m (the negative sign indicates that displacement is toward the back of the plane). His average velocity would be

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane

passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

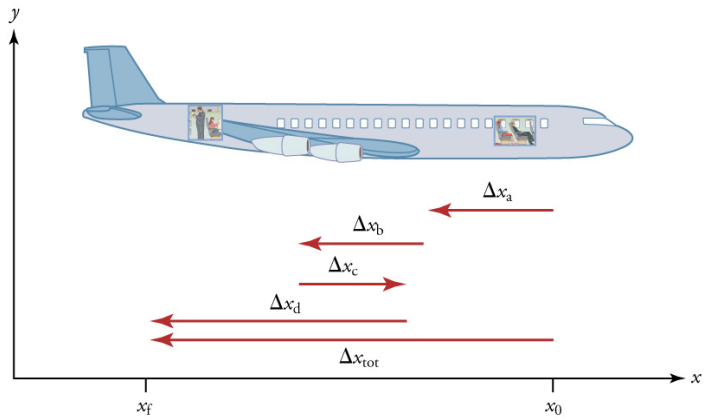


Figure 1. *A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.*

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the *instantaneous velocity or the velocity at a specific instant.* A car's speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) **Instantaneous velocity** $\vec{v}$ is the average velocity at a specific instant in time (or over an infinitesimally small time interval).

Mathematically, finding instantaneous velocity, $\vec{v}$, at a precise instant $\boldsymbol{t}$ can involve taking a limit, a calculus operation beyond the scope of this text. However, under many circumstances, we can find precise values for instantaneous velocity without calculus.

Speed

In everyday language, most people use the terms "speed" and "velocity" interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus *speed is a scalar*. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

Instantaneous speed is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of −3.0 m/s (the minus meaning toward the rear of the plane). At that same time his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is very different from average velocity. **Average speed** is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car's odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus average speed is *not* simply the magnitude of average velocity.

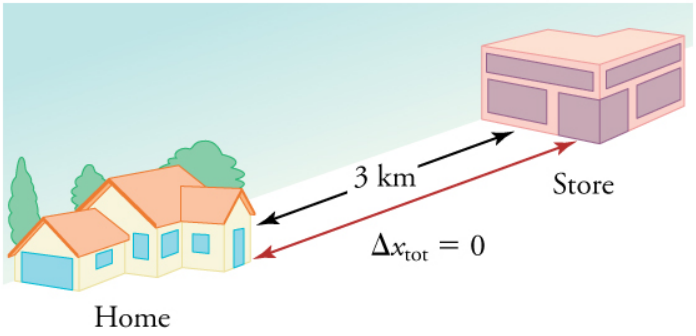


Figure 2. *During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.*

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be very useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in Figure 3. (Note that these graphs depict a very simplified **model** of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we'll probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)

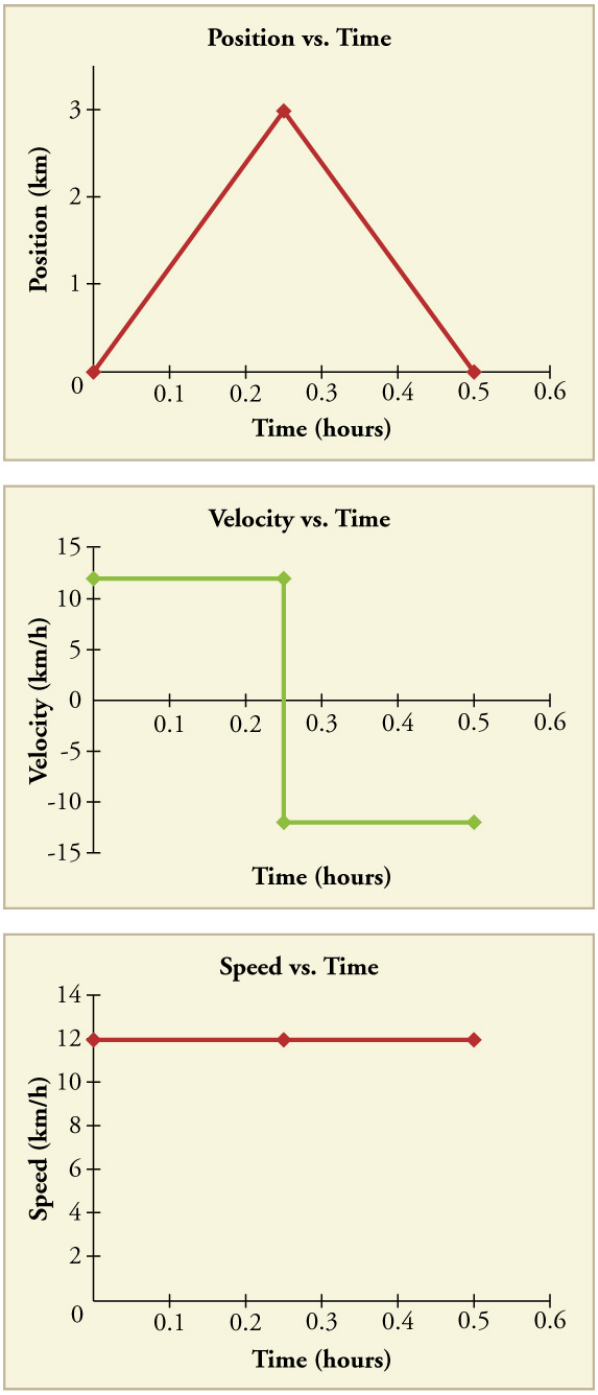


Figure 3. *Position vs. time, velocity vs. time, and speed vs. time on a trip. Note that the velocity for the return trip is negative.*

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION — GETTING A SENSE OF SPEED

If you have spent much time driving, you probably have a good sense of speeds between about 10 and 70 miles per hour. But what are these in meters per second? What do we mean when we say that something is moving at 10 m/s? To get a better sense of what these values really mean, do some observations and calculations on your own:

- calculate typical car speeds in meters per second
- estimate jogging and walking speed by timing yourself; convert the measurements into both m/s and mi/h
- determine the speed of an ant, snail, or falling leaf

Check Your Understanding

1: A commuter train travels from Baltimore to Washington, DC, and back in 1 hour and 45 minutes. The distance between the two stations is approximately 40 miles. What is (a) the average velocity of the train, and (b) the average speed of the train in m/s?

Section Summary

- Time is measured in terms of change, and its SI unit is the second (s). Elapsed time for an event is
$$\Delta t = t_f - t_0$$
where t_f is the final time and t_0 is the initial time. The initial time is often taken to be zero, as if measured with a stopwatch; the elapsed time is then just t.
- Average velocity $\bar{v}$ is defined as displacement divided by the travel time. In symbols, average velocity is
$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}.$$
- The SI unit for velocity is m/s.
- Velocity is a vector and thus has a direction.
- Instantaneous velocity $\vec{v}$ is the velocity at a specific instant or the average velocity for an infinitesimal interval.
- Instantaneous speed is the magnitude of the instantaneous velocity.
- Instantaneous speed is a scalar quantity, as it has no direction specified.

- Average speed is the total distance traveled divided by the elapsed time. (Average speed is *not* the magnitude of the average velocity.) Speed is a scalar quantity; it has no direction associated with it.

Glossary

average speed
distance traveled divided by time during which motion occurs
average velocity
displacement divided by time over which displacement occurs
instantaneous velocity
velocity at a specific instant, or the average velocity over an infinitesimal time interval
instantaneous speed
magnitude of the instantaneous velocity
time
change, or the interval over which change occurs
model
simplified description that contains only those elements necessary to describe the physics of a physical situation
elapsed time
the difference between the ending time and beginning time

Conceptual Questions

1: Give an example (but not one from the text) of a device used to measure time and identify what change in that device indicates a change in time.

2: There is a distinction between average speed and the magnitude of average velocity. Give an example that illustrates the difference between these two quantities.

3: Does a car's odometer measure position or displacement? Does its speedometer measure speed or velocity?

4: If you divide the total distance traveled on a car trip (as determined by the odometer) by the time for the trip, are you calculating the average speed or the magnitude of the average velocity? Under what circumstances are these two quantities the same?

5: How are instantaneous velocity and instantaneous speed related to one another? How do they differ?

Problems & Exercises

1: (a) Calculate Earth's average speed relative to the Sun. (b) What is its average velocity over a period of one year?

2: A helicopter blade spins at exactly 100 revolutions per minute. Its tip is 5.00 m from the center of rotation. (a) Calculate the average speed of the blade tip in the helicopter's frame of reference. (b) What is its average velocity over one revolution?

3: The North American and European continents are moving apart at a rate of about 3 cm/y. At this rate how long will it take them to drift 500 km farther apart than they are at present?

4: Land west of the San Andreas fault in southern California is moving at an average velocity of about 6 cm/y northwest relative to land east of the fault. Los Angeles is west of the fault and may thus someday be at the same latitude as San Francisco, which is east of the fault. How far in the future will this occur if the displacement to be made is 590 km northwest, assuming the motion remains constant?

5: On May 26, 1934, a streamlined, stainless steel diesel train called the Zephyr set the world's nonstop long-distance speed record for trains. Its run from Denver to Chicago took 13 hours, 4 minutes, 58 seconds, and was witnessed by more than a million people along the route. The total distance traveled was 1633.8 km. What was its average speed in km/h and m/s?

6: Tidal friction is slowing the rotation of the Earth. As a result, the orbit of the Moon is increasing in radius at a rate of approximately 4 cm/year. Assuming this to be a constant rate, how many years will pass before the radius of the Moon's orbit increases by 3.84×10^6 m (1%)?

7: A student drove to the university from her home and noted that the odometer reading of her car increased by 12.0 km. The trip took 18.0 min. (a) What was her average speed? (b) If the straight-line distance from her home to the university is 10.3 km in a direction 25.0° south of east, what was her average velocity? (c) If she returned home by the same path 7 h 30 min after she left, what were her average speed and velocity for the entire trip?

8: The speed of propagation of the action potential (an electrical signal) in a nerve cell depends (inversely) on the diameter of the axon (nerve fiber). If the nerve cell connecting the spinal cord to your feet is 1.1 m long, and the nerve impulse speed is 18 m/s, how long does it take for the nerve signal to travel this distance?

9: Conversations with astronauts on the lunar surface were characterized by a kind of echo in which the earthbound person's voice was so loud in the astronaut's space helmet that it was picked up by the astronaut's microphone and transmitted back to Earth. It is reasonable to assume that the echo time equals the time necessary for the radio wave to travel from the Earth to the Moon and back (that is, neglecting any time delays in the electronic equipment). Calculate the distance from Earth to the Moon given that the echo time was 2.56 s and that radio waves travel at the speed of light (3.00×10^8 m/s).

10: A football quarterback runs 15.0 m straight down the playing field in 2.50 s. He is then hit and pushed 3.00 m straight backward in 1.75 s. He breaks the tackle and runs straight forward another 21.0 m in 5.20 s. Calculate his average velocity (a) for each of the three intervals and (b) for the entire motion.

11: The planetary model of the atom pictures electrons orbiting the atomic nucleus much as planets orbit the Sun. In this model you can view hydrogen, the simplest atom, as having a single electron in a circular orbit 1.06×10^{-10} m in diameter. (a) If the average speed of the electron in this orbit is known to be 2.20×10^6 m/s, calculate the **distance** it travels in one second it makes about the nucleus. (b) What is the electron's average **velocity**? Hint: What is the displacement?

Solutions

Check Your Understanding

1: (a) The average velocity of the train is zero because $x_f = x_0$; the train ends up at the same place it starts.

(b) The average speed of the train is calculated below. Note that the train travels 40 miles one way and 40 miles back, for a total distance of 80 miles.

$$\frac{\text{Distance}}{\text{Time}} = \frac{80 \text{ miles}}{105 \text{ minutes}}$$

$$\frac{80 \text{ miles}}{105 \text{ minutes}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{1 \text{ meter}}{3.28 \text{ feet}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 20 \text{ m/s}$$

Problems & Exercises

1: (a) 3.0 x 10^4 m/s (b) 0 m/s

3: 2 x 10^7 years

5: 34.689 m/s = 124.88 km/h

7: (a) 40.0 km/h (b) 34.3 km/h, 25^o S of E (c) Average speed = 3.20 km/h, $\overline{v}$ = 0.

9: 384,000 km

11: (a) 2.20 x 10^6 meters (b) 0 meters

2.4 Acceleration

Summary

- Define and distinguish instantaneous acceleration, average acceleration, and deceleration.
- Calculate acceleration given initial time, initial velocity, and final velocity.

Figure 1. *A plane decelerates, or slows down, as it comes in for landing in St. Maarten. Its acceleration is opposite in direction to its velocity. (credit: Steve Conry, Flickr).*

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the **acceleration**, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

AVERAGE ACCELERATION

Average Acceleration is *the rate at which velocity changes*,

$$\bar{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_0}{t_f - t_0},$$

where $\bar{a}$ is average acceleration, $\vec{v}$ is velocity, and t is time. (The bar over the $\bar{a}$ means *average* acceleration.)

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are m/s^2, meters

per second squared or meters per second per second, which literally means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in *direction*. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

ACCELERATION AS A VECTOR

Acceleration is a vector in the same direction as the *change* in velocity, $\Delta\vec{v}$. Since velocity is a vector, it can change either in magnitude or in direction. Acceleration is therefore a change in either speed or direction, or both.

Keep in mind that although acceleration is in the direction of the *change* in velocity, it is not always in the direction of *motion*. When an object slows down, its acceleration is opposite to the direction of its motion. This is known as **deceleration**.

Figure 2. *A subway train in Sao Paulo, Brazil, decelerates as it comes into a station. It is accelerating in a direction opposite to its direction of motion. (credit: Yusuke Kawasaki, Flickr)*

MISCONCEPTION ALERT: DECELERATION VS. NEGATIVE ACCELERATION

Deceleration always refers to acceleration in the direction opposite to the direction of the velocity. Deceleration always reduces speed. Negative acceleration, however, is acceleration *in the negative direction in the chosen coordinate system*. Negative acceleration may or may not be deceleration, and deceleration may or may not be considered negative acceleration. For example, consider Figure 3.

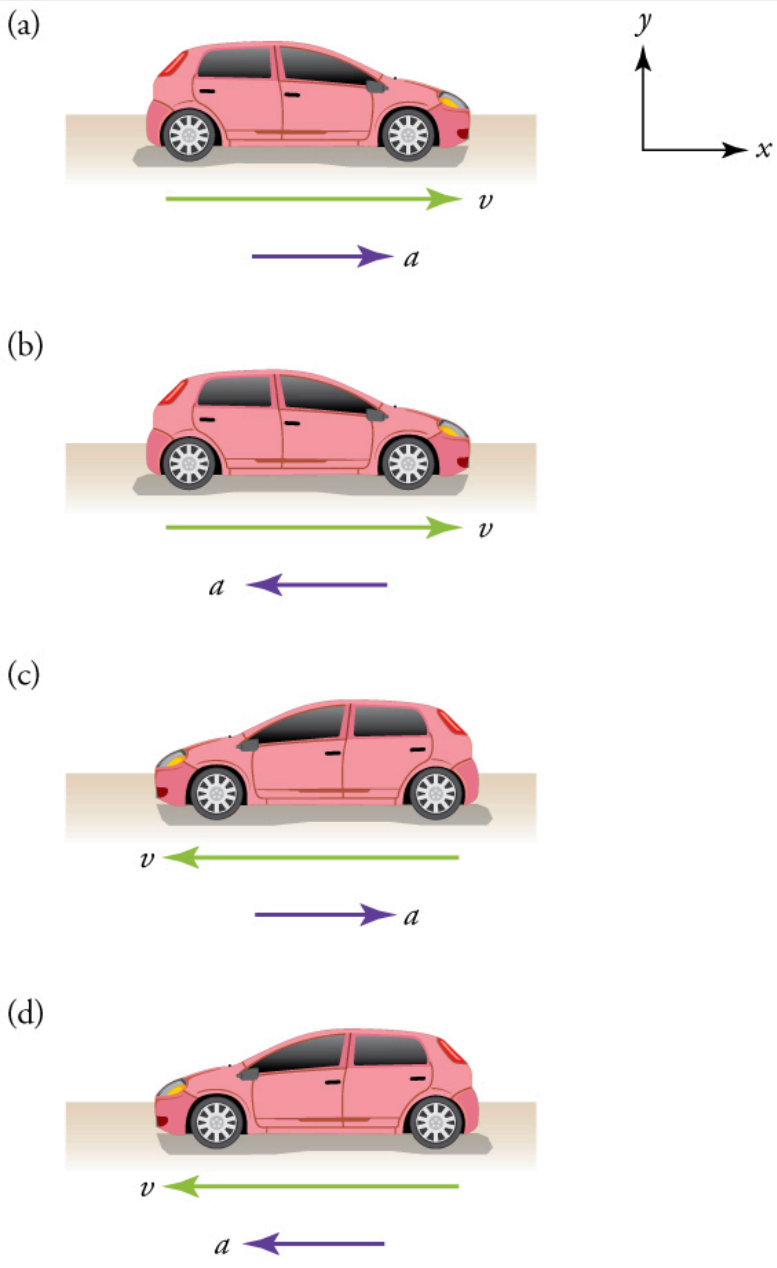


Figure 3. *(a) This car is speeding up as it moves toward the right. It therefore has positive acceleration in our coordinate system. (b) This car is slowing down as it moves toward the right. Therefore, it has negative acceleration in our coordinate system, because its acceleration is toward the left. The car is also decelerating: the direction of its acceleration is opposite to its direction of motion. (c) This car is moving toward the left, but slowing down over time. Therefore, its acceleration is positive in our coordinate system because it is toward the right. However, the car is decelerating because its acceleration is opposite to its motion. (d) This car is speeding up as it moves toward the left. It has negative acceleration because it is accelerating toward the left. However, because its acceleration is in the*

same direction as its motion, it is speeding up (not decelerating).

Example 1: Calculating Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

Strategy

First we draw a sketch and assign a coordinate system to the problem. This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.

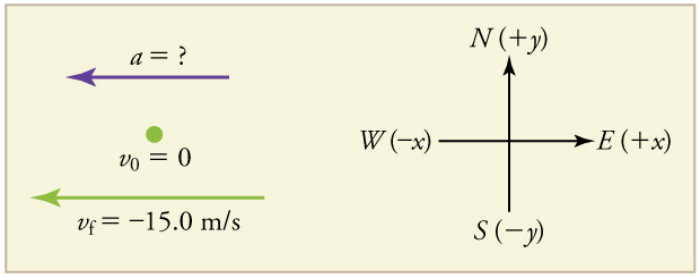


Figure 4.

We can solve this problem by identifying $\Delta\vec{v}$ and Δt from the given information and then calculating the average acceleration directly from the equation

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_0}{t_f - t_0}.$$

Solution

1. Identify the knowns. $\vec{v}_0 = 0$, $\Delta\vec{v} = -15.0\ \text{m/s}$ (the negative sign indicates direction toward the west), $\Delta t = 1.80\ \text{s}$.

2. Find the change in velocity. Since the horse is going from zero to -15.0 m/s, its change in velocity equals its final velocity: $\Delta\vec{v} = \vec{v}_f = -15.0\ \text{m/s}$.

3. Plug in the known values ($\Delta\vec{v}$ and Δt) and solve for the unknown $\bar{a}$.

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{-15.0\ \text{m/s}}{1.80\ \text{s}} = -8.33\ \text{m/s}^2.$$

Discussion

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of 8.33 m/s^2 due west means that the horse increases its velocity by 8.33 m/s due west each second, that is, 8.33 meters per second per second, which we write as 8.33 m/s^2. This is truly an average acceleration, because the ride is not smooth. We shall see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

Instantaneous Acceleration

Instantaneous acceleration $\vec{a}$, or the *acceleration at a specific instant in time*, is obtained by the same process as discussed for instantaneous velocity in Chapter 2.3 Time, Velocity, and Speed—that is, by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. Figure 5 shows graphs of instantaneous acceleration versus time for two very different motions. In Figure 5(a), the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In this case, we should treat this motion as if it had a constant acceleration equal to the average (in this case about 1.8 m/s^2). In Figure 5(b), the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of +3.0 m/s^2 and -2.0 m/s^2, respectively.

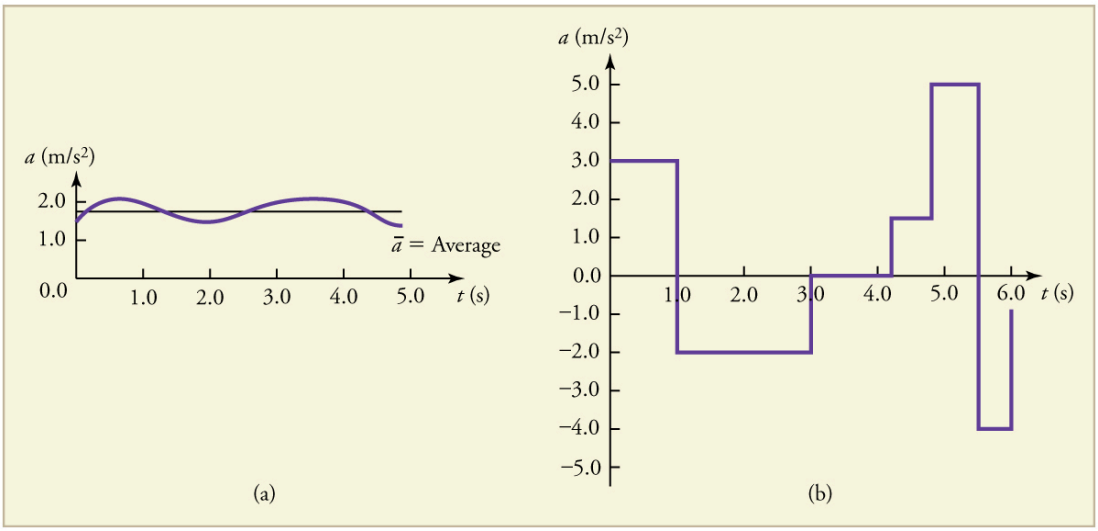


Figure 5. *Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Here the acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.*

The next several examples consider the motion of the subway train shown in Figure 6. In (a) the shuttle

moves to the right, and in (b) it moves to the left. The examples are designed to further illustrate aspects of motion and to illustrate some of the reasoning that goes into solving problems.

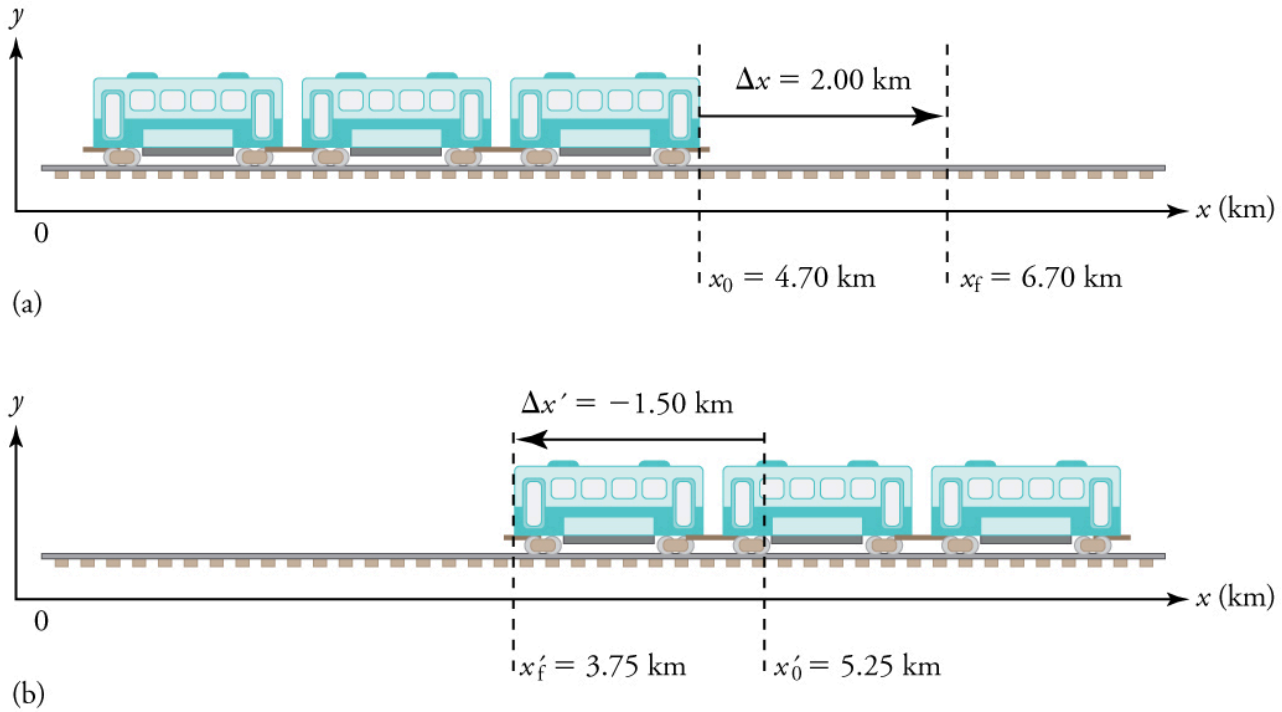


Figure 6. *One-dimensional motion of a subway train considered in Example 2, Example 3, Example 4, Example 5, Example 6, and Example 7. Here we have chosen the x-axis so that + means to the right and − means to the left for displacements, velocities, and accelerations. (a) The subway train moves to the right from x_0 to x_f. Its displacement Δx is +2.0 km. (b) The train moves to the left from x'_0 to x'_f. Its displacement Δx' is -1.5 km. (Note that the prime symbol (') is used simply to distinguish between displacement in the two different situations. The distances of travel and the size of the cars are on different scales to fit everything into the diagram.).*

Example 2: Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of Figure 6?

Strategy

A drawing with a coordinate system is already provided, so we don't need to make a sketch, but we should analyze it to make sure we understand what it is showing. Pay particular attention to the coordinate system. To find displacement, we use the equation $\Delta x = x_f - x_0$. This is straightforward since the initial and final positions are given.

Solution

1. Identify the knowns. In the figure we see that $x_f = 6.70\ \text{km}$ and $x_0 = 4.70\ \text{km}$ for part (a), and $x'_f = 3.75\ \text{km}$ and $x'_0 = 5.25\ \text{km}$ for part (b).

2. Solve for displacement in part (a).

$$\Delta x = x_f - x_0 = 6.70 \text{ km} - 4.70 \text{ km} = +2.00 \text{ km}$$

3. Solve for displacement in part (b).

$$\Delta x' = x'_f - x'_0 = 3.75 \text{ km} - 5.25 \text{ km} = -1.50 \text{ km}$$

Discussion

The direction of the motion in (a) is to the right and therefore its displacement has a positive sign, whereas motion in (b) is to the left and thus has a negative sign.

Example 3: Calculating Distance Traveled with Displacement: A Subway Train

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in Figure 6?

Strategy

To answer this question, think about the definitions of distance and distance traveled, and how they are related to displacement. Distance between two positions is defined to be the magnitude of displacement, which was found in Example 2. Distance traveled is the total length of the path traveled between the two positions. (See Chapter 2.1 Displacement.) In the case of the subway train shown in Figure 6, the distance traveled is the same as the distance between the initial and final positions of the train.

Solution

1. The displacement for part (a) was +2.00 km. Therefore, the distance between the initial and final positions was 2.00 km, and the distance traveled was 2.00 km.

2. The displacement for part (b) was -1.5 km. Therefore, the distance between the initial and final positions was 1.50 km, and the distance traveled was 1.50 km.

Discussion

Distance is a scalar. It has magnitude but no sign to indicate direction.

Example 4: Calculating Acceleration: A Subway Train Speeding Up

Suppose the train in Figure 6(a) accelerates from rest to 30.0 km/h in the first 20.0 s of its motion. What is its average acceleration during that time interval?

Strategy

It is worth it at this point to make a simple sketch:

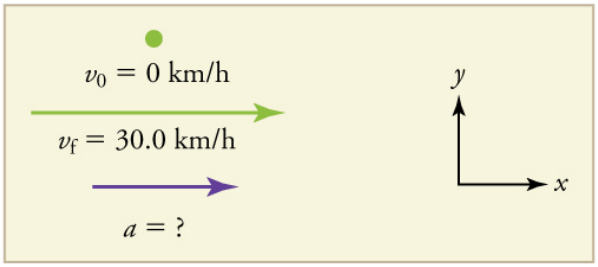


Figure 8.

This problem involves three steps. First we must determine the change in velocity, then we must determine the change in time, and finally we use these values to calculate the acceleration.

Solution

1. Identify the knowns. $\vec{v}_0 = 0$ (the trains starts at rest), $\vec{v}_f = 30.0\ \text{km/h}$, and $\Delta t = 20.0\ \text{s}$.

2. Calculate $\Delta\vec{v}$. Since the train starts from rest, its change in velocity is $\Delta\vec{v} = +30.0\ \text{km/h}$, where the plus sign means velocity to the right.

3. Plug in known values and solve for the unknown, $\bar{a}$.

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{+30.0\ \text{km/h}}{20.0\ \text{s}}$$

4. Since the units are mixed (we have both hours and seconds for time), we need to convert everything into SI units of meters and seconds.

$$\bar{a} = \frac{+30.0\ \text{km/h}}{20.0\ \text{s}} \times \frac{10^3\ \text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{3600\ \text{s}} = 0.417\ \text{m/s}^2$$

Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train starts from rest and ends up with a velocity to the right (also positive). So acceleration is in the same direction as the *change* in velocity, as is always the case.

Example 5: Calculate Acceleration: A Subway Train Slowing Down

Now suppose that at the end of its trip, the train in Figure 6(a) slows to a stop from a speed of 30.0 km/h in 8.00 s. What is its average acceleration while stopping?

Strategy

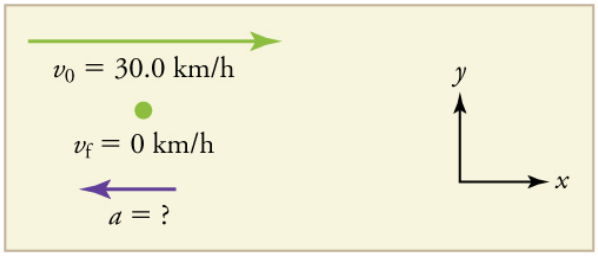


Figure 9.

In this case, the train is decelerating and its acceleration is negative because it is toward the left. As in the previous example, we must find the change in velocity and the change in time and then solve for acceleration.

Solution

1. Identify the knowns. $\vec{v}_0 = 30.0\ \text{km/h}$, $\vec{v}_f = 0\ \text{km/h}$ (the train is stopped, so its velocity is 0), and $\Delta t = 8.00\ \text{s}$.

2. Solve for the change in velocity, $\Delta\vec{v}$

$$\Delta\vec{v} = \vec{v}_f - \vec{v}_0 = 0 - 30.0\ \text{km/h} = -30.0\ \text{km/h}$$

3. Plug in the knowns, $\Delta\vec{v}$ and Δt, and solve for $\bar{a}$.

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{-30.0\ \text{km/h}}{8.00\ \text{s}}$$

4. Convert the units to meters and seconds.

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{-30.0\ \text{km/h}}{8.00\ \text{s}} \times \frac{10^3\ \text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{3600\ \text{s}} = -1.04\ \text{m/s}^2.$$

The minus sign indicates that acceleration is to the left. This sign is reasonable because the train initially has a positive velocity in this problem, and a negative acceleration would oppose the motion. Again, acceleration is in the same direction as the *change* in velocity, which is negative here. This acceleration can be called a deceleration because it has a direction opposite to the velocity.

The graphs of position, velocity, and acceleration vs. time for the trains in Example 4 and Example 5 are displayed in Figure 9**.** (We have taken the velocity to remain constant from 20 to 40 s, after which the train decelerates.)

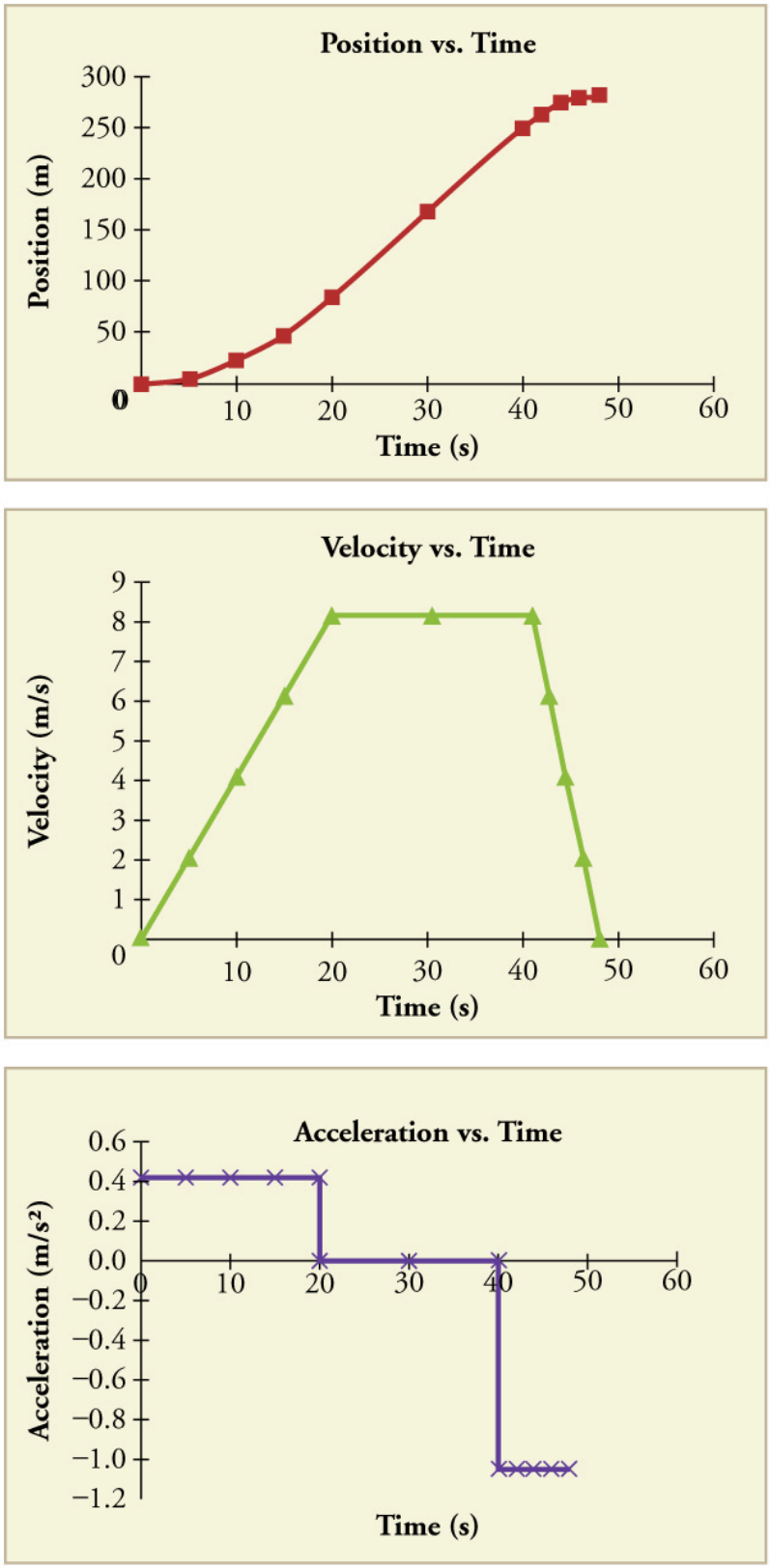


Figure 9. *(a) Position of the train over time. Notice that the train's position changes slowly at the beginning of the journey, then more and more quickly as it picks up speed. Its position then changes more slowly as it slows down at the end of the journey. In the middle of the*

journey, while the velocity remains constant, the position changes at a constant rate. (b) Velocity of the train over time. The train's velocity increases as it accelerates at the beginning of the journey. It remains the same in the middle of the journey (where there is no acceleration). It decreases as the train decelerates at the end of the journey. (c) The acceleration of the train over time. The train has positive acceleration as it speeds up at the beginning of the journey. It has no acceleration as it travels at constant velocity in the middle of the journey. Its acceleration is negative as it slows down at the end of the journey.

Example 6: Calculating Average Velocity: The Subway Train

What is the average velocity of the train in part b of Example 2, and shown again below, if it takes 5.00 min to make its trip?

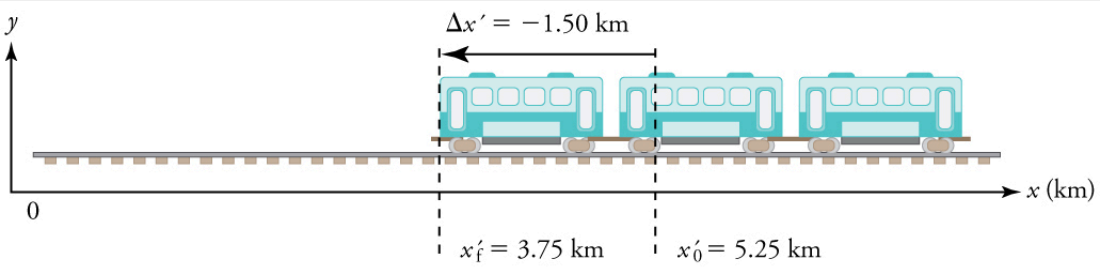


Figure 10.

Strategy

Average velocity is displacement divided by time. It will be negative here, since the train moves to the left and has a negative displacement.

Solution

1. Identify the knowns. $x\prime_f = 3.75\ \text{km}$, $x\prime_0 = 5.25\ \text{km}$, $\Delta t = 5.00\ \text{min}$.

2. Determine displacement, $\Delta x'$. We found $\Delta x'$ to be $-1.5\ \text{km}$ in Example 2.

3. Solve for average velocity.

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \frac{-1.5\ \text{km}}{5.00\ \text{min}}$$

4. Convert units.

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \frac{-1.5\ \text{km}}{5.00\ \text{min}} \times \frac{60\ \text{min}}{1\ \text{h}} = -18.0\ \text{km/h}$$

Discussion

The negative velocity indicates motion to the left.

Example 7: Calculating Deceleration: The Subway Train

Finally, suppose the train in Figure 10 slows to a stop from a velocity of 20.0 km/h in 10.0 s. What is its average acceleration?

Strategy

Once again, let's draw a sketch:

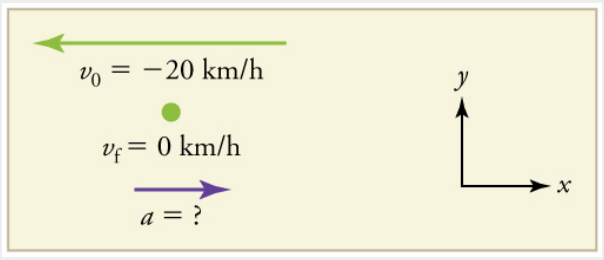


Figure 11.

As before, we must find the change in velocity and the change in time to calculate average acceleration.

Solution

1. Identify the knowns. $\vec{v}_0 = -20 \text{ km/h}, \vec{v}_f = 0 \text{ km/h}, \Delta t = 10.0 \text{ s}.$

2. Calculate $\Delta\vec{v}$. The change in velocity here is actually positive, since

$$\Delta\vec{v} = \Delta\vec{v}_f - \Delta\vec{v}_0 = 0 - (-20 \text{ km/h} = +20 \text{ km/h}).$$

3. Solve for $\bar{a}$.

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{+20.0 \text{ km/h}}{10.0 \text{ s}}$$

4. Convert units

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{+20.0 \text{ km/h}}{10.0 \text{ s}} \times \frac{10^3 \text{ m}}{1 \text{ km}} \times \frac{1 \text{ h}}{3600 \text{ s}} = +0.556 \text{ m/s}^2$$

Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train initially has a negative velocity (to the left) in this problem and a positive acceleration opposes the motion (and so it is to the right). Again, acceleration is in the same direction as the *change* in velocity, which is positive here. As in Example 5, this acceleration can be called a deceleration since it is in the direction opposite to the velocity.

Sign and Direction

Perhaps the most important thing to note about these examples is the signs of the answers. In our chosen coordinate system, plus means the quantity is to the right and minus means it is to the left. This is easy to imagine for displacement and velocity. But it is a little less obvious for acceleration. Most people interpret negative acceleration as the slowing of an object. This was not the case in Example 7, where a positive acceleration slowed a negative velocity. The crucial distinction was that the acceleration was in the opposite direction from the velocity. In fact, a negative acceleration will *increase* a negative velocity. For example, the train moving to the left in Figure 10 is sped up by an acceleration to the left. In that case, both $\vec{v}$ and $\vec{a}$ are negative. The plus and minus signs give the directions of the accelerations. If acceleration has the same sign as the velocity, the object is speeding up. If acceleration has the opposite sign as the velocity, the object is slowing down.

Check Your Understanding

1: An airplane lands on a runway traveling east. Describe its acceleration.

PHET EXPLORATIONS: MOVING MAN SIMULATION

Learn about position, velocity, and acceleration graphs. Move the little man back and forth with the mouse and plot his motion. Set the position, velocity, or acceleration and let the simulation move the man for you.

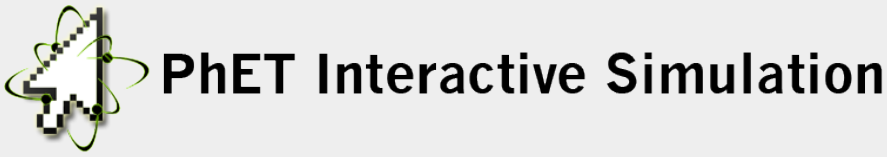


Figure 13. *Moving Man.*

Section Summary

- Acceleration is the rate at which velocity changes. In symbols, **average acceleration** $\bar{a}$ is
$$\bar{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_0}{t_f - t_0}.$$
- The SI unit for acceleration is m/s^2.
- Acceleration is a vector, and thus has a both a magnitude and direction.

- Acceleration can be caused by either a change in the magnitude or the direction of the velocity.
- Instantaneous acceleration $\vec{a}$ is the acceleration at a specific instant in time.
- Deceleration is an acceleration with a direction opposite to that of the velocity.

Glossary

acceleration
the rate of change in velocity; the change in velocity over time
average acceleration
the change in velocity divided by the time over which it changes
instantaneous acceleration
acceleration at a specific point in time
deceleration
acceleration in the direction opposite to velocity; acceleration that results in a decrease in velocity

Conceptual Questions

1: Is it possible for speed to be constant while acceleration is not zero? Give an example of such a situation.

2: Is it possible for velocity to be constant while acceleration is not zero? Explain.

3: Give an example in which velocity is zero yet acceleration is not.

4: If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

5: Plus and minus signs are used in one-dimensional motion to indicate direction. What is the sign of an acceleration that reduces the magnitude of a negative velocity? Of a positive velocity?

Problems & Exercises

1: A cheetah can accelerate from rest to a speed of 30.0 m/s in 7.00 s. What is its acceleration?

2: Professional Application

Dr. John Paul Stapp was U.S. Air Force officer who studied the effects of extreme deceleration on the human body. On December 10, 1954, Stapp rode a rocket sled, accelerating from rest to a top speed of 282 m/s (1015 km/h) in 5.00 s, and was brought jarringly back to rest in only 1.40 s! Calculate his (a) acceleration and (b) deceleration. Express each in multiples of g (9.80 m/s^2) by taking its ratio to the acceleration of gravity.

3: A commuter backs her car out of her garage with an acceleration of 1.40 m/s^2. (a) How long does it take her to reach a speed of 2.00 m/s? (b) If she then brakes to a stop in 0.800 s, what is her deceleration?

4: Assume that an intercontinental ballistic missile goes from rest to a suborbital speed of 6.50 km/s in 60.0 s (the actual speed and time are classified). What is its average acceleration in m/s and in multiples of *g* (9.80 m/s^2)?

Solutions

Check Your Understanding

1: If we take east to be positive, then the airplane has negative acceleration, as it is accelerating toward the west. It is also decelerating: its acceleration is opposite in direction to its velocity.

Problems & Exercises

1: 4.29 m/s^2

3: (a) 1.43 seconds (b) -2.50 m/s^2

2.5 Graphical Analysis of One-Dimensional Motion

Summary

- Describe a straight-line graph in terms of its slope and y-intercept.
- Determine average velocity or instantaneous velocity from a graph of position vs. time.
- Determine average or instantaneous acceleration from a graph of velocity vs. time.
- Derive a graph of velocity vs. time from a graph of position vs. time.
- Derive a graph of acceleration vs. time from a graph of velocity vs. time.

A graph, like a picture, is worth a thousand words. Graphs not only contain numerical information; they also reveal relationships between physical quantities. This section uses graphs of displacement, velocity, and acceleration versus time to illustrate one-dimensional kinematics.

Slopes and General Relationships

First note that graphs in this text have perpendicular axes, one horizontal and the other vertical. When two physical quantities are plotted against one another in such a graph, the horizontal axis is usually considered to be an **independent variable** and the vertical axis a **dependent variable**. If we call the horizontal axis the ***x***-axis and the vertical axis the ***y***-axis, as in Figure 1, a straight-line graph has the general form

$$y = mx + b$$

Here *m* is the **slope,** defined to be the rise divided by the run (as seen in the figure) of the straight line. The letter *b* is used for the ***y*-intercept**, which is the point at which the line crosses the vertical axis.

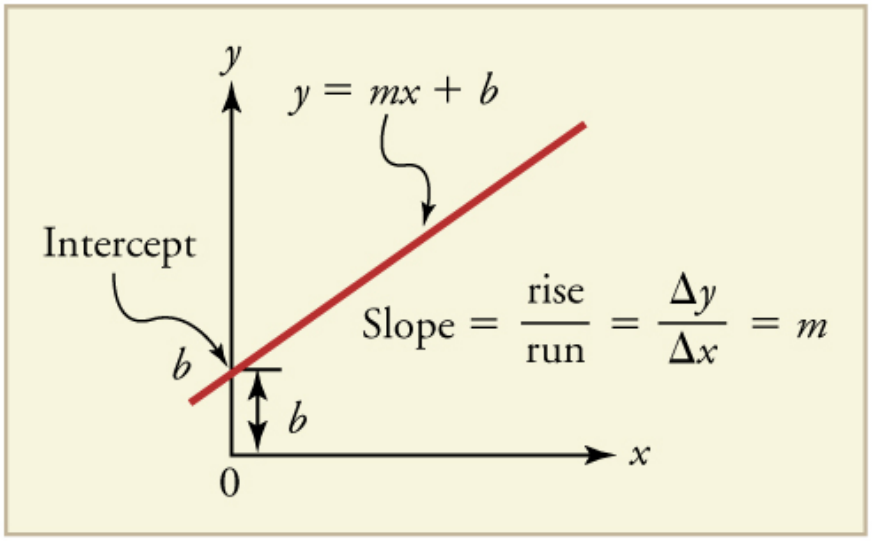


Figure 1. *A straight-line graph. The equation for a straight line is y = mx + b.*

Graph of Displacement vs. Time (*a* = 0, so *v* is constant)

Time is usually an independent variable that other quantities, such as displacement, depend upon. A graph of displacement versus time would, thus, have *x* on the vertical axis and *t* on the horizontal axis. Figure 2 is just such a straight-line graph. It shows a graph of displacement versus time for a jet-powered car on a very flat dry lake bed in Nevada**.**

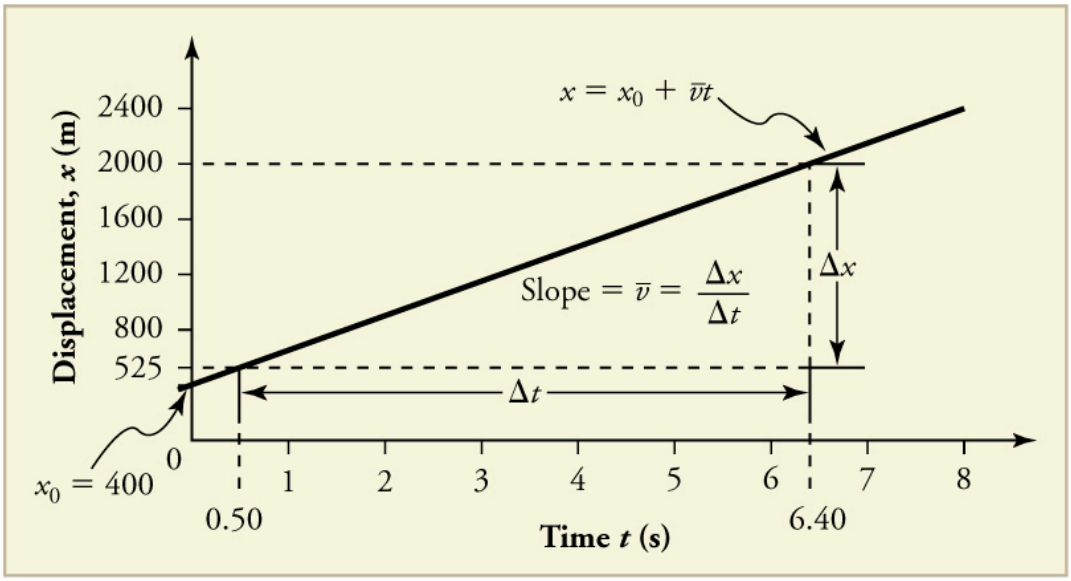


Figure 2. *Graph of displacement versus time for a jet-powered car on the Bonneville Salt Flats.*

Using the relationship between dependent and independent variables, we see that the slope in the graph above is average velocity $\bar{v}$ and the intercept is displacement at time zero—that is, x_0. Substituting these symbols into $y = mx + b$ gives

$$x = \bar{v}t + x_0$$

or

$$x = x_0 + \bar{v}t.$$

Thus a graph of displacement versus time gives a general relationship among displacement, velocity, and time, as well as giving detailed numerical information about a specific situation.

THE SLOPE OF X VS. T

The slope of the graph of displacement *x* vs. time *t* is velocity $\vec{v}$.

$$\text{Slope} = \frac{\Delta x}{\Delta t} = \bar{v}$$

Notice that this equation is the same as that derived algebraically from other motion equations in Chapter 2.6 Motion Equations for Constant Acceleration in One Dimension.

From the figure we can see that the car has a displacement of 25 m at 0.50 s and 2000 m at 6.40 s. Its displacement at other times can be read from the graph; furthermore, information about its velocity and acceleration can also be obtained from the graph.

Example 1: Determining Average Velocity from a Graph of Displacement versus Time: Jet Car

Find the average velocity of the car whose position is graphed in Figure 2.

Strategy

The slope of a graph of *x* vs. *t* is average velocity, since slope equals rise over run. In this case, rise = change in displacement and run = change in time, so that

$$\text{Slope} = \frac{\Delta x}{\Delta t} = \bar{v}$$

Since the slope is constant here, any two points on the graph can be used to find the slope. (Generally speaking, it is most accurate to use two widely separated points on the straight line. This is because any error in reading data from the graph is proportionally smaller if the interval is larger.)

Solution

1. Choose two points on the line. In this case, we choose the points labeled on the graph: (6.4 s, 2000 m) and (0.50 s, 525 m). (Note, however, that you could choose any two points.)

2. Substitute the *x* and *t* values of the chosen points into the equation. Remember in calculating change (Δ) we always use final value minus initial value.

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{2000\ \text{m} - 525\ \text{m}}{6.4\ \text{s} - 0.50\ \text{s}}$$

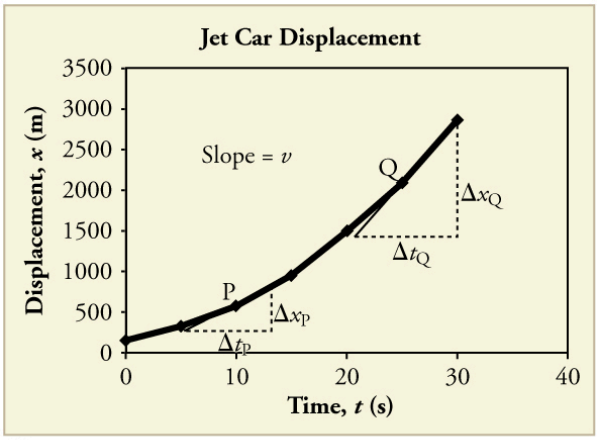
$$= 250\ \text{m/s}$$

Discussion

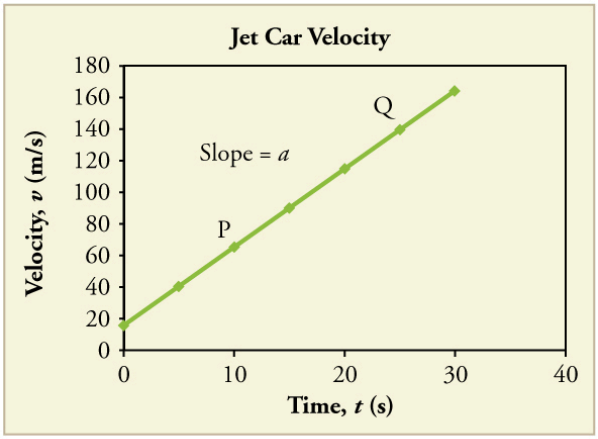
This is an impressively large land speed (900 km/h, or about 560 mi/h): much greater than the typical highway speed limit of 60 mi/h (27 m/s or 96 km/h), but considerably shy of the record of 343 m/s (1234 km/h or 766 mi/h) set in 1997.

Graphs of Motion when α is constant but $\alpha \neq 0$

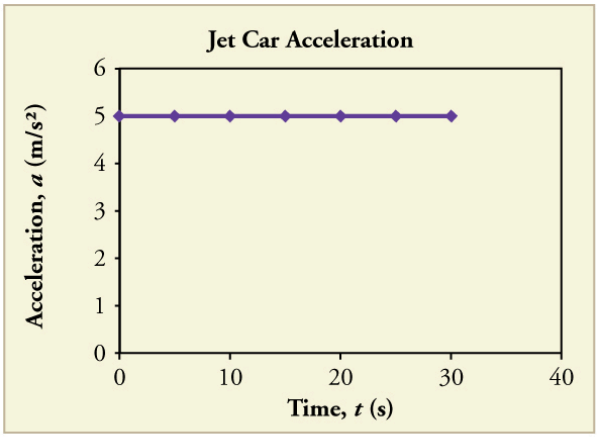
The graphs in Figure 3 below represent the motion of the jet-powered car as it accelerates toward its top speed, but only during the time when its acceleration is constant. Time starts at zero for this motion (as if measured with a stopwatch), and the displacement and velocity are initially 200 m and 15 m/s, respectively.



(a)



(b)



(c)

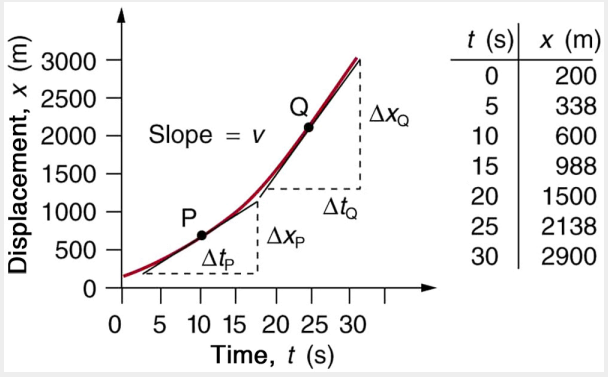
Figure 3. *Graphs of motion of a jet-powered car during the time span when its acceleration is constant. (a) The slope of an x vs. t graph is velocity. This is shown at two points, and the instantaneous velocities obtained are plotted in the next graph. Instantaneous velocity at any point is the slope of the tangent at that point. (b) The slope of the [latex]\vec{v}[/latex] vs. t graph is constant for this part of the motion, indicating constant acceleration. (c) Acceleration has the constant value of 5.0 m/s^2 over the time interval plotted.*

Figure 4. *A U.S. Air Force jet car speeds down a track. (credit: Matt Trostle, Flickr).*

The graph of displacement versus time in Figure 3(a) is a curve rather than a straight line. The slope of the curve becomes steeper as time progresses, showing that the velocity is increasing over time. The slope at any point on a displacement-versus-time graph is the instantaneous velocity at that point. It is found by drawing a straight line tangent to the curve at the point of interest and taking the slope of this straight line. Tangent lines are shown for two points in Figure 3(a). If this is done at every point on the curve and the values are plotted against time, then the graph of velocity versus time shown in Figure 3(b) is obtained. Furthermore, the slope of the graph of velocity versus time is acceleration, which is shown in Figure 3(c).

Example 2: Determining Instantaneous Velocity from the Slope at a Point: Jet Car

Calculate the velocity of the jet car at a time of 25 s by finding the slope of the *x* vs. *t* graph in the graph below.



t (s)	x (m)
0	200
5	338
10	600
15	988
20	1500
25	2138
30	2900

Figure 5. *The slope of an x vs. t graph is velocity. This is shown at two points. Instantaneous velocity at any point is the slope of the tangent at that point.*

Strategy

The slope of a curve at a point is equal to the slope of a straight line tangent to the curve at that point. This principle is illustrated in Figure 5, where Q is the point at $t = 25\ \text{s}$.

Solution

1. Find the tangent line to the curve at $t = 25\ \text{s}$.

2. Determine the endpoints of the tangent. These correspond to a position of 1300 m at time 19 s and a position of 3120 m at time 32 s.

3. Plug these endpoints into the equation to solve for the slope, $\vec{v}$.

$$\text{Slope} = \vec{v}_Q = \frac{\Delta x_Q}{\Delta t_Q} = \frac{3120\ \text{m} - 1300\ \text{m}}{32\ \text{s} - 19\ \text{s}}$$
$$= \frac{1820\ \text{m}}{13\ \text{s}} = 140\ \text{m/s}$$

Discussion

This is the value given in this figure's table for $\vec{v}$ at $t = 25\ \text{s}$. The value of 140 m/s for $\vec{v}_Q$ is plotted in Figure 5. The entire graph of $\vec{v}$ vs. t can be obtained in this fashion.

Carrying this one step further, we note that the slope of a velocity versus time graph is acceleration. Slope is rise divided by run; on a $\vec{v}$ vs. t graph, rise = change in velocity $\Delta\vec{v}$ and run = change in time Δt.

THE SLOPE OF V VS. T

The slope of a graph of velocity $\vec{v}$ vs. time t is acceleration $\vec{a}$.

$$\text{Slope} = \frac{\Delta\vec{v}}{\Delta t} = \vec{a}$$

Since the velocity versus time graph in Figure 3(b) is a straight line, its slope is the same everywhere, implying that acceleration is constant. Acceleration versus time is graphed in Figure 3(c).

Additional general information can be obtained from Figure 5 and the expression for a straight line, *y* = *mx* + *b*.

In this case, the vertical axis *y* is *V*, the intercept *b* is $\vec{v}_0$, the slope *m* is $\vec{a}$, and the horizontal axis *x* is *t*. Substituting these symbols yields

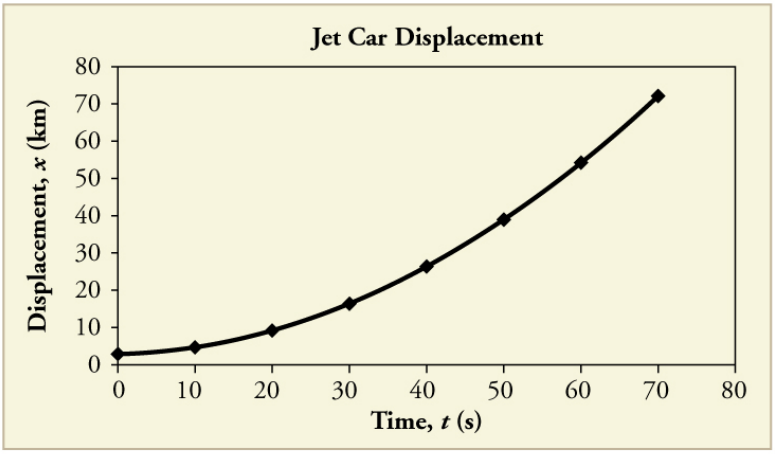
$$\vec{v} = \vec{v}_0 + \vec{a}t$$

A general relationship for velocity, acceleration, and time has again been obtained from a graph. Notice that this equation was also derived algebraically from other motion equations in Chapter 2.6 Motion Equations for Constant Acceleration in One Dimension.

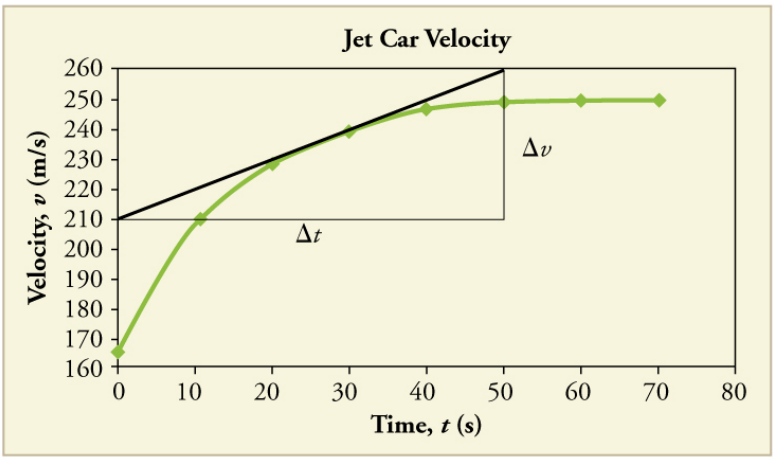
It is not accidental that the same equations are obtained by graphical analysis as by algebraic techniques. In fact, an important way to *discover* physical relationships is to measure various physical quantities and then make graphs of one quantity against another to see if they are correlated in any way. Correlations imply physical relationships and might be shown by smooth graphs such as those above. From such graphs, mathematical relationships can sometimes be postulated. Further experiments are then performed to determine the validity of the hypothesized relationships.

Graphs of Motion Where Acceleration is Not Constant

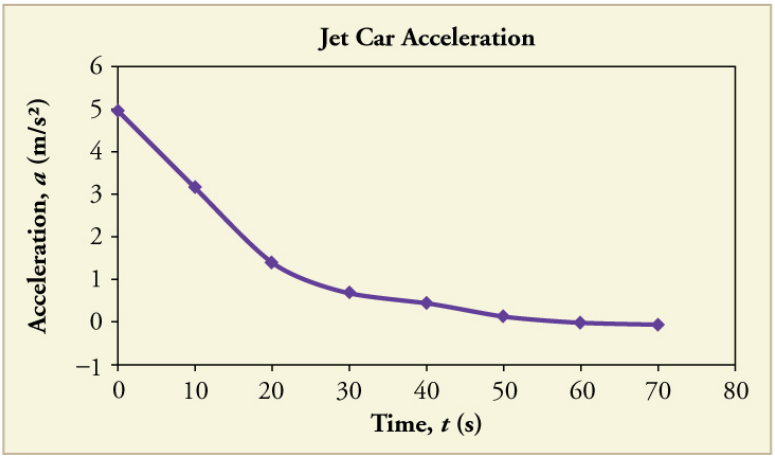
Now consider the motion of the jet car as it goes from 165 m/s to its top velocity of 250 m/s, graphed in Figure 6. Time again starts at zero, and the initial displacement and velocity are 2900 m and 165 m/s, respectively. (These were the final displacement and velocity of the car in the motion graphed in Figure 3.) Acceleration gradually decreases from 5.0 m/s^2 to zero when the car hits 250 m/s. The slope of the x vs. t graph increases until t = 55 s, after which time the slope is constant. Similarly, velocity increases until 55 s and then becomes constant, since acceleration decreases to zero at 55 s and remains zero afterward.



(a)



(b)



(c)

Figure 6. *Graphs of motion of a jet-powered car as it reaches its top velocity. This motion begins where the motion in Figure 3 ends. (a) The slope of this graph is velocity; it is plotted in the next graph. (b) The velocity gradually approaches its top value. The slope of this graph is acceleration; it is plotted in the final graph. (c) Acceleration gradually declines to zero when velocity becomes constant.*

Example 3: Calculating Acceleration from a Graph of Velocity versus Time

Calculate the acceleration of the jet car at a time of 25 s by finding the slope of the $\vec{v}$ vs. t graph in Figure 6(b).

Strategy

The slope of the curve at t = 25 s is equal to the slope of the line tangent at that point, as illustrated in Figure 6(b).

Solution

Determine endpoints of the tangent line from the figure, and then plug them into the equation to solve for slope, $\vec{a}$.

$$\text{Slope} = \frac{\Delta \vec{v}}{\Delta t} = \frac{260 \text{ m/s} - 210 \text{ m/s}}{51 \text{ s} - 1.0 \text{ s}}$$

$$\vec{a} = \frac{50 \text{ m/s}}{50 \text{ s}} = 1.0 \text{ m/s}^2$$

Discussion

Note that this value for $\vec{a}$ is consistent with the value plotted in Figure 6(c) at $t = 25$ s.

A graph of displacement versus time can be used to generate a graph of velocity versus time, and a graph of velocity versus time can be used to generate a graph of acceleration versus time. We do this by finding the slope of the graphs at every point. If the graph is linear (i.e., a line with a constant slope), it is easy to find the slope at any point and you have the slope for every point. Graphical analysis of motion can be used to describe both specific and general characteristics of kinematics. Graphs can also be used for other topics in physics. An important aspect of exploring physical relationships is to graph them and look for underlying relationships.

Check Your Understanding

1: A graph of velocity vs. time of a ship coming into a harbor is shown below. (a) Describe the motion of the ship based on the graph. (b)What would a graph of the ship's acceleration look like?

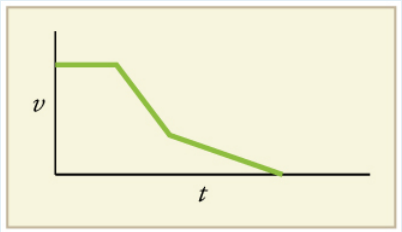


Figure 7.

Section Summary

- Graphs of motion can be used to analyze motion.
- Graphical solutions yield identical solutions to mathematical methods for deriving motion equations.
- The slope of a graph of displacement ***x*** vs. time ***t*** is velocity $\vec{v}$.
- The slope of a graph of velocity $\vec{v}$ vs. time ***t*** graph is acceleration $\vec{a}$.
- Average velocity, instantaneous velocity, and acceleration can all be obtained by analyzing graphs.

Glossary

independent variable
: the variable that the dependent variable is measured with respect to; usually plotted along the *x*-axis

dependent variable
: the variable that is being measured; usually plotted along the *y*-axis

slope
: the difference in *y*-value (the rise) divided by the difference in *x*-value (the run) of two points on a straight line

y-intercept
: the *y*-value when $x = 0$, or when the graph crosses the *y*-axis

Conceptual Questions

1: (a) Explain how you can use the graph of position versus time in Figure 8 to describe the change in velocity over time. Identify (b) the time (t_a, t_b, t_c, t_d, or t_e) at which the instantaneous velocity is greatest, (c) the time at which it is zero, and (d) the time at which it is negative.

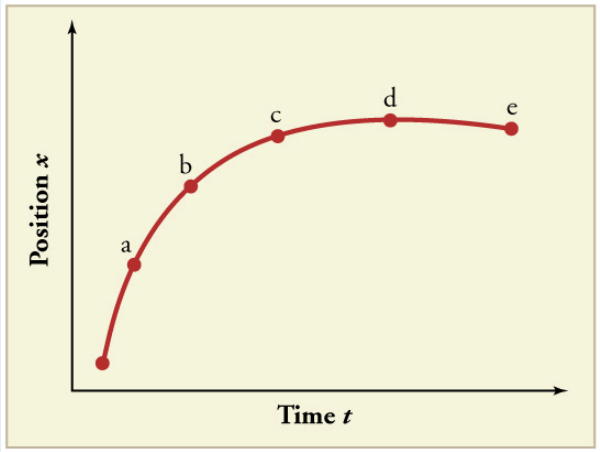


Figure 8.

2: (a) Sketch a graph of velocity versus time corresponding to the graph of displacement versus time given in Figure 9. (b) Identify the time or times (t_a, t_b, t_c, etc.) at which the instantaneous velocity is greatest. (c) At which times is it zero? (d) At which times is it negative?

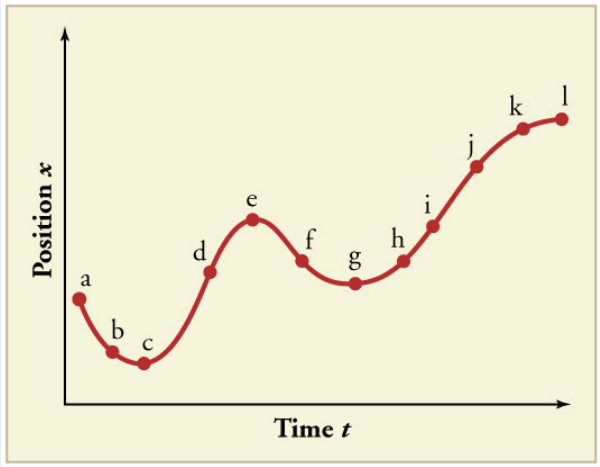


Figure 9.

3: (a) Explain how you can determine the acceleration over time from a velocity versus time graph such as the one in Figure 10. (b) Based on the graph, how does acceleration change over time?

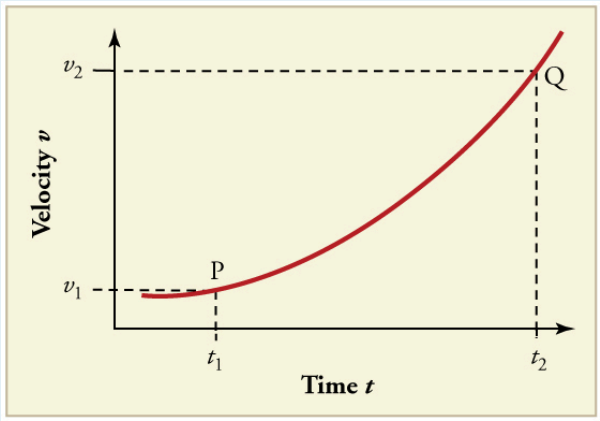


Figure 10.

4: (a) Sketch a graph of acceleration versus time corresponding to the graph of velocity versus time given in Figure 11. (b) Identify the time or times (t_a, t_b, t_c, etc.) at which the acceleration is greatest. (c) At which times is it zero? (d) At which times is it negative?

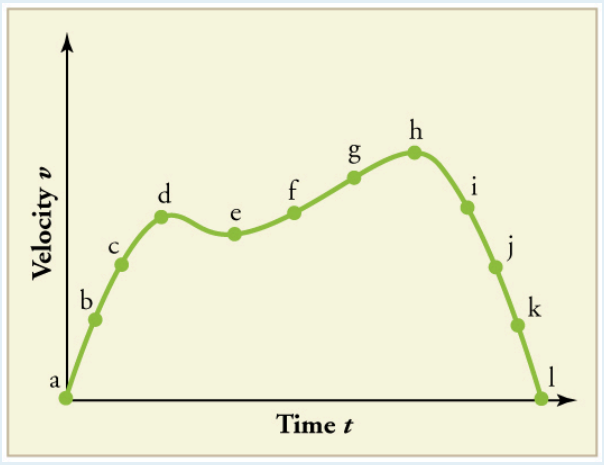


Figure 11.

5: Consider the velocity vs. time graph of a person in an elevator shown in Figure 12. Suppose the elevator is initially at rest. It then accelerates for 3 seconds, maintains that velocity for 15 seconds, then decelerates for 5 seconds until it stops. The acceleration for the entire trip is not constant so we cannot use the equations of motion from Chapter 2.6 Motion Equations for Constant Acceleration in One Dimension for the complete trip. (We could, however, use them in the three individual sections where acceleration is a constant.) Sketch graphs of (a) position vs. time and (b) acceleration vs. time for this trip.

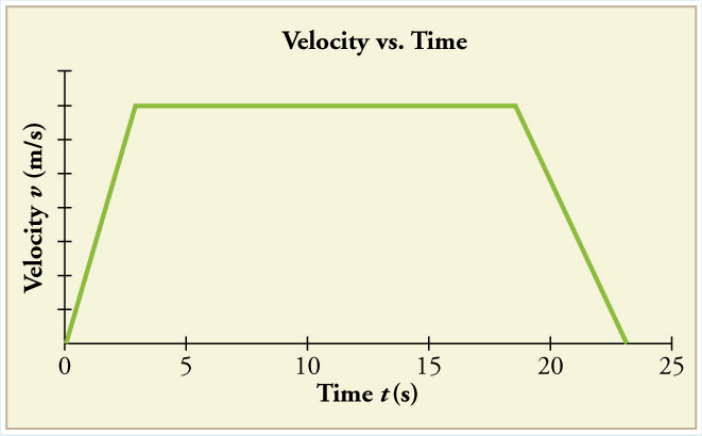


Figure 12.

6: A cylinder is given a push and then rolls up an inclined plane. If the origin is the starting point, sketch the position, velocity, and acceleration of the cylinder vs. time as it goes up and then down the plane.

Problems & Exercises

Note: There is always uncertainty in numbers taken from graphs. If your answers differ from expected values, examine them to see if they are within data extraction uncertainties estimated by you.

1: (a) By taking the slope of the curve in Figure 13, verify that the velocity of the jet car is 115 m/s at t = 20 s. (b) By taking the slope of the curve at any point in Figure 14, verify that the jet car's acceleration is 5.0 m/s^2.

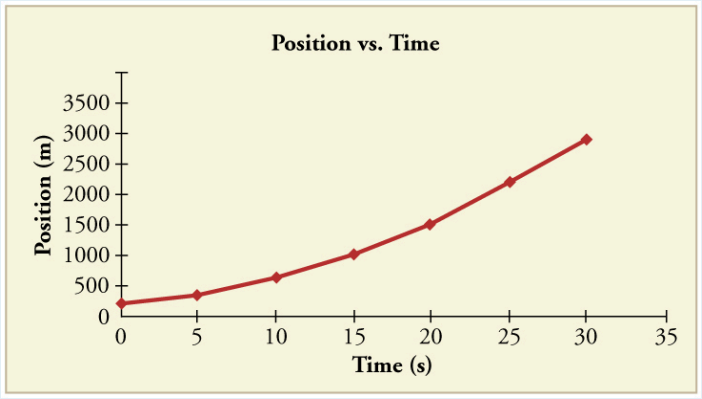


Figure 13.

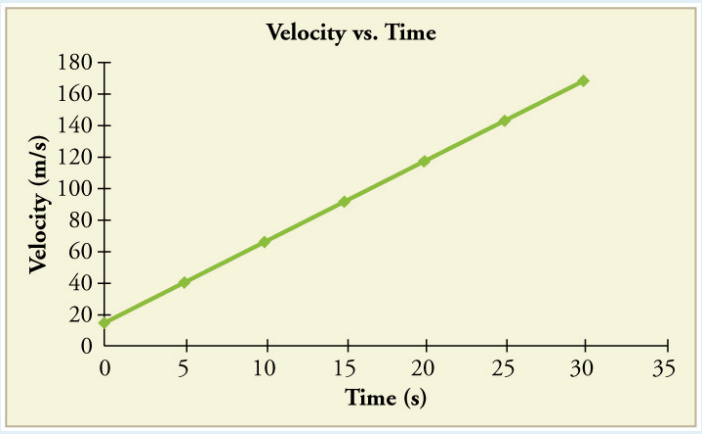


Figure 14.

2: Using approximate values, calculate the slope of the curve in Figure 15 to verify that the velocity at t = 10.0 s is 0.208 m/s. Assume all values are known to 3 significant figures.

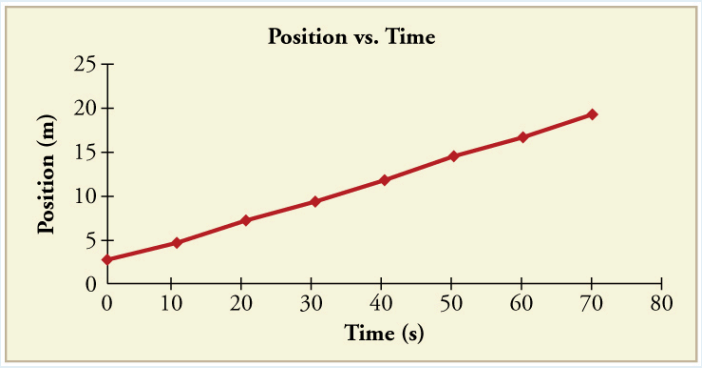


Figure 15.

3: Using approximate values, calculate the slope of the curve in Figure 15 to verify that the velocity at t = 30.0 s is 0.238 m/s. Assume all values are known to 3 significant figures.

4: By taking the slope of the curve in Figure 16, verify that the acceleration is 3.2 m/s^2 at t = 10 s.

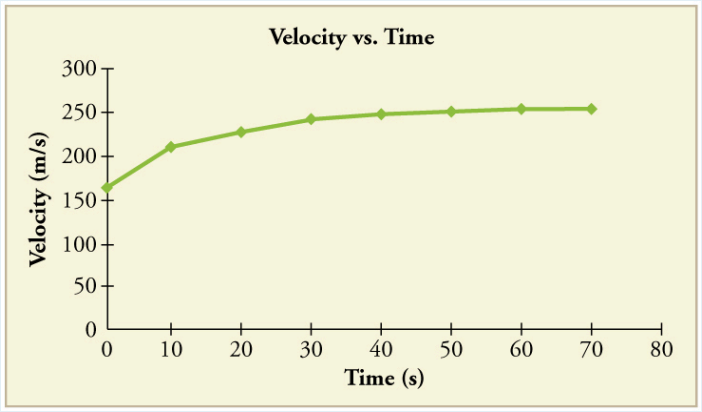


Figure 16.

5: Construct the displacement graph for the subway shuttle train as shown in Chapter 2.4 Figure 7(a). Your graph should show the position of the train, in kilometers, from t = 0 to 20 s. You will need to use the information on acceleration and velocity given in the examples for this figure.

6: (a) Take the slope of the curve in Figure 17 to find the jogger's velocity at t = 2.5 s. (b) Repeat at 7.5 s. These values must be consistent with the graph in Figure 18.

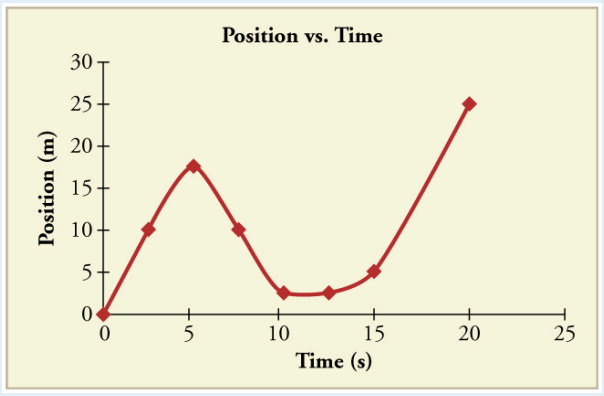


Figure 17.

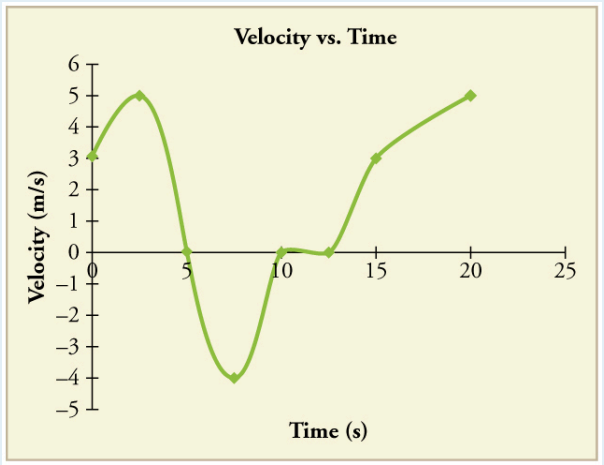


Figure 18.

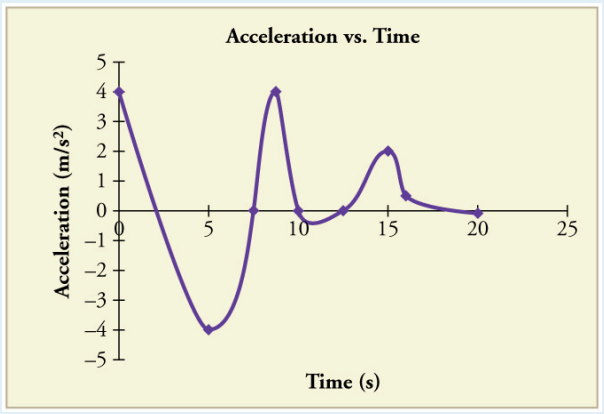


Figure 19.

7: A graph of *v(t)* is shown for a world-class track sprinter in a 100-m race. (See Figure 20). (a) What is his average velocity for the first 4 s? (b) What is his instantaneous velocity at t = 5 s? (c) What is his average acceleration between 0 and 4 s? (d) What is his time for the race?

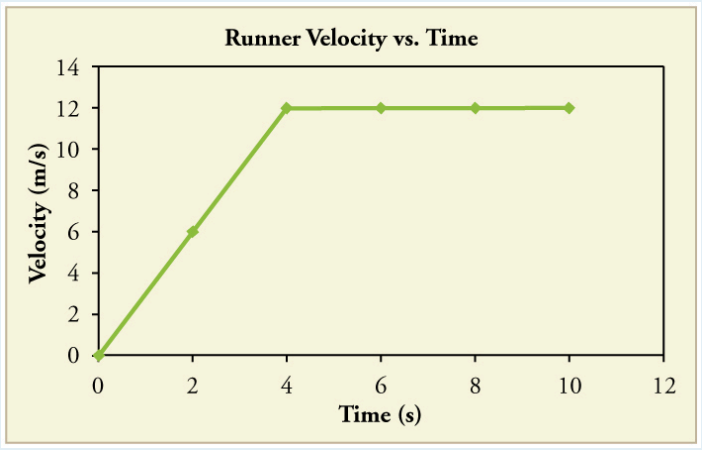


Figure 20.

8: Figure 21 shows the displacement graph for a particle for 5 s. Draw the corresponding velocity and acceleration graphs.

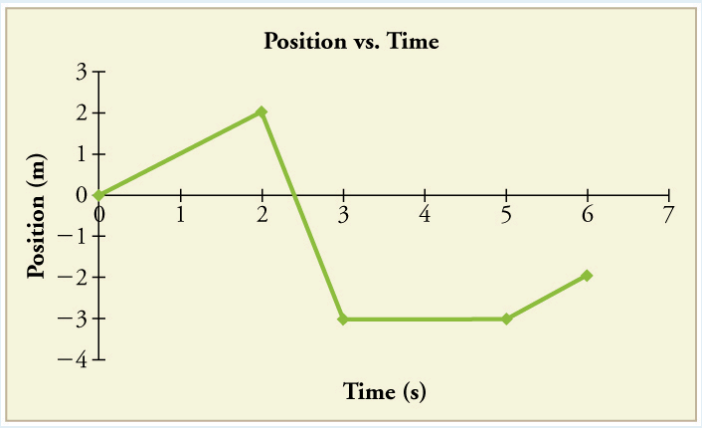


Figure 21.

Solutions

Check Your Understanding

1: (a) The ship moves at constant velocity and then begins to decelerate at a constant rate. At some point, its deceleration rate decreases. It maintains this lower deceleration rate until it stops moving.

(b) A graph of acceleration vs. time would show zero acceleration in the first leg, large and constant negative acceleration in the second leg, and constant negative acceleration.

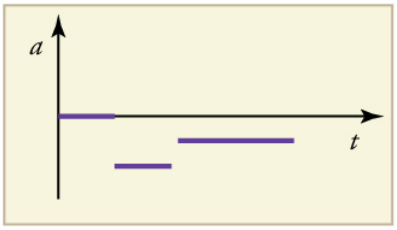


Figure 22.

Problems & Exercises

1: (a) 115 m/s (b) 5.0 m/s^2

3: $\vec{v} = \dfrac{(11.7 - 6.95) \times 10^3 \text{ m}}{40.0 - 20.0 \text{ s}} = 238 \text{ m/s}$

5:

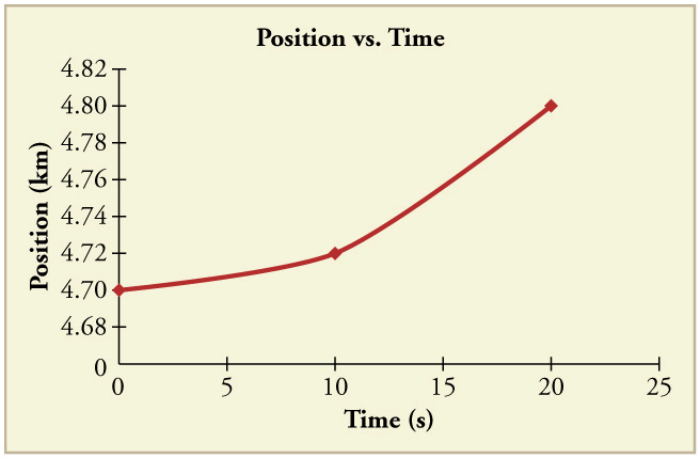


Figure 23.

7: (a) 6 m/s (b) 12 m/s (c) 3 m/s^2 (d) 10 s

2.6 Motion Equations for Constant Acceleration in One Dimension

Summary

- Calculate displacement of an object moving at a constant velocity, given initial position and velocity.
- Calculate final velocity of an accelerating object, given initial velocity, acceleration, and time.
- Calculate displacement and final position of an accelerating object, given initial position, initial velocity, time, and acceleration.

Figure 1. *Kinematic equations can help us describe and predict the motion of moving objects such as these kayaks racing in Newbury, England. (credit: Barry Skeates, Flickr).*

We might know that the greater the acceleration of, say, a car moving away from a stop sign, the greater the displacement in a given time. But we have not developed a specific equation that relates acceleration and displacement. In this section, we develop some convenient equations for kinematic relationships, starting from the definitions of displacement, velocity, and acceleration already covered.

Notation: *t*, *x*, *v*, *a*

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is $\Delta t = t_f - t_0$, taking $t_0 = 0$ means that $\Delta t = t_f$, the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, x_0 *is the initial position* and v_0 *is the initial velocity*. We put no subscripts on the final values. That is, *t is the final time*, *x is the final*

position, and *v is the final velocity*. This gives a simpler expression for elapsed time—now, $\Delta t = t$. It also simplifies the expression for displacement, which is now $\Delta x = x - x_0$. Also, it simplifies the expression for change in velocity, which is now $\Delta v = v - v_0$. To summarize, using the simplified notation, with the initial time taken to be zero,

$$\Delta t = t$$
$$\Delta x = x - x_0$$
$$\Delta \vec{v} = \vec{v} - \vec{v}_0$$

where *the subscript 0 denotes an initial value and the absence of a subscript denotes a final value* in whatever motion is under consideration.

We now make the important assumption that *acceleration is constant*. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal. That is,

$$\bar{a} = \vec{a} = \text{constant},$$

so we use the symbol $\vec{a}$ for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor degrade the accuracy of our treatment. For one thing, acceleration *is* constant in a great number of situations. Furthermore, in many other situations we can accurately describe motion by assuming a constant acceleration equal to the average acceleration for that motion. Finally, in motions where acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, the motion can be considered in separate parts, each of which has its own constant acceleration.

SOLVING FOR DISPLACEMENT (Δx) AND FINAL POSITION (x) FROM AVERAGE VELOCITY WHEN ACCELERATION (a) IS CONSTANT

To get our first two new equations, we start with the definition of average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t}.$$

Substituting the simplified notation for Δx and Δt yields

$$\bar{v} = \frac{x - x_0}{t}.$$

Solving for x yields

$$x = x_0 + \bar{v}t,$$

where the average velocity is

$$\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2} (\text{constant } \vec{a}).$$

The equation $\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2}$ reflects the fact that, when acceleration is constant, *v* is just the simple average of the initial and final velocities. For example, if you steadily increase your velocity (that is, with constant acceleration) from 30 to 60 km/h, then your average velocity during this steady increase is 45 km/h. Using the equation $\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2}$ to check this, we see that

$$\bar{v} = \frac{v_0 + v}{2} = \frac{30 \text{ km/h} + 60 \text{ km/h}}{2} = 45 \text{ km/h},$$

which seems logical.

Example 1: Calculating Displacement: How Far does the Jogger Run?

A jogger runs down a straight stretch of road with an average velocity of 4.00 m/s for 2.00 min. What is his final position, taking his initial position to be zero?

Strategy

Draw a sketch.

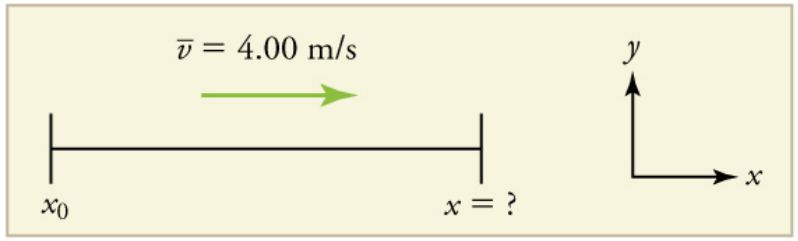


Figure 2.

The final position *x* is given by the equation

$$x = x_0 + \bar{v}t.$$

To find *x*, we identify the values of x_0, $\bar{v}$, and *t from the statement of the problem and substitute them into the equation.*

Solution

1. Identify the knowns. $\bar{v} = 4.00 \text{ m/s}$, Δt = 2.00 min, and x_0 = 0 m.

2. Enter the known values into the equation.

$$x = x_0 + \bar{v}t = 0 + 4.00\ \text{m/s} \times 120\ \text{s} = 480\ \text{m}$$

Discussion

Velocity and final position are both positive, which means they are in the same direction.

The equation $x = x_0 + \bar{v}t$ gives insight into the relationship between displacement, average velocity, and time. It shows, for example, that displacement is a linear function of average velocity. (By linear function, we mean that displacement depends on $\bar{v}$ rather than on $\bar{v}$ raised to some other power, such as $\bar{v}^2$. When graphed, linear functions look like straight lines with a constant slope.) On a car trip, for example, we will get twice as far in a given time if we average 90 km/h than if we average 45 km/h.

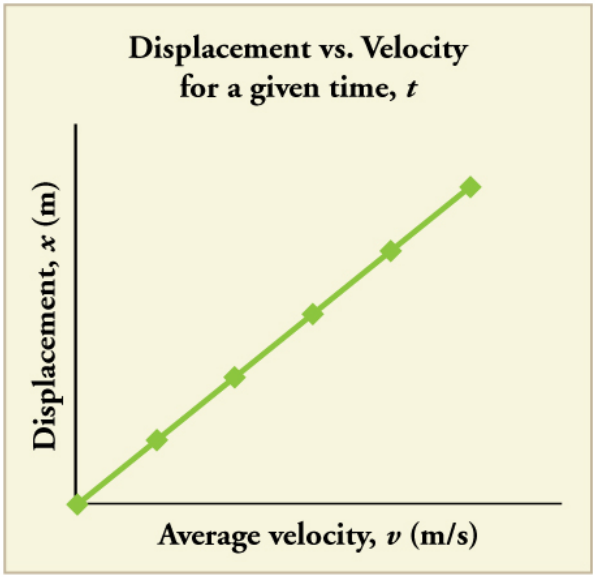


Figure 3. *There is a linear relationship between displacement and average velocity. For a given time t, an object moving twice as fast as another object will move twice as far as the other object.*

SOLVING FOR FINAL VELOCITY

We can derive another useful equation by manipulating the definition of acceleration.

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

Substituting the simplified notation for $\Delta\vec{v}$ and Δt gives us

$$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t} \text{ (constant } \vec{a}\text{)}.$$

Solving for $\vec{v}$ yields

$$\vec{v} = \vec{v}_0 + \vec{a}t \text{ (constant } \vec{a}\text{)}.$$

Example 2: Calculating Final Velocity: An Airplane Slowing Down after Landing

An airplane lands with an initial velocity of 70.0 m/s and then decelerates at 1.50 m/s^2 for 40.0 s. What is its final velocity?

Strategy

Draw a sketch. We draw the acceleration vector in the direction opposite the velocity vector because the plane is decelerating.

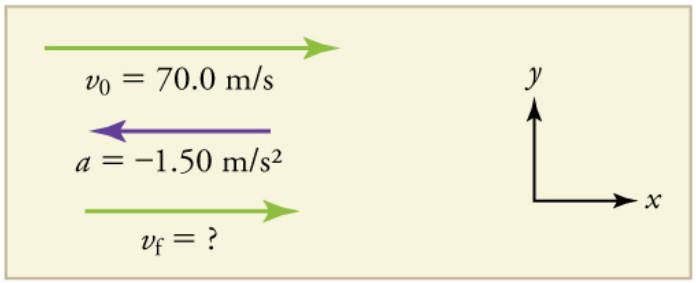


Figure 4.

Solution

1. Identify the knowns. $\vec{v}_0$ = 70.0 m/s, $\vec{a}$ = -1.50 m/s^2, t = 40.0 s.

2. Identify the unknown. In this case, it is final velocity, $\vec{v}_f$.

3. Determine which equation to use. We can calculate the final velocity using the equation $\vec{v} = \vec{v}_0 + \vec{a}t$.

4. Plug in the known values and solve,

$$\vec{v} = \vec{v}_0 + \vec{a}t = 70.0 \text{ m/s} + (-1.50 \text{ m/s}^2)(40.0 \text{ s}) = 10.0 \text{ m/s}$$

Discussion

The final velocity is much less than the initial velocity, as desired when slowing down, but still positive. With jet engines, reverse thrust could be maintained long enough to stop the plane and start moving it backward. That would be indicated by a negative final velocity, which is not the case here.

Figure 5. *The airplane lands with an initial velocity of 70.0 m/s and slows to a final velocity of 10.0 m/s before heading for the terminal. Note that the acceleration is negative because its direction is opposite to its velocity, which is positive.*

In addition to being useful in problem solving, the equation $\vec{v} = \vec{v}_0 + \vec{a}t$ gives us insight into the relationships among velocity, acceleration, and time. From it we can see, for example, that

- final velocity depends on how large the acceleration is and how long it lasts
- if the acceleration is zero, then the final velocity equals the initial velocity ($\vec{v}_f = \vec{v}_0$), as expected (i.e., velocity is constant)
- if $\vec{a}$ is negative, then the final velocity is less than the initial velocity

(All of these observations fit our intuition, and it is always useful to examine basic equations in light of our intuition and experiences to check that they do indeed describe nature accurately.)

MAKING CONNECTIONS: REAL WORLD CONNECTION

Figure 6. *The Space Shuttle Endeavor blasts off from the Kennedy Space Center in February 2010. (credit: Matthew Simantov, Flickr).*

An intercontinental ballistic missile (ICBM) has a larger average acceleration than the Space Shuttle

and achieves a greater velocity in the first minute or two of flight (actual ICBM burn times are classified—short-burn-time missiles are more difficult for an enemy to destroy). But the Space Shuttle obtains a greater final velocity, so that it can orbit the earth rather than come directly back down as an ICBM does. The Space Shuttle does this by accelerating for a longer time.

SOLVING FOR FINAL POSITION WHEN VELOCITY IS NOT CONSTANT (a ≠ 0)

We can combine the equations above to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

$$\vec{v} = \vec{v}_0 + \vec{a}t.$$

Adding $\vec{v}_0$ to each side of this equation and dividing by 2 gives

$$\frac{\vec{v}_0 + \vec{v}}{2} = \vec{v}_0 + \frac{1}{2}at$$

Since $\frac{\vec{v}_0 + v}{2} = \bar{v}$ for constant acceleration, then

$$\bar{v} = \vec{v}_0 + \frac{1}{2}\vec{a}t.$$

Now we substitute this expression for $\bar{v}$ into the equation for displacement, $x = x_0 + \bar{v}t$, yielding

$$x = x_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \ (\text{constant } \ \vec{a}).$$

Example 3: Calculating Displacement of an Accelerating Object: Dragsters

Dragsters can achieve average accelerations of 26.0 m/s^2. Suppose such a dragster accelerates from rest at this rate for 5.56 s. How far does it travel in this time?

Figure 7. *U.S. Army Top Fuel pilot Tony "The Sarge" Schumacher begins a race with a controlled burnout. (credit: Lt. Col. William Thurmond. Photo Courtesy of U.S. Army.).*

Strategy

Draw a sketch.

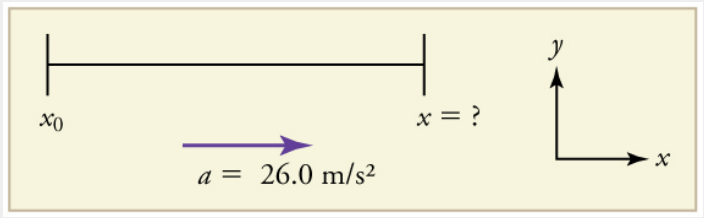


Figure 8.

We are asked to find displacement, which is x if we take x_0 to be zero. (Think about it like the starting line of a race. It can be anywhere, but we call it 0 and measure all other positions relative to it.) We can use the equation $x = x_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ once we identify $\vec{v}_0$, $\vec{a}$, and t from the statement of the problem.

Solution

1. Identify the knowns. Starting from rest means that $\vec{v}_0 = 0$, $\vec{a}$ is given as 26.0 m/s^2 and t is given as 5.56 s.

2. Plug the known values into the equation to solve for the unknown x:

$$x = x_0 + v_0 t + \frac{1}{2}\vec{a}t^2.$$

Since the initial position and velocity are both zero, this simplifies to

$$x = \frac{1}{2}\vec{a}t^2.$$

Substituting the identified values of $\vec{a}$ and t gives

$$x = \frac{1}{2}(26.0 \text{ m/s}^2)(5.56 \text{ s})^2,$$

yielding

$$x = 402 \text{ m}.$$

Discussion

If we convert 402 m to miles, we find that the distance covered is very close to one quarter of a mile, the standard distance for drag racing. So the answer is reasonable. This is an impressive displacement in only 5.56 s, but top-notch dragsters can do a quarter mile in even less time than this.

What else can we learn by examining the equation $x = x_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$? We see that:

- displacement depends on the square of the elapsed time when acceleration is not zero. In Example 3, the dragster covers only one fourth of the total distance in the first half of the elapsed time
- if acceleration is zero, then the initial velocity equals average velocity $\vec{v}_0 = \bar{v}$ and $x = x_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ becomes $x = x_0 + \vec{v}_0 t$

SOLVING FOR FINAL VELOCITY WHEN VELOCITY IS NOT CONSTANT (a ≠ 0)

A fourth useful equation can be obtained from another algebraic manipulation of previous equations.

If we solve $\vec{v} = \vec{v}_0 + \vec{a}t$ for *t*, we get

$$t = \frac{\vec{v} - \vec{v}_0}{\vec{a}}.$$

Substituting this and $\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2}$ into $x = x_0 + \bar{v}t$, we get

$$\vec{v}^2 = \vec{v}_0{}^2 + 2\vec{a}(x - x_0) \text{ (constant } \vec{a}).$$

Example 4: Calculating Final Velocity: Dragsters

Calculate the final velocity of the dragster in Example 3 without using information about time.

Strategy

Draw a sketch.

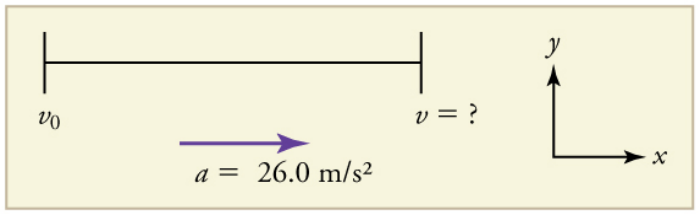


Figure 9.

The equation $\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}(x - x_0)$ is ideally suited to this task because it relates velocities, acceleration, and displacement, and no time information is required.

Solution

1. Identify the known values. We know that \vec{v}_0 = 0, since the dragster starts from rest. Then we note that $x - x_0 = 402$ m (this was the answer in Example 3). Finally, the average acceleration was given to be $\vec{a} = 26.0\ \text{m/s}^2$.

2. Plug the knowns into the equation $\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}(x - x_0)$ and solve for $\vec{v}$.

$$\vec{v}^2 = 0 + 2(26.0\ \text{m/s}^2)(402\ \text{m}).$$

Thus

$$\vec{v}^2 = 2.09 \times 10^4\ \text{m}^2/\text{s}^2$$

To get $\vec{v}$, we take the square root:

$$\vec{v} = \sqrt{2.09 \times 10^4\ \text{m}^2/\text{s}^2} = 145\ \text{m/s}.$$

Discussion

145 m/s is about 522 km/h or about 324 mi/h, but even this breakneck speed is short of the record for the quarter mile. Also, note that a square root has two values; we took the positive value to indicate a velocity in the same direction as the acceleration.

An examination of the equation $\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}(x - x_0)$ can produce further insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts

- For a fixed deceleration, a car that is going twice as fast doesn't simply stop in twice the distance—it takes much further to stop. (This is why we have reduced speed zones near schools.)

Putting Equations Together

In the following examples, we further explore one-dimensional motion, but in situations requiring slightly more algebraic manipulation. The examples also give insight into problem-solving techniques. The box below provides easy reference to the equations needed.

SUMMARY OF KINEMATIC EQUATIONS(CONSTANT *a*)

$$x = x_0 + \bar{v}t$$

$$\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2}$$

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$x = x_0 + v_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}^2 = \vec{v}_0{}^2 + 2\vec{a}(x - x_0)$$

Example 5: Calculating Displacement: How Far Does a Car Go When Coming to a Halt?

On dry concrete, a car can decelerate at a rate of 7.00 m/s^2, whereas on wet concrete it can decelerate at only 5.00 m/s^2. Find the distances necessary to stop a car moving at 30.0 m/s (about 110 km/h) (a) on dry concrete and (b) on wet concrete. (c) Repeat both calculations, finding the displacement from the point where the driver sees a traffic light turn red, taking into account his reaction time of 0.500 s to get his foot on the brake.

Strategy

Draw a sketch.

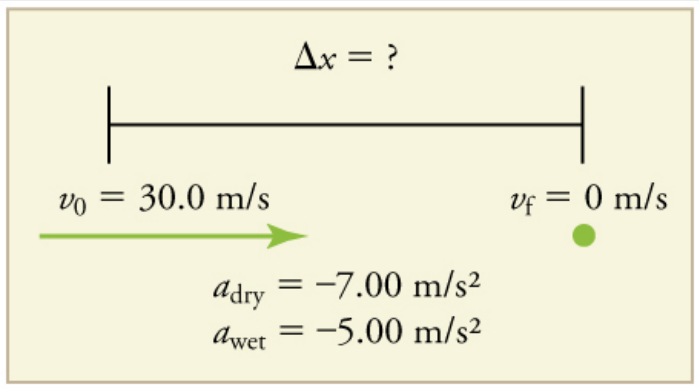


Figure 10.

In order to determine which equations are best to use, we need to list all of the known values and identify exactly what we need to solve for. We shall do this explicitly in the next several examples, using tables to set them off.

Solution for (a)

1. Identify the knowns and what we want to solve for. We know that v_0 = 30.0 m/s; v=*0*; $\vec{a}$ = -7.00 m/s^2 ($\vec{a}$ is negative because it is in a direction opposite to velocity). We take x_0 to be 0. We are looking for displacement Δx, or $x - x_0$.

2. Identify the equation that will help up solve the problem. The best equation to use is

$$\vec{v}^2 = \vec{v}_0{}^2 + 2\vec{a}(x - x_0)\cdot$$

This equation is best because it includes only one unknown, *x*. We know the values of all the other variables in this equation. (There are other equations that would allow us to solve for *x*, but they require us to know the stopping time, *t*, which we do not know. We could use them but it would entail additional calculations.)

3. Rearrange the equation to solve for x.

$$x - x_0 = \frac{\vec{v}^2 - \vec{v}_0{}^2}{2\vec{a}}$$

4. Enter known values.

$$\mathrm{x} - 0 = \frac{0^2 - (30.0\ \mathrm{m/s})^2}{2(-7.00\ \mathrm{m/s}^2)}$$

Thus,

$$x = 64.3\ \mathrm{m\ on\ dry\ concrete.}$$

Solution for (b)

This part can be solved in exactly the same manner as Part A. The only difference is that the deceleration is -5.00 m/s^2. The result is

$$x_{wet} = 90.0\ \mathrm{m\ on\ wet\ concrete.}$$

Solution for (c)

Once the driver reacts, the stopping distance is the same as it is in Parts A and B for dry and wet concrete. So to answer this question, we need to calculate how far the car travels during the reaction time, and then add that to the stopping time. It is reasonable to assume that the velocity remains constant during the driver's reaction time.

1. Identify the knowns and what we want to solve for. We know that $\bar{v} = 30.0\ \text{m/s}$; $t_{reaction} = 0.500\ \text{s}$; $a_{reaction} = 0$. We take $x_{0-reaction}$ to be 0. We are looking for $x_{reaction}$.

2. Identify the best equation to use.

$x = x_0 + \bar{v}t$ works well because the only unknown value is x, which is what we want to solve for.

3. Plug in the knowns to solve the equation.

$$x = 0 + (30.0\ \text{m/s})(0.500\ \text{s}) = 15.0\ \text{m}.$$

This means the car travels 15.0 m while the driver reacts, making the total displacements in the two cases of dry and wet concrete 15.0 m greater than if he reacted instantly.

4. Add the displacement during the reaction time to the displacement when braking.

$$x_{braking} + x_{reaction} = x_{total}$$

(a) 64.3 m + 15.0 m = 79.3 m when dry
(b) 90.0 m + 15.0 m = 105 m when wet

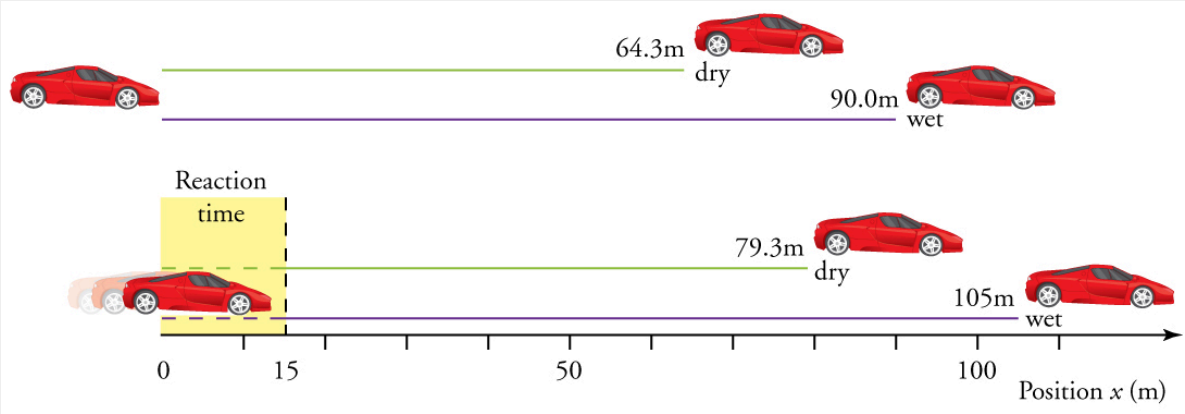


Figure 11. *The distance necessary to stop a car varies greatly, depending on road conditions and driver reaction time. Shown here are the braking distances for dry and wet pavement, as calculated in this example, for a car initially traveling at 30.0 m/s. Also shown are the total distances traveled from the point where the driver first sees a light turn red, assuming a 0.500 s reaction time.*

Discussion

The displacements found in this example seem reasonable for stopping a fast-moving car. It should take longer to stop a car on wet rather than dry pavement. It is interesting that reaction time adds significantly to the displacements. But more important is the general approach to solving problems. We identify the knowns and the quantities to be determined and then find an appropriate equation. There is often more than one way to solve a problem. The various parts of this example can in fact be solved by other methods, but the solutions presented above are the shortest.

Example 6: Calculating Time: A Car Merges into Traffic

Suppose a car merges into freeway traffic on a 200-m-long ramp. If its initial velocity is 10.0 m/s and it accelerates at 2.00 m/s^2, how long does it take to travel the 200 m up the ramp? (Such information might be useful to a traffic engineer.)

Strategy

Draw a sketch.

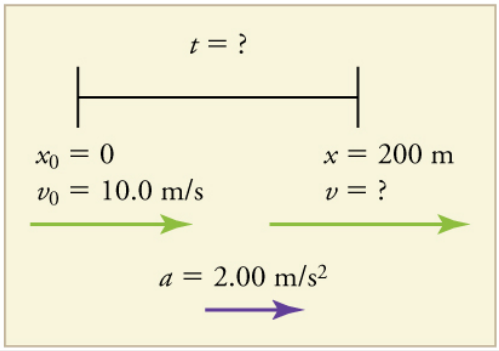


Figure 12.

We are asked to solve for the time *t*. As before, we identify the known quantities in order to choose a convenient physical relationship (that is, an equation with one unknown, *t*).

Solution

1. Identify the knowns and what we want to solve for. We know that $\vec{v}_0 = 10 \text{ m/s}$; $\vec{a} = 2.00 \text{ m/s}^2$; and *x* = 200 m.

2. We need to solve for *t*. Choose the best equation. $x = x_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ works best because the only unknown in the equation is the variable t for which we need to solve.

3. We will need to rearrange the equation to solve for *t*. In this case, it will be easier to plug in the knowns first.

$$200 \text{ m} = 0 \text{ m} + (10.0 \text{ m/s})t + \frac{1}{2}(2.00 \text{ m/s}^2)t^2$$

4. Simplify the equation. The units of meters (m) cancel because they are in each term. We can get the units of seconds (s) to cancel by taking *t* = *t* s, where *t* is the magnitude of time and s is the unit. Doing so leaves

$$200 = 10t + t^2.$$

5. Use the quadratic formula to solve for *t*.

(a) Rearrange the equation to get 0 on one side of the equation.

$$t^2 + 10t - 200 = 0$$

This is a quadratic equation of the form

$$at^2 + bt + c = 0,$$

where the constants are a = 1.00, b = 10.0, and c = -200.

(b) Its solutions are given by the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This yields two solutions for t, which are

$$t = 10.0 \text{ and } -20.0.$$

In this case, then, the time is $t = t$ in seconds, or

$$t = 10.0 \text{ s and } -20.0 \text{ s}.$$

A negative value for time is unreasonable, since it would mean that the event happened 20 s before the motion began. We can discard that solution. Thus,

$$t = 10.0 \text{ s}.$$

Discussion

Whenever an equation contains an unknown squared, there will be two solutions. In some problems both solutions are meaningful, but in others, such as the above, only one solution is reasonable. The 10.0 s answer seems reasonable for a typical freeway on-ramp.

With the basics of kinematics established, we can go on to many other interesting examples and applications. In the process of developing kinematics, we have also glimpsed a general approach to problem solving that produces both correct answers and insights into physical relationships. Chapter 2.7 Problem-Solving Basics discusses problem-solving basics and outlines an approach that will help you succeed in this invaluable task.

MAKING CONNECTIONS: TAKE-HOME EXPERIMENT–BREAKING NEWS

We have been using SI units of meters per second squared to describe some examples of acceleration or deceleration of cars, runners, and trains. To achieve a better feel for these numbers, one can measure the braking deceleration of a car doing a slow (and safe) stop. Recall that, for average acceleration, $\bar{a} = \frac{\Delta \vec{v}}{\Delta t}$. While traveling in a car, slowly apply the brakes as you come up to a stop sign. Have a passenger note the initial speed in miles per hour and the time taken (in seconds) to stop. From this, calculate the deceleration in miles per hour per second. Convert this to meters per second squared and compare with other decelerations mentioned in this chapter. Calculate the distance traveled in braking.

Check Your Understanding

1: A manned rocket accelerates at a rate of 20 m/s^2 during launch. How long does it take the rocket to reach a velocity of 400 m/s?

Section Summary

- To simplify calculations we take acceleration to be constant, so that $\bar{a} = a$ at all times.
- We also take initial time to be zero.
- Initial position and velocity are given a subscript 0; final values have no subscript. Thus,

$$\Delta t = t$$
$$\Delta x = x - x_0$$
$$\Delta \vec{v} = \vec{v} - \vec{v}_0$$

- The following kinematic equations for motion with constant $\vec{a}$ are useful:

$$x = x_0 + \bar{v}t$$
$$\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2}$$
$$\vec{v} = \vec{v}_0 + \vec{a}t$$
$$x = x_0 + v_0 t + \frac{1}{2}\vec{a}t^2$$
$$\vec{v}^2 = {\vec{v}_0}^2 + 2\vec{a}(x - x_0)$$

- In vertical motion, *y* is substituted for *x*.

Problems & Exercises

1: An Olympic-class sprinter starts a race with an acceleration of 4.50 m/s^2. (a) What is her speed 2.40 s later? (b) Sketch a graph of her position vs. time for this period.

2: A well-thrown ball is caught in a well-padded mitt. If the deceleration of the ball is 2.10 × 10^4 m/s^2, and 1.85 ms (1 ms =10^{-3} s) elapses from the time the ball first touches the mitt until it stops, what was the initial velocity of the ball?

3: A bullet in a gun is accelerated from the firing chamber to the end of the barrel at an average rate of 6.20 × 10^5 m/s^2 for 8.10 × 10^{-4} s. What is its muzzle velocity (that is, its final velocity)?

4: (a) A light-rail commuter train accelerates at a rate of 1.35 m/s^2. How long does it take to reach its top speed of 80.0 km/h, starting from rest? (b) The same train ordinarily decelerates at a rate of 1.65 m/s^2. How

long does it take to come to a stop from its top speed? (c) In emergencies the train can decelerate more rapidly, coming to rest from 80.0 km/h in 8.30 s. What is its emergency deceleration in m/s^2?

5: While entering a freeway, a car accelerates from rest at a rate of 2.40 m/s^2 for 12.0 s. (a) Draw a sketch of the situation. (b) List the knowns in this problem. (c) How far does the car travel in those 12.0 s? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, check your units, and discuss whether the answer is reasonable. (d) What is the car's final velocity? Solve for this unknown in the same manner as in part (c), showing all steps explicitly.

6: At the end of a race, a runner decelerates from a velocity of 9.00 m/s at a rate of 2.00 m/s^2. (a) How far does she travel in the next 5.00 s? (b) What is her final velocity? (c) Evaluate the result. Does it make sense?

7: Professional Application:

Blood is accelerated from rest to 30.0 cm/s in a distance of 1.80 cm by the left ventricle of the heart. (a) Make a sketch of the situation. (b) List the knowns in this problem. (c) How long does the acceleration take? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking your units. (d) Is the answer reasonable when compared with the time for a heartbeat?

8: In a slap shot, a hockey player accelerates the puck from a velocity of 8.00 m/s to 40.0 m/s in the same direction. If this shot takes 3.33×10^{-2} s, calculate the distance over which the puck accelerates.

9: A powerful motorcycle can accelerate from rest to 26.8 m/s (100 km/h) in only 3.90 s. (a) What is its average acceleration? (b) How far does it travel in that time?

10: Freight trains can produce only relatively small accelerations and decelerations. (a) What is the final velocity of a freight train that accelerates at a rate of 0.0500 m/s^2 for 8.00 min, starting with an initial velocity of 4.00 m/s? (b) If the train can slow down at a rate of 0.550 m/s^2, how long will it take to come to a stop from this velocity? (c) How far will it travel in each case?

11: A fireworks shell is accelerated from rest to a velocity of 65.0 m/s over a distance of 0.250 m. (a) How long did the acceleration last? (b) Calculate the acceleration.

12: A swan on a lake gets airborne by flapping its wings and running on top of the water. (a) If the swan must reach a velocity of 6.00 m/s to take off and it accelerates from rest at an average rate of 0.350 m/s^2, how far will it travel before becoming airborne? (b) How long does this take?

13: Professional Application:

A woodpecker's brain is specially protected from large decelerations by tendon-like attachments inside the skull. While pecking on a tree, the woodpecker's head comes to a stop from an initial velocity of 0.600 m/s in a distance of only 2.00 mm. (a) Find the acceleration in m/s^2 and in multiples of g (g=9.80 m/s^2). (b) Calculate the stopping time. (c) The tendons cradling the brain stretch, making its stopping distance 4.50 mm (greater than the head and, hence, less deceleration of the brain). What is the brain's deceleration, expressed in multiples of g?

14: An unwary football player collides with a padded goalpost while running at a velocity of 7.50 m/s and

comes to a full stop after compressing the padding and his body 0.350 m. (a) What is his deceleration? (b) How long does the collision last?

15: In World War II, there were several reported cases of airmen who jumped from their flaming airplanes with no parachute to escape certain death. Some fell about 20,000 feet (6000 m), and some of them survived, with few life-threatening injuries. For these lucky pilots, the tree branches and snow drifts on the ground allowed their deceleration to be relatively small. If we assume that a pilot's speed upon impact was 123 mph (54 m/s), then what was his deceleration? Assume that the trees and snow stopped him over a distance of 3.0 m.

16: Consider a grey squirrel falling out of a tree to the ground. (a) If we ignore air resistance in this case (only for the sake of this problem), determine a squirrel's velocity just before hitting the ground, assuming it fell from a height of 3.0 m. (b) If the squirrel stops in a distance of 2.0 cm through bending its limbs, compare its deceleration with that of the airman in the previous problem.

17: An express train passes through a station. It enters with an initial velocity of 22.0 m/s and decelerates at a rate of 0.150 m/s^2 as it goes through. The station is 210 m long. (a) How long is the nose of the train in the station? (b) How fast is it going when the nose leaves the station? (c) If the train is 130 m long, when does the end of the train leave the station? (d) What is the velocity of the end of the train as it leaves?

18: Dragsters can actually reach a top speed of 145 m/s in only 4.45 s—considerably less time than given in Example 3 and Example 4. (a) Calculate the average acceleration for such a dragster. (b) Find the final velocity of this dragster starting from rest and accelerating at the rate found in (a) for 402 m (a quarter mile) without using any information on time. (c) Why is the final velocity greater than that used to find the average acceleration? *Hint*: Consider whether the assumption of constant acceleration is valid for a dragster. If not, discuss whether the acceleration would be greater at the beginning or end of the run and what effect that would have on the final velocity.

19: A bicycle racer sprints at the end of a race to clinch a victory. The racer has an initial velocity of 11.5 m/s and accelerates at the rate of 0.500 m/s^2 for 7.00 s. (a) What is his final velocity? (b) The racer continues at this velocity to the finish line. If he was 300 m from the finish line when he started to accelerate, how much time did he save? (c) One other racer was 5.00 m ahead when the winner started to accelerate, but he was unable to accelerate, and traveled at 11.8 m/s until the finish line. How far ahead of him (in meters and in seconds) did the winner finish?

20: In 1967, New Zealander Burt Munro set the world record for an Indian motorcycle, on the Bonneville Salt Flats in Utah, with a maximum speed of 183.58 mi/h. The one-way course was 5.00 mi long. Acceleration rates are often described by the time it takes to reach 60.0 mi/h from rest. If this time was 4.00 s, and Burt accelerated at this rate until he reached his maximum speed, how long did it take Burt to complete the course?

21: (a) A world record was set for the men's 100-m dash in the 2008 Olympic Games in Beijing by Usain Bolt of Jamaica. Bolt "coasted" across the finish line with a time of 9.69 s. If we assume that Bolt accelerated for 3.00 s to reach his maximum speed, and maintained that speed for the rest of the race, calculate his maximum speed and his acceleration. (b) During the same Olympics, Bolt also set the world record in the 200-m dash with a time of 19.30 s. Using the same assumptions as for the 100-m dash, what was his maximum speed for this race?

Solutions

Check Your Understanding

1: To answer this, choose an equation that allows you to solve for time *t*, given only $\vec{a}$, $\vec{v}_0$, and $\vec{v}$.

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

Rearrange to solve for *t*.

$$t = \frac{\vec{v} - \vec{v}_0}{\vec{a}} = \frac{400 \text{ m/s} - 0 \text{ m/s}}{20 \text{ m/s}^2} = 20 \text{ s}$$

Problems & Exercises

1: (a) 10.8 m/s

(b)

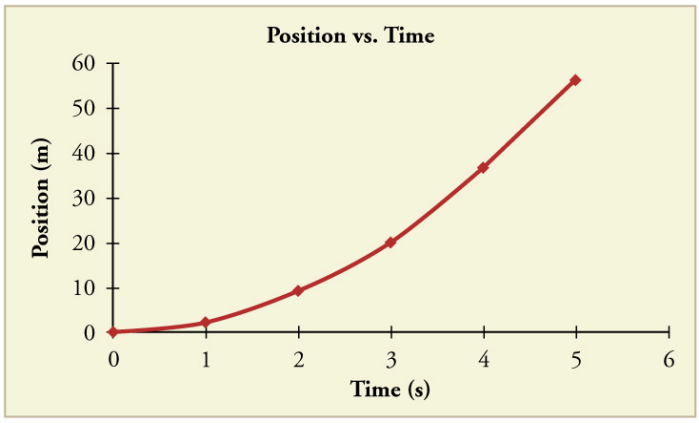


Figure 13.

2: 38.9 m/s (about 87 miles per hour)
4: (a) 16.5 s (b) 13.5 (c) -2.68 m/s^2
6: (a) 20.0 m (b) -1.00 m/s [/latex] (c) This result does not really make sense. If the runner starts at 9.00 m/s and decelerates at 2.00 m/s^2, then she will have stopped after 4.50 s. If she continues to decelerate, she will be running backwards.
8: 0.799 m
10: (a) 28.0 m/s (b) 50.9 s (c) 7.68 km to accelerate and 713 m to decelerate
12: (a) 51.4 m (b) 17.1 s
14: (a) -80.4 m/s^2 (b) 9.33 x 10^{-2} s
16: (a) 7.7 m/s (b) -15 x 10^2 m/s^2. This is about 3 times the deceleration of the pilots, who were falling from thousands of meters high!
18: (a) 32.6 m/s^2 (b) 162 m/s (c) $v > v_{max}$v_{max}}" title="{v>v_{max}}" class="latex mathjax">, because the assumption of constant acceleration is not valid for a dragster. A dragster changes gears, and would have a greater acceleration in first gear than second gear than third gear, etc. The acceleration would be greatest at the beginning, so it would not be accelerating at 32.6 m/

s^2 during the last few meters, but substantially less, and the final velocity would be less than 162 m/s.
20: 104 s
21: (a) $\vec{v}$ = 12.2 m/s; $\vec{a}$ = 4.07 m/s^2 (b) $\vec{v}$ = 11.2 m/s

2.7 Problem-Solving Basics for One-Dimensional Kinematics

Summary

- Apply problem-solving steps and strategies to solve problems of one-dimensional kinematics.
- Apply strategies to determine whether or not the result of a problem is reasonable, and if not, determine the cause.

Problem-solving skills are obviously essential to success in a quantitative course in physics. More importantly, the ability to apply broad physical principles, usually represented by equations, to specific situations is a very powerful form of knowledge. It is much more powerful than memorizing a list of facts. Analytical skills and problem-solving abilities can be applied to new situations, whereas a list of facts cannot be made long enough to contain every possible circumstance. Such analytical skills are useful both for solving problems in this text and for applying physics in everyday and professional life.

Problem-Solving Steps

While there is no simple step-by-step method that works for every problem, the following general procedures facilitate problem solving and make it more meaningful. A certain amount of creativity and insight is required as well.

Step 1

Examine the situation to determine which physical principles are involved. It often helps to *draw a simple sketch* at the outset. You will also need to decide which direction is positive and note that on your sketch. Once you have identified the physical principles, it is much easier to find and apply the equations representing those principles. Although finding the correct equation is essential, keep in mind that equations represent physical principles, laws of nature, and relationships among physical quantities. Without a conceptual understanding of a problem, a numerical solution is meaningless.

Step 2

Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Many problems are stated very succinctly and require some inspection to determine what is known. A sketch can also be very useful at this point. Formally identifying the knowns is of particular importance in applying physics to real-world situations. Remember, “stopped” means velocity is zero, and we often can take initial time and position as zero.

Step 3

Identify exactly what needs to be determined in the problem (identify the unknowns). In complex problems, especially, it is not always obvious what needs to be found or in what sequence. Making a list can help.

Step 4

Find an equation or set of equations that can help you solve the problem. Your list of knowns and unknowns can help here. It is easiest if you can find equations that contain only one unknown—that is, all of the other variables are known, so you can easily solve for the unknown. If the equation contains more than one unknown, then an additional equation is needed to solve the problem. In some problems, several unknowns must be determined to get at the one needed most. In such problems it is especially important to keep physical principles in mind to avoid going astray in a sea of equations. You may have to use two (or more) different equations to get the final answer.

Step 5

Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units. This step produces the numerical answer; it also provides a check on units that can help you find errors. If the units of the answer are incorrect, then an error has been made. However, be warned that correct units do not guarantee that the numerical part of the answer is also correct.

Step 6

Check the answer to see if it is reasonable: Does it make sense? This final step is extremely important—the goal of physics is to accurately describe nature. To see if the answer is reasonable, check both its magnitude and its sign, in addition to its units. Your judgment will improve as you solve more and more physics problems, and it will become possible for you to make finer and finer judgments regarding whether nature is adequately described by the answer to a problem. This step brings the problem back to its conceptual meaning. If you can judge whether the answer is reasonable, you have a deeper understanding of physics than just being able to mechanically solve a problem.

When solving problems, we often perform these steps in different order, and we also tend to do several steps simultaneously. There is no rigid procedure that will work every time. Creativity and insight grow with experience, and the basics of problem solving become almost automatic. One way to get practice is to work out the text's examples for yourself as you read. Another is to work as many end-of-section problems as possible, starting with the easiest to build confidence and progressing to the more difficult. Once you become involved in physics, you will see it all around you, and you can begin to apply it to situations you encounter outside the classroom, just as is done in many of the applications in this text.

Unreasonable Results

Physics must describe nature accurately. Some problems have results that are unreasonable because one premise is unreasonable or because certain premises are inconsistent with one another. The physical principle applied correctly then produces an unreasonable result. For example, if a person starting a foot race accelerates at 0.40 m/s^2 for 100 s, his final speed will be 40 m/s (about 150 km/h)—clearly unreasonable because the time of 100 s is an unreasonable premise. The physics is correct in a sense, but there is more to describing nature than just manipulating equations correctly. Checking the result of a problem to see if it is reasonable does more than help uncover errors in problem solving—it also builds intuition in judging whether nature is being accurately described.

Use the following strategies to determine whether an answer is reasonable and, if it is not, to determine what is the cause.

Step 1

Solve the problem using strategies as outlined and in the format followed in the worked examples in the text. In the example given in the preceding paragraph, you would identify the givens as the acceleration and time and use the equation below to find the unknown final velocity. That is,

$$\vec{v} = \vec{v}_0 + \vec{a}t = 0 + (0.40 \text{ m/s}^2)(100 \text{ s}) = 40 \text{ m/s}$$

Step 2

Check to see if the answer is reasonable. Is it too large or too small, or does it have the wrong sign, improper units, …? In this case, you may need to convert meters per second into a more familiar unit, such as miles per hour.

$$\frac{40 \text{ m}}{\text{s}} \times \frac{3.28 \text{ ft}}{\text{m}} \times \frac{1 \text{ mi}}{5280 \text{ ft}} \times \frac{60 \text{ s}}{\text{min}} \times \frac{60 \text{ min}}{1 \text{ h}}$$

This velocity is about four times greater than a person can run—so it is too large.

Step 3

If the answer is unreasonable, look for what specifically could cause the identified difficulty. In the example of the runner, there are only two assumptions that are suspect. The acceleration could be too great or the time too long. First look at the acceleration and think about what the number means. If someone accelerates at 0.40 m/s^2, their velocity is increasing by 0.4 m/s each second. Does this seem reasonable? If so, the time must be too long. It is not possible for someone to accelerate at a constant rate of 0.40 m/s^2 for 100 s (almost two minutes).

Section Summary

- *The six basic problem solving steps for physics are:*

  Step 1. Examine the situation to determine which physical principles are involved.

  Step 2. Make a list of what is given or can be inferred from the problem as stated (identify the knowns).

  Step 3. Identify exactly what needs to be determined in the problem (identify the unknowns).

  Step 4. Find an equation or set of equations that can help you solve the problem.

  Step 5. Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units.

  Step 6. Check the answer to see if it is reasonable: Does it make sense?

Conceptual Questions

1: What information do you need in order to choose which equation or equations to use to solve a problem? Explain.

2: What is the last thing you should do when solving a problem? Explain.

2.8 Falling Objects

Summary

- Describe the effects of gravity on objects in motion.
- Describe the motion of objects that are in free fall.
- Calculate the position and velocity of objects in free fall.

Falling objects form an interesting class of motion problems. For example, we can estimate the depth of a vertical mine shaft by dropping a rock into it and listening for the rock to hit the bottom. By applying the kinematics developed so far to falling objects, we can examine some interesting situations and learn much about gravity in the process.

Gravity

The most remarkable and unexpected fact about falling objects is that, if air resistance and friction are negligible, then in a given location all objects fall toward the center of Earth with the *same constant acceleration, independent of their mass*. This experimentally determined fact is unexpected, because we are so accustomed to the effects of air resistance and friction that we expect light objects to fall slower than heavy ones. You can watch the video of the hammer and feather drop on the Moon by visiting NASA at https://www.history.nasa.gov/alsj/a15/video15.html#closeout3

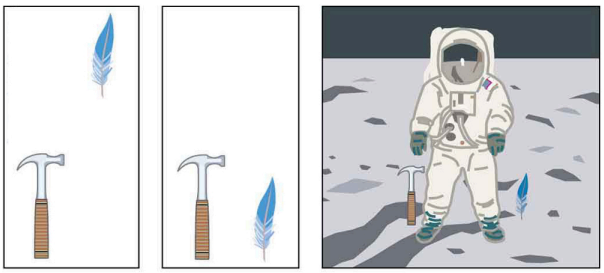


Figure 1. *A hammer and a feather will fall with the same constant acceleration if air resistance is considered negligible. This is a general characteristic of gravity not unique to Earth, as astronaut David R. Scott demonstrated on the Moon in 1971, where the acceleration due to gravity is only 1.67 m/s^2.*

In the real world, air resistance can cause a lighter object to fall slower than a heavier object of the same size. A tennis ball will reach the ground after a hard baseball dropped at the same time. (It might be

difficult to observe the difference if the height is not large.) Air resistance opposes the motion of an object through the air, while friction between objects—such as between clothes and a laundry chute or between a stone and a pool into which it is dropped—also opposes motion between them. For the ideal situations of these first few chapters, an object *falling without air resistance or friction* is defined to be in **free-fall**.

The force of gravity causes objects to fall toward the center of Earth. The acceleration of free-falling objects is therefore called the **acceleration due to gravity**. The acceleration due to gravity is *constant*, which means we can apply the kinematics equations to any falling object where air resistance and friction are negligible. This opens a broad class of interesting situations to us. The acceleration due to gravity is so important that its magnitude is given its own symbol, *g*. It is constant at any given location on Earth and has the average value

$$g = 9.80 \text{ m/s}^2$$

Although *g* varies from 9.78 m/s^2 to 9.83 m/s^2, depending on latitude, altitude, underlying geological formations, and local topography, the average value of 9.80 m/s^2 will be used in this text unless otherwise specified. The direction of the acceleration due to gravity is *downward (towards the center of Earth)*. In fact, its direction *defines* what we call vertical. Note that whether the acceleration $\vec{a}$ in the kinematic equations has the value +*g* or -*g* depends on how we define our coordinate system. If we define the upward direction as positive, then $\vec{a}$ = -*g* = -9.80 m/s^2, and if we define the downward direction as positive, then $\vec{a}$ = *g* = 9.80 m/s^2.

One-Dimensional Motion Involving Gravity

The best way to see the basic features of motion involving gravity is to start with the simplest situations and then progress toward more complex ones. So we start by considering straight up and down motion with no air resistance or friction. These assumptions mean that the velocity (if there is any) is vertical. If the object is dropped, we know the initial velocity is zero. Once the object has left contact with whatever held or threw it, the object is in free-fall. Under these circumstances, the motion is one-dimensional and has constant acceleration of magnitude *g*. We will also represent vertical displacement with the symbol *y* and use *x* for horizontal displacement.

KINEMATIC EQUATIONS FOR OBJECTS IN FREE FALL WHERE ACCELERATION = -g

$$\vec{v} = \vec{v}_0 - gt$$

$$y = y_0 + \vec{v}_0 t - \frac{1}{2} g t^2$$

$$\vec{v}^2 = v_0^2 - 2g(y - y_0)$$

Example 1: Calculating Position and Velocity of a Falling Object: A Rock Thrown Upward

A person standing on the edge of a high cliff throws a rock straight up with an initial velocity of 13.0 m/s. The rock misses the edge of the cliff as it falls back to earth. Calculate the position and velocity of the rock 1.00 s, 2.00 s, and 3.00 s after it is thrown, neglecting the effects of air resistance.

Strategy

Draw a sketch.

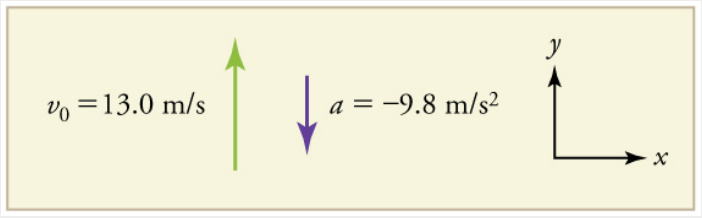


Figure 2.

We are asked to determine the position *y* at various times. It is reasonable to take the initial position y_0 to be zero. This problem involves one-dimensional motion in the vertical direction. We use plus and minus signs to indicate direction, with up being positive and down negative. Since up is positive, and the rock is thrown upward, the initial velocity must be positive too. The acceleration due to gravity is downward, so *a* is negative. It is crucial that the initial velocity and the acceleration due to gravity have opposite signs. Opposite signs indicate that the acceleration due to gravity opposes the initial motion and will slow and eventually reverse it.

Since we are asked for values of position and velocity at three times, we will refer to these as y_1 and v_1; y_2 and v_2; and y_3 and v_3.

Solution for Position y_1

1. Identify the knowns. We know that $y_0 = 0$; $\vec{v}_0 = 13.0$ m/s; $a = -g = -9.80$ m/s^2; and $t = 1.00$ s.

2. Identify the best equation to use. We will use $y = y_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ because it includes only one unknown, *y* (or y_1, here), *which is the value we want to find.*

3. Plug in the known values and solve for y_1.

$$y_1 = 0 + 13.0 \text{ m/s} \times 1.00 \text{ s} + \frac{1}{2} \times -9.80 \text{ m/s}^2 \times 1.00 \text{ s}^2 = 8.10 \text{ m}$$

Discussion

The rock is 8.10 m above its starting point at $t = 1.00$ s, since $y_1 > y_0$. It could be *moving* up or down; the only way to tell is to calculate $\vec{v}_1$ and find out if it is positive or negative.

Solution for Velocity v_1

1. Identify the knowns. We know that $y_0 = 0$; $\vec{v}_0 = 13.0$ m/s; $\vec{a} = -g = -9.80$ m/s^2; and $t = 1.00$ s. We also know from the solution above that $y_1 = 8.10$ m.

2. Identify the best equation to use. The most straightforward is $\vec{v} = \vec{v}_0 - gt$ (from $\vec{v} = \vec{v}_0 + \vec{a}t$, where $\vec{a}$ = gravitational acceleration = -*g*).

3. Plug in the knowns and solve.

$$\vec{v}_1 = \vec{v}_0 - gt = 13.0 \text{ m/s} - 9.80 \text{ m/s}^2 \times 1.00 \text{ s} = 3.20 \text{ m/s}$$

Discussion

The positive value for $\vec{v}_1$ means that the rock is still heading upward at *t* = 1.00 s. However, it has slowed from its original 13.0 m/s, as expected.

Solution for Remaining Times

The procedures for calculating the position and velocity at *t* = 2.00 s and 3.00 s are the same as those above. The results are summarized in Table 1 and illustrated in Figure 3.

Time, *t*	Position, *y*	Velocity, $\vec{v}$	Acceleration, $\vec{a}$
1.00 s	8.10 m	3.20 m/s	−9.80 m/s^2
2.00 s	6.40 m	−6.60 m/s	−9.80 m/s^2
3.00 s	−5.10 m	−16.4 m/s	−9.80 m/s^2

Table 1. Results.

Graphing the data helps us understand it more clearly.

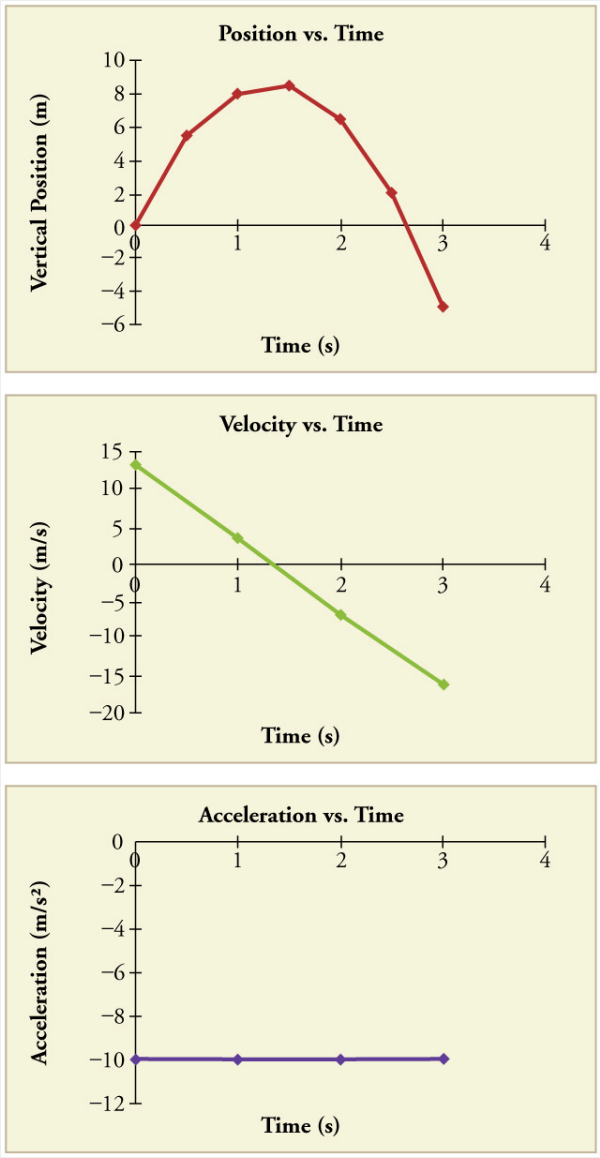


Figure 3. *Vertical position, vertical velocity, and vertical acceleration vs. time for a rock thrown vertically up at the edge of a cliff. Notice that velocity changes linearly with time and that acceleration is constant. Misconception Alert! Notice that the position vs. time graph shows vertical position only. It is easy to get the impression that the graph shows some horizontal motion—the shape of the graph looks like the path of a projectile. But this is not the case; the horizontal axis is time, not space. The actual path of the rock in space is straight up, and straight down.*

Discussion

The interpretation of these results is important. At 1.00 s the rock is above its starting point and heading upward, since y_1 and v_1 are both positive. At 2.00 s, the rock is still above its starting point,

but the negative velocity means it is moving downward. At 3.00 s, both y_3 and v_3 are negative, meaning the rock is below its starting point and continuing to move downward. Notice that when the rock is at its highest point (at 1.5 s), its velocity is zero, but its acceleration is still -9.80 m/s^2. Its acceleration is -9.80 m/s^2 for the whole trip—while it is moving up and while it is moving down. Note that the values for y are the positions (or displacements) of the rock, not the total distances traveled. Finally, note that free-fall applies to upward motion as well as downward. Both have the same acceleration—the acceleration due to gravity, which remains constant the entire time. Astronauts training in the famous Vomit Comet, for example, experience free-fall while arcing up as well as down, as we will discuss in more detail later.

MAKING CONNECTIONS: TAKE HOME EXPERIMENT—REACTION TIME

A simple experiment can be done to determine your reaction time. Have a friend hold a ruler between your thumb and index finger, separated by about 1 cm. Note the mark on the ruler that is right between your fingers. Have your friend drop the ruler unexpectedly, and try to catch it between your two fingers. Note the new reading on the ruler. Assuming acceleration is that due to gravity, calculate your reaction time. How far would you travel in a car (moving at 30 m/s) if the time it took your foot to go from the gas pedal to the brake was twice this reaction time?

Example 2: Calculating Velocity of a Falling Object: A Rock Thrown Down

What happens if the person on the cliff throws the rock straight down, instead of straight up? To explore this question, calculate the velocity of the rock when it is 5.10 m below the starting point, and has been thrown downward with an initial speed of 13.0 m/s.

Strategy

Draw a sketch.

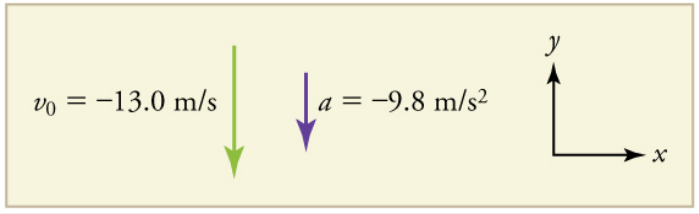


Figure 4.

Since up is positive, the final position of the rock will be negative because it finishes below the

starting point at $y_0 = 0$. Similarly, the initial velocity is downward and therefore negative, as is the acceleration due to gravity. We expect the final velocity to be negative since the rock will continue to move downward.

Solution

1. Identify the knowns. $y_0 = 0$; $y_1 = -5.10$ m; $\vec{v}_0 = -13.0\ \text{m/s}$; $\vec{a} = -g = -9.80\ \text{m/s}^2$.

2. Choose the kinematic equation that makes it easiest to solve the problem. The equation $\vec{v}^2 = \vec{v}_0{}^2 + 2\vec{a}(y - y_0)$ works well because the only unknown in it is $\vec{v}$. (We will plug y_1 in for y.)

3. Enter the known values

$$v^2 = -13.0\ \text{m/s}^2 + 2 \times -9.80\ \text{m/s}^2 \times -5.10\ \text{m} - 0\ \text{m} = 268.96\ \ \text{m}^2/\text{s}^2,$$

where we have retained extra significant figures because this is an intermediate result.

Taking the square root, and noting that a square root can be positive or negative, gives

$$v = \pm 16.4\ \text{m/s}.$$

The negative root is chosen to indicate that the rock is still heading down. Thus,

$$v = -16.4\ \text{m/s}$$

Discussion

Note that *this is exactly the same velocity the rock had at this position when it was thrown straight upward with the same initial speed*. (See Example 1 and Figure 5(a).) This is not a coincidental result. Because we only consider the acceleration due to gravity in this problem, the *speed* of a falling object depends only on its initial speed and its vertical position relative to the starting point. For example, if the velocity of the rock is calculated at a height of 8.10 m above the starting point (using the method from Example 1) when the initial velocity is 13.0 m/s straight up, a result of ±3.20 m/s is obtained. Here both signs are meaningful; the positive value occurs when the rock is at 8.10 m and heading up, and the negative value occurs when the rock is at 8.10 m and heading back down. It has the same *speed* but the opposite direction.

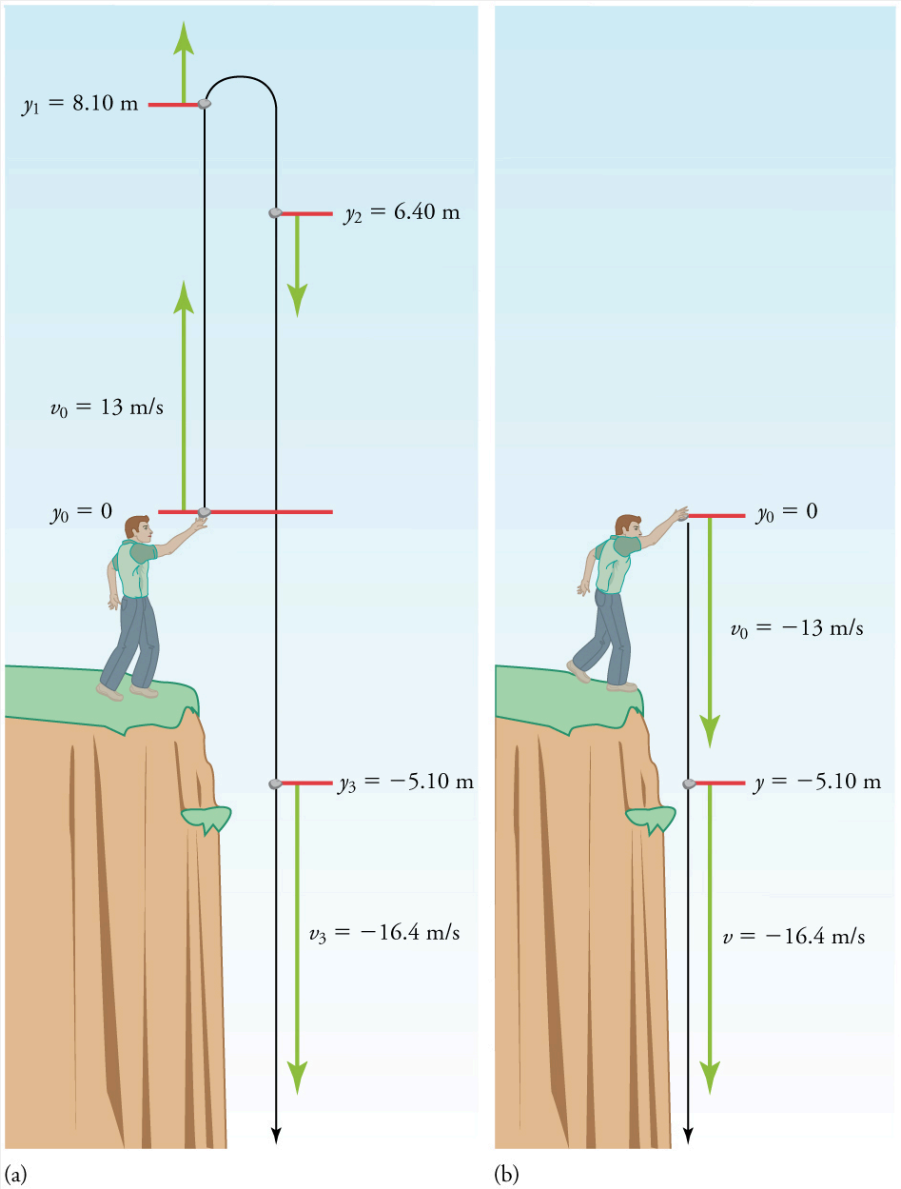


Figure 5. *(a) A person throws a rock straight up, as explored in Example 1. The arrows are velocity vectors at 0, 1.00, 2.00, and 3.00 s. (b) A person throws a rock straight down from a cliff with the same initial speed as before, as in Example 2. Note that at the same distance below the point of release, the rock has the same velocity in both cases.*

Another way to look at it is this: In Example 1, the rock is thrown up with an initial velocity of 13.0 m/s. It rises and then falls back down. When its position is $y = 0$ on its way back down, its velocity is -13.0 m/s. That is, it has the same speed on its way down as on its way up. We would then expect its velocity at a position of $y = -5.10$ m to be the same whether we have thrown it upwards at +13.0 m/s or thrown it downwards at -13.0 m/s. The velocity of the rock on its way down from $y = 0$ is the same whether we have thrown it up or down to start with, as long as the speed with which it was initially thrown is the same.

Example 3: Find g from Data on a Falling Object

The acceleration due to gravity on Earth differs slightly from place to place, depending on topography (e.g., whether you are on a hill or in a valley) and subsurface geology (whether there is dense rock like iron ore as opposed to light rock like salt beneath you.) The precise acceleration due to gravity can be calculated from data taken in an introductory physics laboratory course. An object, usually a metal ball for which air resistance is negligible, is dropped and the time it takes to fall a known distance is measured. See, for example, Figure 6. Very precise results can be produced with this method if sufficient care is taken in measuring the distance fallen and the elapsed time.

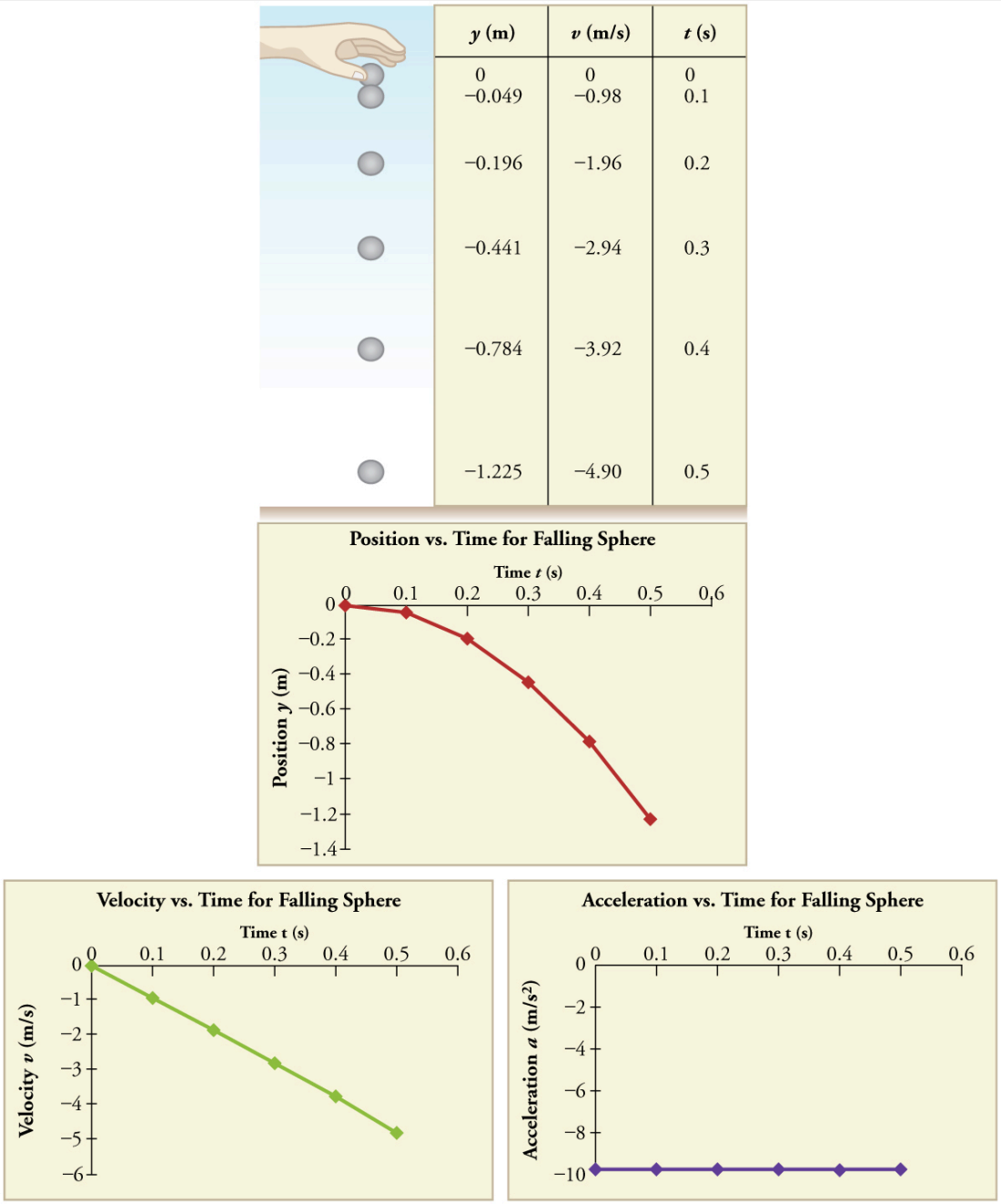


Figure 6. *Positions and velocities of a metal ball released from rest when air resistance is negligible. Velocity is seen to increase linearly with time while displacement increases with time squared. Acceleration is a constant and is equal to gravitational acceleration.*

Suppose the ball falls 1.0000 m in 0.45173 s. Assuming the ball is not affected by air resistance, what is the precise acceleration due to gravity at this location?

Strategy

Draw a sketch.

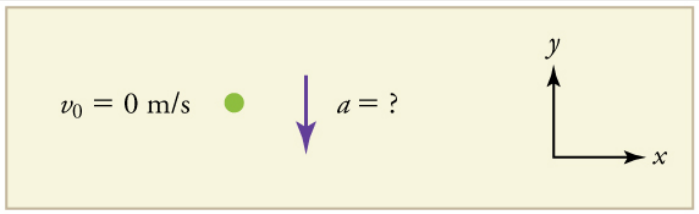


Figure 7.

We need to solve for acceleration $\vec{a}$. Note that in this case, displacement is downward and therefore negative, as is acceleration.

Solution

1. Identify the knowns. $y_0 = 0$; $y = -1.0000$ m; $t = 0.45173$ s; $\vec{v}_0 = 0$.

2. Choose the equation that allows you to solve for $\vec{a}$ using the known values.

$$y = y_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

3. Substitute 0 for $\vec{v}_0$ and rearrange the equation to solve for $\vec{a}$. Substituting 0 for $\vec{v}_0$ yields

$$y = y_0 + \frac{1}{2}\vec{a}t^2$$

Solving for $\vec{a}$ gives

$$\vec{a} = \frac{2(y - y_0)}{t^2}.$$

4. Substitute known values yields

$$\vec{a} = \frac{2(-1.0000 \text{ m} - 0)}{(0.45173 \text{ s})^2} = -9.8010 \text{ m/s}^2,$$

so, because $\vec{a}$ = -g with the directions we have chosen,

$$g = 9.8010 \text{ m/s}^2.$$

Discussion

The negative value for $\vec{a}$ indicates that the gravitational acceleration is downward, as expected. We expect the value to be somewhere around the average value of 9.80 m/s^2, so 9.8010 m/s^2 makes sense. Since the data going into the calculation are relatively precise, this value for g is more precise than the average value of 9.80 m/s^2; it represents the local value for the acceleration due to gravity.

Check Your Understanding

1: A chunk of ice breaks off a glacier and falls 30.0 meters before it hits the water. Assuming it falls freely (there is no air resistance), how long does it take to hit the water?

PHET EXPLORATIONS: EQUATION GRAPHER

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. $y = bx$) to see how they add to generate the polynomial curve.

PhET Interactive Simulation

Figure 8. *Equation Grapher*.

Section Summary

- An object in free-fall experiences constant acceleration if air resistance is negligible.
- On Earth, all free-falling objects have an acceleration due to gravity g, which averages
$$g = 9.80 \text{ m/s}^2.$$
- Whether the acceleration a should be taken as $+g$ or $-g$ is determined by your choice of coordinate system. If you choose the upward direction as positive, $\vec{a} = -g = -9.80 \text{ m/s}^2$ is negative. In the opposite case, $\vec{a} = +g = 9.80 \text{ m/s}^2$ is positive. Since acceleration is constant, the kinematic equations above can be applied with the appropriate $+g$ or $-g$ substituted for $\vec{a}$.
- For objects in free-fall, up is normally taken as positive for displacement, velocity, and acceleration.

Glossary

free-fall
the state of movement that results from gravitational force only

acceleration due to gravity
acceleration of an object as a result of gravity

Conceptual Questions

1: What is the acceleration of a rock thrown straight upward on the way up? At the top of its flight? On the way down?

2: An object that is thrown straight up falls back to Earth. This is one-dimensional motion. (a) When is its

velocity zero? (b) Does its velocity change direction? (c) Does the acceleration due to gravity have the same sign on the way up as on the way down?

3: Suppose you throw a rock nearly straight up at a coconut in a palm tree, and the rock misses on the way up but hits the coconut on the way down. Neglecting air resistance, how does the speed of the rock when it hits the coconut on the way down compare with what it would have been if it had hit the coconut on the way up? Is it more likely to dislodge the coconut on the way up or down? Explain.

4: If an object is thrown straight up and air resistance is negligible, then its speed when it returns to the starting point is the same as when it was released. If air resistance were not negligible, how would its speed upon return compare with its initial speed? How would the maximum height to which it rises be affected?

5: The severity of a fall depends on your speed when you strike the ground. All factors but the acceleration due to gravity being the same, how many times higher could a safe fall on the Moon be than on Earth (gravitational acceleration on the Moon is about 1/6 that of the Earth)?

6: How many times higher could an astronaut jump on the Moon than on Earth if his takeoff speed is the same in both locations (gravitational acceleration on the Moon is about 1/6 of g on Earth)?

Problems & Exercises

Assume air resistance is negligible unless otherwise stated.

1: Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, and (d) 2.00 s for a ball thrown straight up with an initial velocity of 15.0 m/s. Take the point of release to be $y_0 = 0$.

2: Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, (d) 2.00, and (e) 2.50 s for a rock thrown straight down with an initial velocity of 14.0 m/s from the Verrazano Narrows Bridge in New York City. The roadway of this bridge is 70.0 m above the water.

3: A basketball referee tosses the ball straight up for the starting tip-off. At what velocity must a basketball player leave the ground to rise 1.25 m above the floor in an attempt to get the ball?

4: A rescue helicopter is hovering over a person whose boat has sunk. One of the rescuers throws a life preserver straight down to the victim with an initial velocity of 1.40 m/s and observes that it takes 1.8 s to reach the water. (a) List the knowns in this problem. (b) How high above the water was the preserver released? Note that the downdraft of the helicopter reduces the effects of air resistance on the falling life preserver, so that an acceleration equal to that of gravity is reasonable.

5: A dolphin in an aquatic show jumps straight up out of the water at a velocity of 13.0 m/s. (a) List the knowns in this problem. (b) How high does his body rise above the water? To solve this part, first note that the final velocity is now a known and identify its value. Then identify the unknown, and discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking units, and discuss whether the answer is reasonable. (c) How long is the dolphin in the air? Neglect any effects due to his size or orientation.

6: A swimmer bounces straight up from a diving board and falls feet first into a pool. She starts with a

velocity of 4.00 m/s, and her takeoff point is 1.80 m above the pool. (a) How long are her feet in the air? (b) What is her highest point above the board? (c) What is her velocity when her feet hit the water?

7: (a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

8: A very strong, but inept, shot putter puts the shot straight up vertically with an initial velocity of 11.0 m/s. How long does he have to get out of the way if the shot was released at a height of 2.20 m, and he is 1.80 m tall?

9: You throw a ball straight up with an initial velocity of 15.0 m/s. It passes a tree branch on the way up at a height of 7.00 m. How much additional time will pass before the ball passes the tree branch on the way back down?

10: A kangaroo can jump over an object 2.50 m high. (a) Calculate its vertical speed when it leaves the ground. (b) How long is it in the air?

11: Standing at the base of one of the cliffs of Mt. Arapiles in Victoria, Australia, a hiker hears a rock break loose from a height of 105 m. He can't see the rock right away but then does, 1.50 s later. (a) How far above the hiker is the rock when he can see it? (b) How much time does he have to move before the rock hits his head?

12: An object is dropped from a height of 75.0 m above ground level. (a) Determine the distance traveled during the first second. (b) Determine the final velocity at which the object hits the ground. (c) Determine the distance traveled during the last second of motion before hitting the ground.

13: There is a 250-m-high cliff at Half Dome in Yosemite National Park in California. Suppose a boulder breaks loose from the top of this cliff. (a) How fast will it be going when it strikes the ground? (b) Assuming a reaction time of 0.300 s, how long will a tourist at the bottom have to get out of the way after hearing the sound of the rock breaking loose (neglecting the height of the tourist, which would become negligible anyway if hit)? The speed of sound is 335 m/s on this day.

14: A ball is thrown straight up. It passes a 2.00-m-high window 7.50 m off the ground on its path up and takes 0.312 s to go past the window. What was the ball's initial velocity? Hint: First consider only the distance along the window, and solve for the ball's velocity at the bottom of the window. Next, consider only the distance from the ground to the bottom of the window, and solve for the initial velocity using the velocity at the bottom of the window as the final velocity.

15: Suppose you drop a rock into a dark well and, using precision equipment, you measure the time for the sound of a splash to return. (a) Neglecting the time required for sound to travel up the well, calculate the distance to the water if the sound returns in 2.0000 s. (b) Now calculate the distance taking into account the time for sound to travel up the well. The speed of sound is 332.00 m/s in this well.

16: A steel ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.45 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 0.0800 ms (8.00×10^{-5} s). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

17: A coin is dropped from a hot-air balloon that is 300 m above the ground and rising at 10.0 m/s upward. For the coin, find (a) the maximum height reached, (b) its position and velocity 4.00 s after being released, and (c) the time before it hits the ground.

18: A soft tennis ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.10 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 3.50 ms (3.50×10^{-3} s). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

Solutions

Check Your Understanding

1: We know that initial position $y_0 = 0$, final position $y = -30.0\ \text{m}$, and $\vec{a} = -g = -9.80\ \text{m/s}^2$. We can then use the equation $y = y_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ to solve for t. Inserting $\vec{a} = -g$, we obtain

$$y = 0 + 0 - \frac{1}{2}gt^2$$

$$t^2 = \frac{2y}{-g}$$

$$t = \pm\sqrt{\frac{2y}{-g}} = \pm\sqrt{\frac{2(-30.0\ \text{m})}{-9.80\ \ \text{m/s}^2}} = \pm\sqrt{6.12\ \text{s}^2} = 2.47\ \text{s} \approx 2.5\ \text{s}$$

where we take the positive value as the physically relevant answer. Thus, it takes about 2.5 seconds for the piece of ice to hit the water.

Problems & Exercises

1: (a) y_1 = 6.28 m; $\vec{v}_1$ = 10.1 m/s (b) y_2 = 10.1 m; $\vec{v}_2$ = 5.20 m/s (c) y_3 = 11.5 m; $\vec{v}_3$ = 0.300 m/s (d) y_4 = 10.4 m; $\vec{v}_4$ = -4.60 m/s

3: $\vec{v}_0 = 4.95\ \text{m/s}$

5: (a) $\vec{a} = -9.80\ \text{m/s}^2; \vec{v}_0 = 13.0\ \text{m/s}$ y_0 = 0 m (b) $\vec{v} = 0\ \text{m/s}$. Unknown is distance y to top of trajectory, where velocity is zero. Use equation $\vec{v}^2 = {\vec{v}_0}^2 + 2\vec{a}(y - y_0)$ because it contains all known values except for y, so we can solve for y. Solving for y gives

$$\vec{v}^2 - {\vec{v}_0}^2 = 2\vec{a}(y - y_0)$$

$$\frac{\vec{v}^2 - \vec{v}_0{}^2}{2\vec{a}} = y - y_0$$

$$y = y_0 + \frac{\vec{v}^2 - \vec{v}_0{}^2}{2\vec{a}}$$

$$= \frac{(0\ \text{m/s})^2 - (13.0\ \text{m/s})^2}{2(-9.80\ \text{m/s}^2)} = 8.62\ \text{m}$$

Dolphins measure about 2 meters long and can jump several times their length out of the water, so this is a reasonable result. (c) 2.65 s

7: (a) 8.26 m (b) 0.717 s

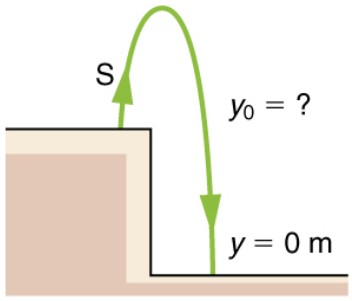


Figure 9.

9: 1.91 s

11: (a) 94.0 m (b) 3.13 s

13: (a) -70.0 m/s (downward) (b) 6.10 s

15: (a) 19.6 m (b) 18.5 m

17: (a) 305 m (b) 262 m, -29.2 m/s (c) 8.91 s

Chapter 3 Two-Dimensional Kinematics

Figure 1. *Everyday motion that we experience is, thankfully, rarely as tortuous as a rollercoaster ride like this—the Dragon Khan in Spain's Universal Port Aventura Amusement Park. However, most motion is in curved, rather than straight-line, paths. Motion along a curved path is two- or three-dimensional motion, and can be described in a similar fashion to one-dimensional motion. (credit: Boris23/Wikimedia Commons).*

The arc of a basketball, the orbit of a satellite, a bicycle rounding a curve, a swimmer diving into a pool, blood gushing out of a wound, and a puppy chasing its tail are but a few examples of motions along curved paths. In fact, most motions in nature follow curved paths rather than straight lines. Motion along a curved path on a flat surface or a plane (such as that of a ball on a pool table or a skater on an ice rink) is two-dimensional, and thus described by two-dimensional kinematics. Motion not confined to a plane, such as a car following a winding mountain road, is described by three-dimensional kinematics. Both two- and three-dimensional kinematics are simple extensions of the one-dimensional kinematics developed for straight-line motion in the previous chapter. This simple extension will allow us to apply physics to many more situations, and it will also yield unexpected insights about nature.

3.1 Introduction to Kinematics in Two Dimensions

Summary

- Observe that motion in two dimensions consists of horizontal and vertical components.
- Understand the independence of horizontal and vertical vectors in two-dimensional motion.

Figure 1. *Walkers and drivers in a city like New York are rarely able to travel in straight lines to reach their destinations. Instead, they must follow roads and sidewalks, making two-dimensional, zigzagged paths. (credit: Margaret W. Carruthers).*

Two-Dimensional Motion: Walking in a City

Suppose you want to walk from one point to another in a city with uniform square blocks, as pictured in Figure 2.

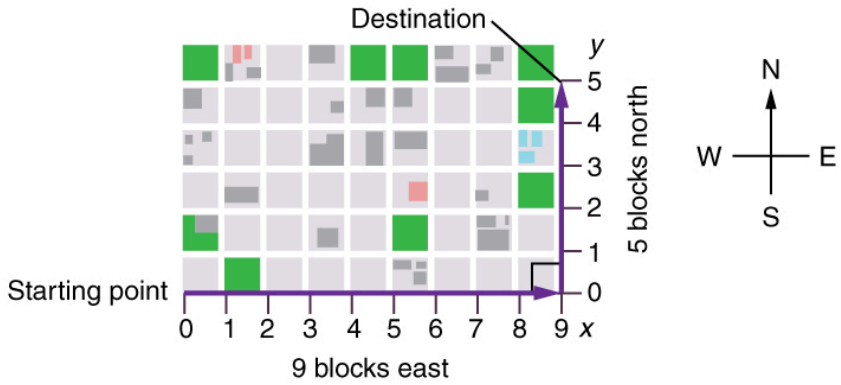


Figure 2. *A pedestrian walks a two-dimensional path between two points in a city. In this scene, all blocks are square and are the same size.*

The straight-line path that a helicopter might fly is blocked to you as a pedestrian, and so you are forced to take a two-dimensional path, such as the one shown. You walk 14 blocks in all, 9 east followed by 5 north. What is the straight-line distance?

An old adage states that the shortest distance between two points is a straight line. The two legs of the trip and the straight-line path form a right triangle, and so the Pythagorean theorem, $a^2 + b^2 = c^2$, can be used to find the straight-line distance.

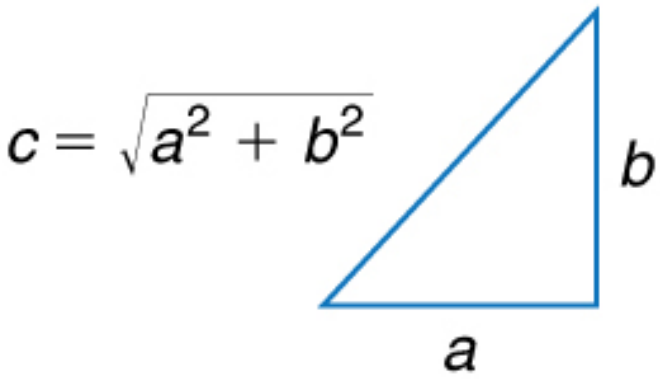


Figure 3. *The Pythagorean theorem relates the length of the legs of a right triangle, labeled a and b, with the hypotenuse, labeled c. The relationship is given by:* $a^2+b^2=c^2$. *This can be rewritten, solving for c:* $c = \sqrt{(a^2+b^2)}$.

The hypotenuse of the triangle is the straight-line path, and so in this case its length in units of city blocks is $\sqrt{(9\text{ blocks})^2 + (5\text{ blocks})^2} = 10.3\text{ blocks}$, considerably shorter than the 14 blocks you walked. (Note that we are using three significant figures in the answer. Although it appears that “9” and “5” have only one significant digit, they are discrete numbers. In this case “9 blocks” is the same as “9.0 or 9.00 blocks.” We have decided to use three significant figures in the answer in order to show the result more precisely.)

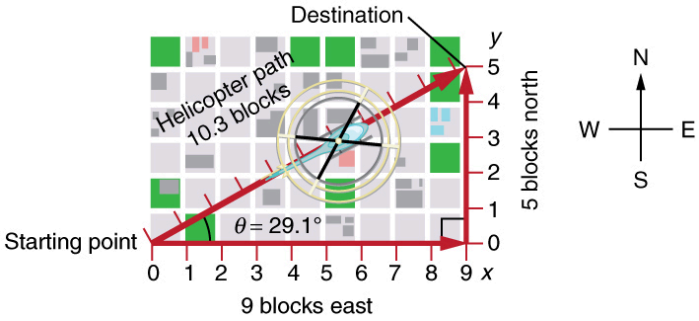


Figure 4. *The straight-line path followed by a helicopter between the two points is shorter than the 14 blocks walked by the pedestrian. All blocks are square and the same size.*

The fact that the straight-line distance (10.3 blocks) in Figure 4 is less than the total distance walked (14 blocks) is one example of a general characteristic of vectors. (Recall that **vectors** are quantities that have both magnitude and direction.)

As for one-dimensional kinematics, we use arrows to represent vectors. The length of the arrow is proportional to the vector's magnitude. The arrow's length is indicated by hash marks in Figure 2 and Figure 4. The arrow points in the same direction as the vector. For two-dimensional motion, the path of an object can be represented with three vectors: one vector shows the straight-line path between the initial and final points of the motion, one vector shows the horizontal component of the motion, and one vector shows the vertical component of the motion. The horizontal and vertical components of the motion add together to give the straight-line path. For example, observe the three vectors in Figure 4. The first represents a 9-block displacement east. The second represents a 5-block displacement north. These vectors are added to give the third vector, with a 10.3-block total displacement. The third vector is the straight-line path between the two points. Note that in this example, the vectors that we are adding are perpendicular to each other and thus form a right triangle. This means that we can use the Pythagorean theorem to calculate the magnitude of the total displacement. (Note that we cannot use the Pythagorean theorem to add vectors that are not perpendicular. We will develop techniques for adding vectors having any direction, not just those perpendicular to one another, in Chapter 3.2 Vector Addition and Subtraction: Graphical Methods and Chapter 3.3 Vector Addition and Subtraction: Analytical Methods.)

The Independence of Perpendicular Motions

The person taking the path shown in Figure 4 walks east and then north (two perpendicular directions). How far he or she walks east is only affected by his or her motion eastward. Similarly, how far he or she walks north is only affected by his or her motion northward.

INDEPENDENCE OF MOTION

The horizontal and vertical components of two-dimensional motion are independent of each other. Any motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

This is true in a simple scenario like that of walking in one direction first, followed by another. It is also true of more complicated motion involving movement in two directions at once. For example, let's compare the motions of two baseballs. One baseball is dropped from rest. At the same instant, another is thrown horizontally from the same height and follows a curved path. A stroboscope has captured the positions of the balls at fixed time intervals as they fall.

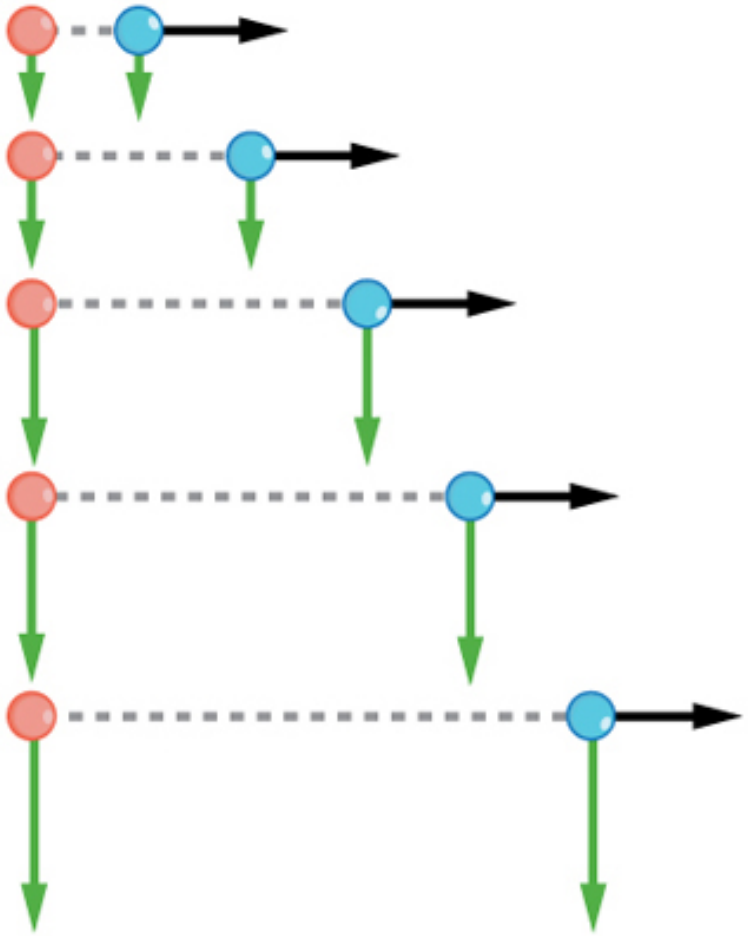


Figure 5. *This shows the motions of two identical balls—one falls from rest, the other has an initial horizontal velocity. Each subsequent position is an equal time interval. Arrows represent horizontal and vertical velocities at each position. The ball on the right has an initial horizontal velocity, while the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls. This shows that the vertical and horizontal motions are independent.*

It is remarkable that for each flash of the strobe, the vertical positions of the two balls are the same. This similarity implies that the vertical motion is independent of whether or not the ball is moving horizontally. (Assuming no air resistance, the vertical motion of a falling object is influenced by gravity only, and not by any horizontal forces.) Careful examination of the ball thrown horizontally shows that it travels the same horizontal distance between flashes. This is due to the fact that there are no additional forces on the ball in the horizontal direction after it is thrown. This result means that the horizontal velocity is constant, and affected neither by vertical motion nor by gravity (which is vertical). Note that this case is true only for ideal conditions. In the real world, air resistance will affect the speed of the balls in both directions.

The two-dimensional curved path of the horizontally thrown ball is composed of two independent one-dimensional motions (horizontal and vertical). The key to analyzing such motion, called *projectile motion*, is to *resolve* (break) it into motions along perpendicular directions. Resolving two-dimensional

motion into perpendicular components is possible because the components are independent. We shall see how to resolve vectors in Chapter 3.2 Vector Addition and Subtraction: Graphical Methods and Chapter 3.3 Vector Addition and Subtraction: Analytical Methods. We will find such techniques to be useful in many areas of physics.

PHET EXPLORATIONS: LADYBUG MOTION IN 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

Figure 6. *Ladybug Motion 2D*

Summary

- The shortest path between any two points is a straight line. In two dimensions, this path can be represented by a vector with horizontal and vertical components.
- The horizontal and vertical components of a vector are independent of one another. Motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Glossary

vector
a quantity that has both magnitude and direction; an arrow used to represent quantities with both magnitude and direction

3.2 Vector Addition and Subtraction: Graphical Methods

Summary

- Understand the rules of vector addition, subtraction, and multiplication.
- Apply graphical methods of vector addition and subtraction to determine the displacement of moving objects.

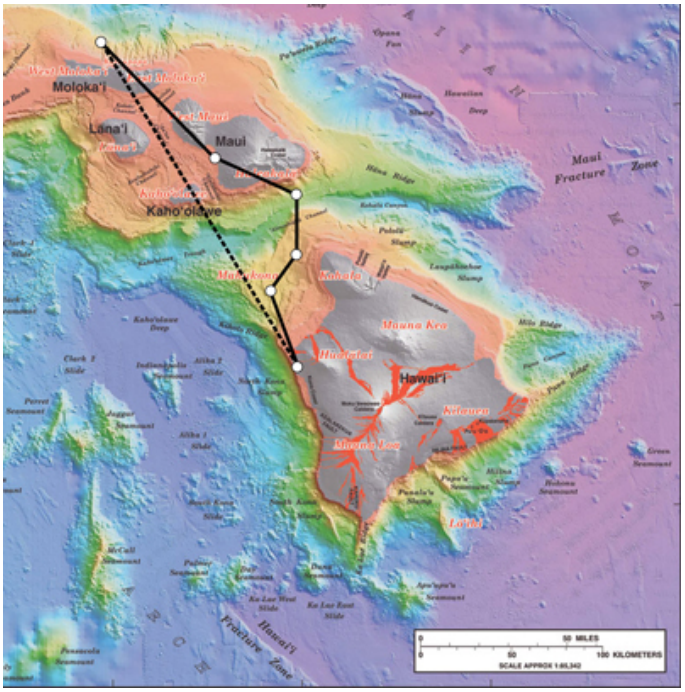


Figure 1. *Displacement can be determined graphically using a scale map, such as this one of the Hawaiian Islands. A journey from Hawai'i to Moloka'i has a number of legs, or journey segments. These segments can be added graphically with a ruler to determine the total two-dimensional displacement of the journey. (credit: US Geological Survey).*

Vectors in Two Dimensions

A **vector** is a quantity that has magnitude and direction. Displacement, velocity, acceleration, and force, for example, are all vectors. In one-dimensional, or straight-line, motion, the direction of a vector can be given simply by a plus or minus sign. In two dimensions (2-d), however, we specify the direction

of a vector relative to some reference frame (i.e., coordinate system), using an arrow having length proportional to the vector's magnitude and pointing in the direction of the vector.

Figure 2 shows such a *graphical representation of a vector*, using as an example the total displacement for the person walking in a city considered in Chapter 3.1 Kinematics in Two Dimensions: An Introduction. We shall use the notation that a symbol with an arrow over it, such as $\vec{D}$, stands for a vector. Its magnitude is represented by the symbol in italics, D, and its direction by θ.

VECTORS IN THIS TEXT

In this text, we will represent a vector with an arrow over a symbol. For example, we will represent the quantity force with the vector $\vec{F}$, which has both magnitude and direction. The magnitude of the vector will be represented by a variable in italics, such as F, and the direction of the variable will be given by an angle θ.

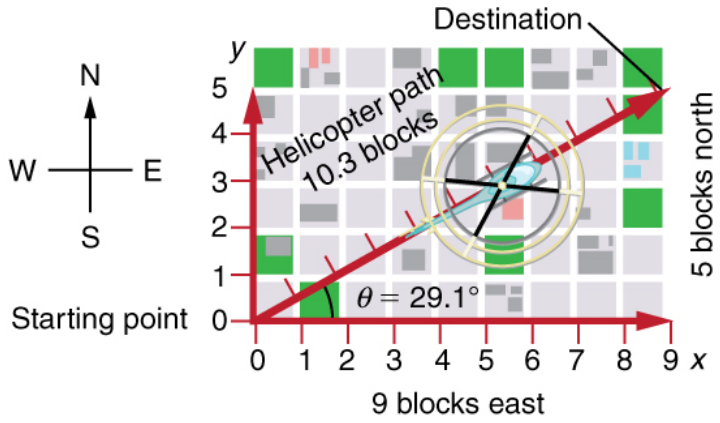


Figure 2. *A person walks 9 blocks east and 5 blocks north. The displacement is 10.3 blocks at an angle 29.1° north of east.*

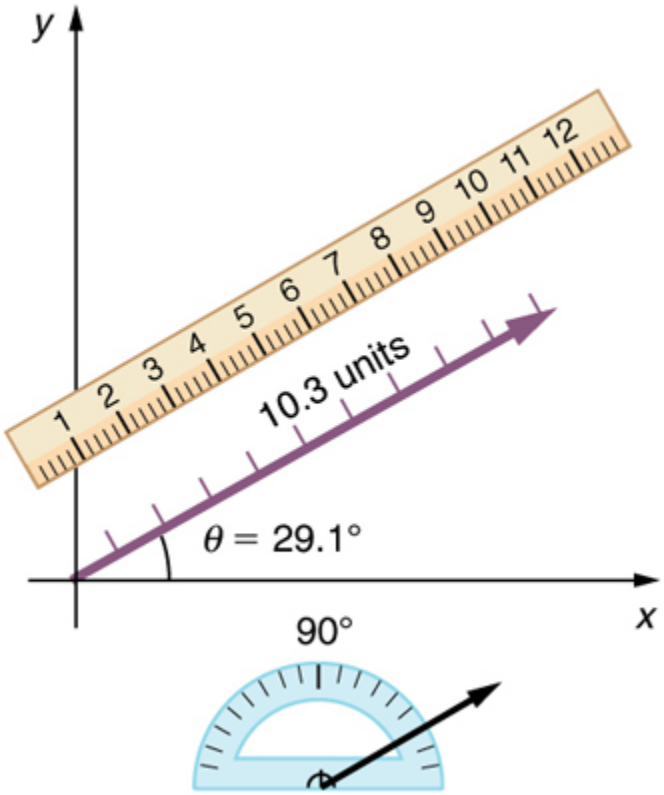


Figure 3. *To describe the resultant vector for the person walking in a city considered in Figure 2 graphically, draw an arrow to represent the total displacement vector D. Using a protractor, draw a line at an angle θ relative to the east-west axis. The length D of the arrow is proportional to the vector's magnitude and is measured along the line with a ruler. In this example, the magnitude D of the vector is 10.3 units, and the direction θ is 29.1° north of east.*

Vector Addition: Head-to-Tail Method

The **head-to-tail method** is a graphical way to add vectors, described in Figure 4 below and in the steps following. The **tail** of the vector is the starting point of the vector, and the **head** (or tip) of a vector is the final, pointed end of the arrow.

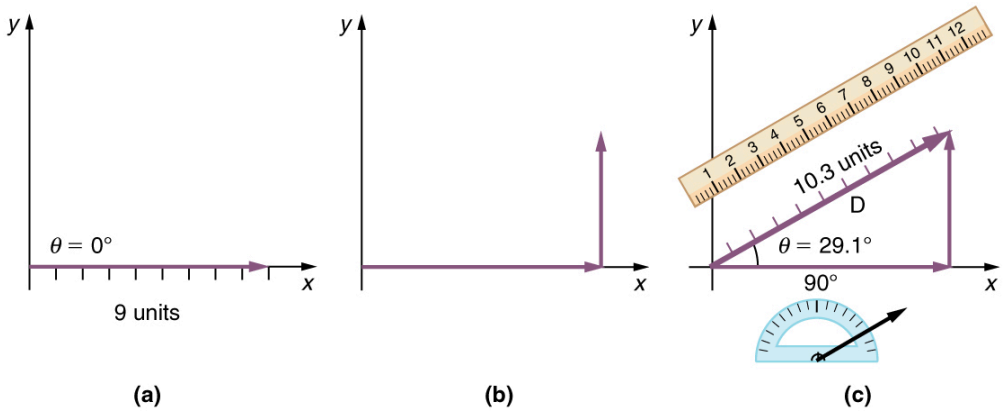


Figure 4. *Head-to-Tail Method: The head-to-tail method of graphically adding vectors is illustrated for the two displacements of the person walking in a city considered in Figure 2. (a) Draw a vector representing the displacement to the east. (b) Draw a vector representing the displacement to the north. The tail of this vector should originate from the head of the first, east-pointing vector. (c) Draw a line from the tail of the east-pointing vector to the head of the north-pointing vector to form the sum or* ***resultant vector D****. The length of the arrow* ***D*** *is proportional to the vector's magnitude and is measured to be 10.3 units . Its direction, described as the angle with respect to the east (or horizontal axis) θ is measured with a protractor to be 29.1°.*

Step 1. *Draw an arrow to represent the first vector (9 blocks to the east) using a ruler and protractor.*

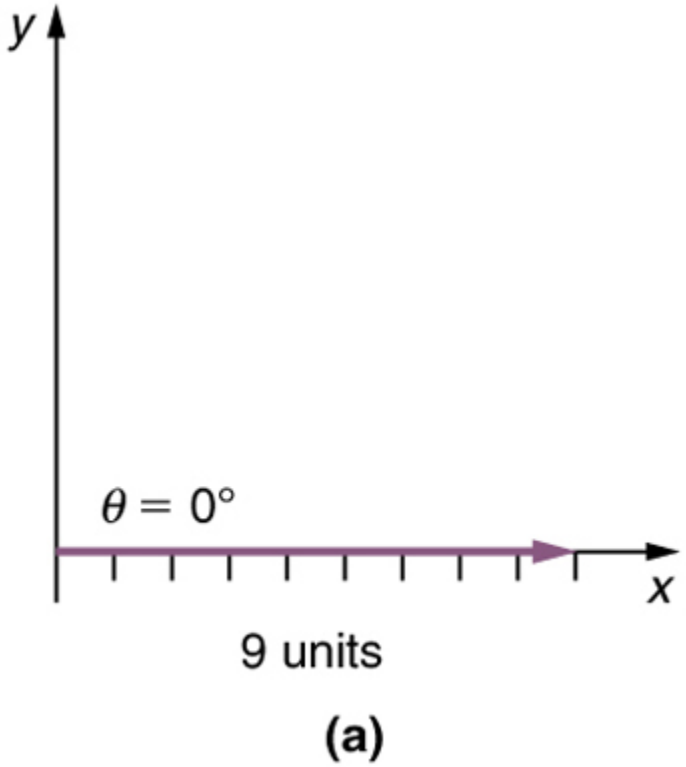


Figure 5.

Step 2. Now draw an arrow to represent the second vector (5 blocks to the north). *Place the tail of the second vector at the head of the first vector.*

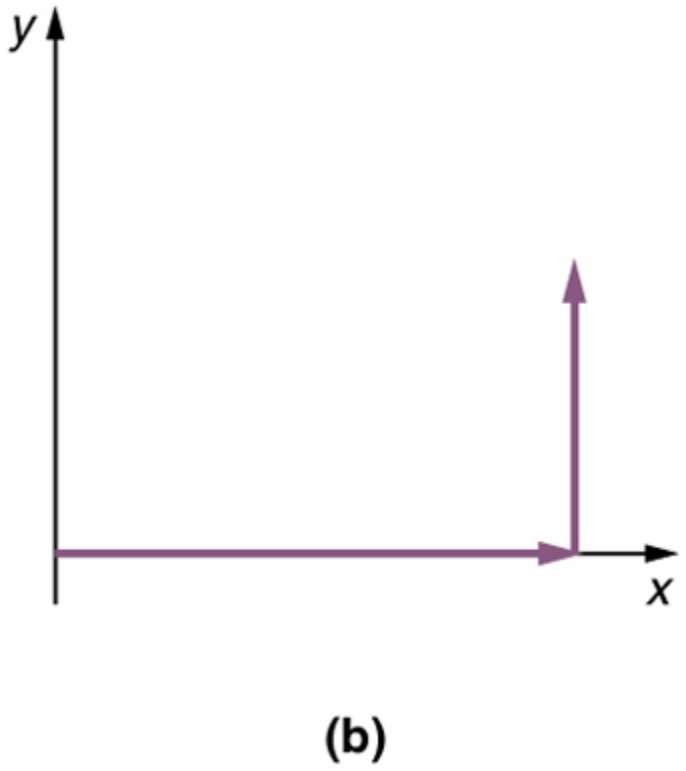


Figure 6.

Step 3. *If there are more than two vectors, continue this process for each vector to be added. Note that in our example, we have only two vectors, so we have finished placing arrows tip to tail.*

Step 4. *Draw an arrow from the tail of the first vector to the head of the last vector*. This is the **resultant**, or the sum, of the other vectors.

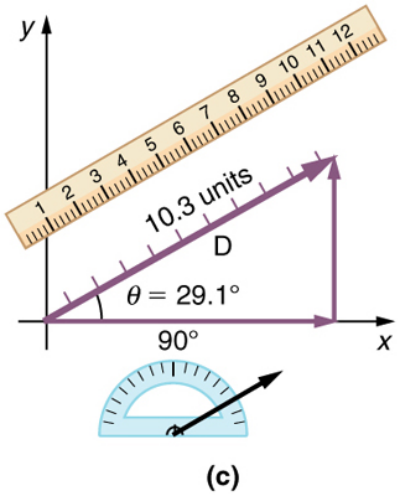


Figure 7.

Step 5. To get the **magnitude** of the resultant, *measure its length with a ruler. (Note that in most calculations, we will use the Pythagorean theorem to determine this length.)*

Step 6. To get the **direction** of the resultant, *measure the angle it makes with the reference frame using*

a protractor. (Note that in most calculations, we will use trigonometric relationships to determine this angle.)

The graphical addition of vectors is limited in accuracy only by the precision with which the drawings can be made and the precision of the measuring tools. It is valid for any number of vectors.

Example 1: Adding Vectors Graphically Using the Head-to-Tail Method: A Women Takes a Walk

Use the graphical technique for adding vectors to find the total displacement of a person who walks the following three paths (displacements) on a flat field. First, she walks 25.0 m in a direction 49.0° north of east. Then, she walks 23.0 m heading 15.0° north of east. Finally, she turns and walks 32.0 m in a direction 68.0° south of east.

Strategy

Represent each displacement vector graphically with an arrow, labeling the first $\vec{A}$, the second $\vec{B}$, and the third $\vec{C}$, making the lengths proportional to the distance and the directions as specified relative to an east-west line. The head-to-tail method outlined above will give a way to determine the magnitude and direction of the resultant displacement, denoted $\vec{R}$.

Solution

(1) Draw the three displacement vectors.

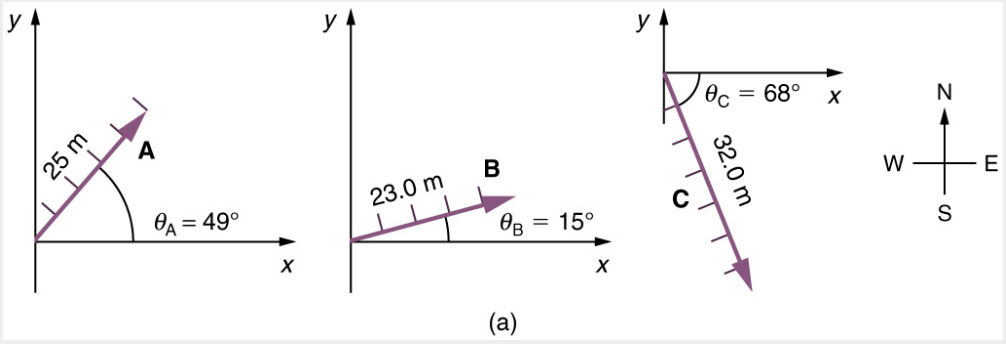


Figure 8.

(2) Place the vectors head to tail retaining both their initial magnitude and direction.

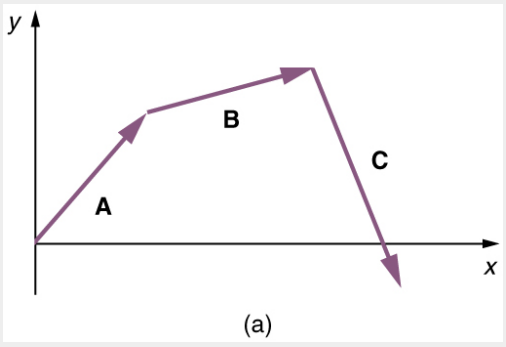


Figure 9.

(3) Draw the resultant vector, $\vec{\mathrm{R}}$.

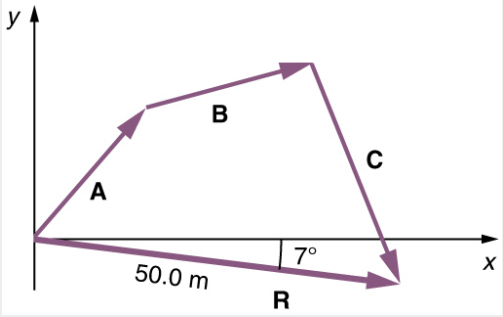


Figure 10.

(4) Use a ruler to measure the magnitude of $\vec{\mathrm{R}}$ and a protractor to measure the direction of $\vec{\mathrm{R}}$. While the direction of the vector can be specified in many ways, the easiest way is to measure the angle between the vector and the nearest horizontal or vertical axis. Since the resultant vector is south of the eastward pointing axis, we flip the protractor upside down and measure the angle between the eastward axis and the vector.

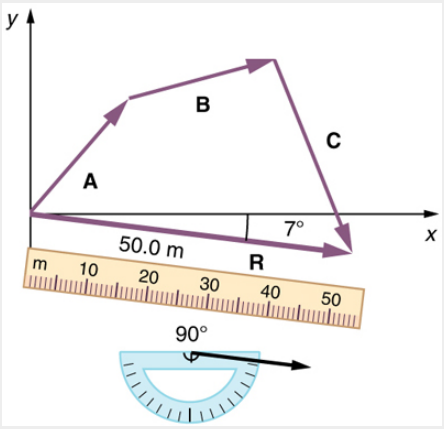


Figure 11.

In this case, the total displacement R is seen to have a magnitude of 50.0 m and to lie in a direction 7.0° south of east. By using its magnitude and direction, this vector can be expressed as R = 50.0 m and θ = 7.0° south of east.

Discussion

The head-to-tail graphical method of vector addition works for any number of vectors. It is also important to note that the resultant is independent of the order in which the vectors are added. Therefore, we could add the vectors in any order as illustrated in Figure 12 and we will still get the same solution.

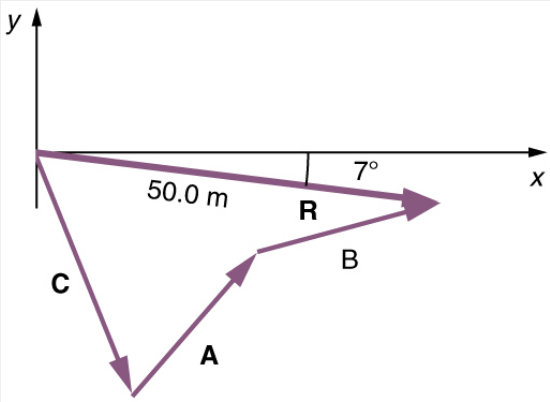


Figure 12.

Here, we see that when the same vectors are added in a different order, the result is the same. This characteristic is true in every case and is an important characteristic of vectors. Vector addition is **commutative**. Vectors can be added in any order.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}.$$

(This is true for the addition of ordinary numbers as well—you get the same result whether you add 2 + 3 or 3 + 2, for example).

Vector Subtraction

Vector subtraction is a straightforward extension of vector addition. To define subtraction (say we want to subtract $\vec{B}$ from $\vec{A}$, written $\vec{A} - \vec{B}$, we must first define what we mean by subtraction. The *negative* of a vector $\vec{B}$ is defined to be $-\vec{B}$; that is, graphically *the negative of any vector has the same magnitude but the opposite direction*, as shown in Figure 13. In other words, $\vec{B}$ has the same length as $-\vec{B}$, but points in the opposite direction. Essentially, we just flip the vector so it points in the opposite direction.

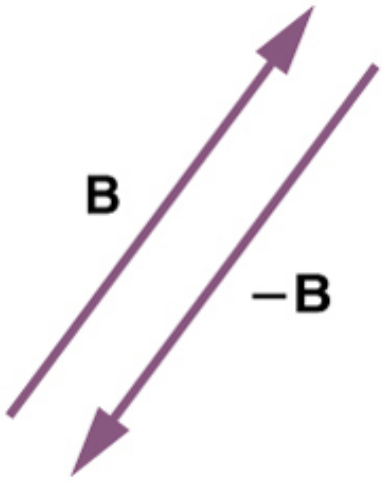


Figure 13. *The negative of a vector is just another vector of the same magnitude but pointing in the opposite direction. So B is the negative of -B; it has the same length but opposite direction.*

The *subtraction* of vector $\vec{\mathrm{B}}$ from vector $\vec{\mathrm{A}}$ is then simply defined to be the addition of $-\vec{\mathrm{B}}$ to $\vec{\mathrm{A}}$. Note that vector subtraction is the addition of a negative vector. The order of subtraction does not affect the results.

$$\vec{\mathrm{A}} - \vec{\mathrm{B}} = \vec{\mathrm{A}} + (-\vec{\mathrm{B}})$$

This is analogous to the subtraction of scalars (where, for example, 5 – 2 = 5 + (-2)). Again, the result is independent of the order in which the subtraction is made. When vectors are subtracted graphically, the techniques outlined above are used, as the following example illustrates.

Example 2: Subtracting Vectors Graphically: A Women Sailing a Boat

A woman sailing a boat at night is following directions to a dock. The instructions read to first sail 27.5 m in a direction 66.0° north of east from her current location, and then travel 30.0 m in a direction 112° north of east (or 22.0° west of north). If the woman makes a mistake and travels in the *opposite* direction for the second leg of the trip, where will she end up? Compare this location with the location of the dock.

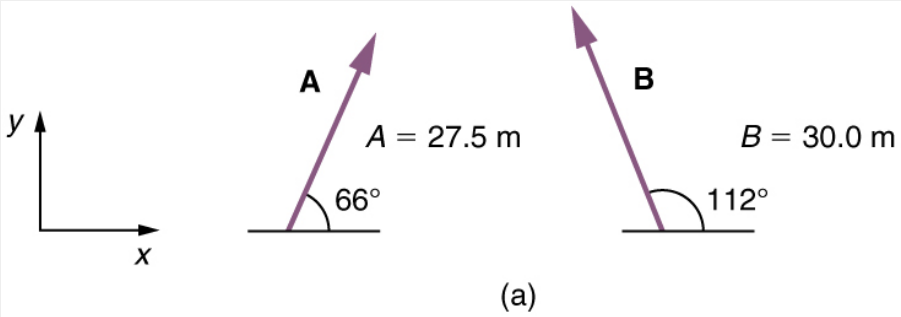


Figure 14.

Strategy

We can represent the first leg of the trip with a vector $\vec{\mathrm{A}}$, and the second leg of the trip with a vector $\vec{\mathrm{B}}$. The dock is located at a location $\vec{\mathrm{A}} + \vec{\mathrm{B}}$. If the woman mistakenly travels in the *opposite* direction for the second leg of the journey, she will travel a distance $\vec{\mathrm{B}}$ (30.0 m) in the direction 180° – 112° = 68° south of east. We represent this as $-\vec{\mathrm{B}}$, as shown below. The vector $-\vec{\mathrm{B}}$ has the same magnitude as$\vec{\mathrm{B}}$ but is in the opposite direction. Thus, she will end up at a location $\vec{\mathrm{A}} + (-\vec{\mathrm{B}})$, or $\vec{\mathrm{A}} - \vec{\mathrm{B}}$.

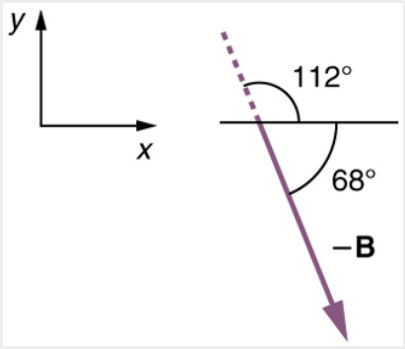


Figure 15.

We will perform vector addition to compare the location of the dock, $\vec{\mathrm{A}} + \vec{\mathrm{B}}$, with the location at which the woman mistakenly arrives, $\vec{\mathrm{A}} + (-\vec{\mathrm{B}})$.

Solution

(1) To determine the location at which the woman arrives by accident, draw vectors $\vec{\mathrm{A}}$ and $-\vec{\mathrm{B}}$.

(2) Place the vectors head to tail.

(3) Draw the resultant vector $\vec{\mathrm{R}}$.

(4) Use a ruler and protractor to measure the magnitude and direction of $\vec{\mathrm{R}}$.

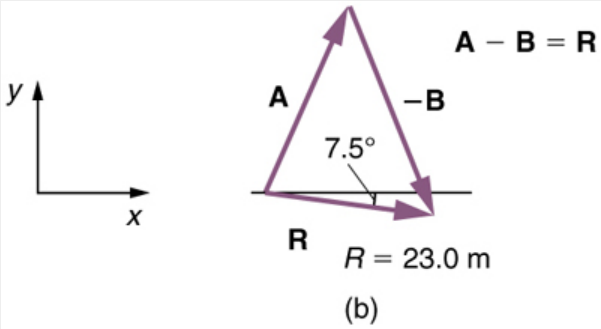


Figure 16.

In this case, R=23.0 m and θ=7.5° south of east.

(5) To determine the location of the dock, we repeat this method to add vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$. We obtain the resultant vector $\vec{\mathrm{R}}'$:

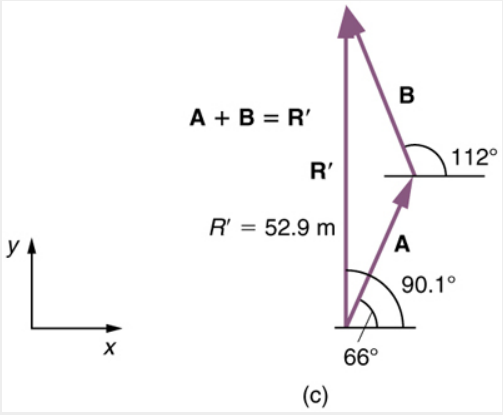


Figure 17.

In this case R=52.9 m and θ=90.1° north of east.

We can see that the woman will end up a significant distance from the dock if she travels in the opposite direction for the second leg of the trip.

Discussion

Because subtraction of a vector is the same as addition of a vector with the opposite direction, the graphical method of subtracting vectors works the same as for addition.

Multiplication of Vectors and Scalars

If we decided to walk three times as far on the first leg of the trip considered in the preceding example, then we would walk 3 × 27.5 m, or 82.5 m, in a direction 66.0° north of east. This is an example of multiplying a vector by a positive **scalar**. Notice that the magnitude changes, but the direction stays the same.

If the scalar is negative, then multiplying a vector by it changes the vector's magnitude and gives the new vector the *opposite* direction. For example, if you multiply by –2, the magnitude doubles but the direction changes. We can summarize these rules in the following way: When vector $\vec{A}$ is multiplied by a scalar c,

- the magnitude of the vector becomes the absolute value of $c\vec{A}$,
- if c is positive, the direction of the vector does not change,
- if c is negative, the direction is reversed.

In our case, $c = 3$ and $A = 27.5$ m. Vectors are multiplied by scalars in many situations. Note that division is the inverse of multiplication. For example, dividing by 2 is the same as multiplying by the value (1/2). The rules for multiplication of vectors by scalars are the same for division; simply treat the divisor as a scalar between 0 and 1.

Resolving a Vector into Components

In the examples above, we have been adding vectors to determine the resultant vector. In many cases, however, we will need to do the opposite. We will need to take a single vector and find what other vectors added together produce it. In most cases, this involves determining the perpendicular **components** of a single vector, for example the *x– and y*-components, or the north-south and east-west components.

For example, we may know that the total displacement of a person walking in a city is 10.3 blocks in a direction 29.0° north of east and want to find out how many blocks east and north had to be walked. This method is called *finding the components (or parts)* of the displacement in the east and north directions, and it is the inverse of the process followed to find the total displacement. It is one example of finding the components of a vector. There are many applications in physics where this is a useful thing to do. We will see this soon in Chapter 3.4 Projectile Motion, and much more when we cover **forces** in Chapter 4 Dynamics: Newton's Laws of Motion. Most of these involve finding components along perpendicular axes (such as north and east), so that right triangles are involved. The analytical techniques presented in Chapter 3.3 Vector Addition and Subtraction: Analytical Methods are ideal for finding vector components.

PHET EXPLORATIONS: MAZE GAME

Learn about position, velocity, and acceleration in the "Arena of Pain". Use the green arrow to move the ball. Add more walls to the arena to make the game more difficult. Try to make a goal as fast as you can.

PhET Interactive Simulation

Figure 18. *Maze Game*

Summary

- The **graphical method of adding vectors** $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ involves drawing vectors on a graph and adding them using the head-to-tail method. The resultant vector $\vec{\mathrm{R}}$ is defined such that $\vec{\mathrm{A}} + \vec{\mathrm{B}} = \vec{\mathrm{R}}$. The magnitude and direction of $\vec{\mathrm{R}}$ are then determined with a ruler and protractor, respectively.
- The **graphical method of subtracting vector** $\vec{\mathrm{B}}$ from $\vec{\mathrm{A}}$ involves adding the opposite of vector $\vec{\mathrm{B}}$, which is defined as $-\vec{\mathrm{B}}$. In this case, $\vec{\mathrm{A}} - \vec{\mathrm{B}} = \vec{\mathrm{A}} + (-\vec{\mathrm{B}}) = \vec{\mathrm{R}}$. Then, the head-to-tail method of addition is followed in the usual way to obtain the resultant vector $\vec{\mathrm{R}}$.
- Addition of vectors is **commutative** such that $\vec{\mathrm{A}} + \vec{\mathrm{B}} = \vec{\mathrm{B}} + \vec{\mathrm{A}}$.
- The **head-to-tail method** of adding vectors involves drawing the first vector on a graph and then placing the tail of each subsequent vector at the head of the previous vector. The resultant vector is then drawn from the tail of the first vector to the head of the final vector.
- If a vector $\vec{\mathrm{A}}$ is multiplied by a scalar quantity *c*, the magnitude of the product is given by $c\vec{\mathrm{A}}$. If *c* is positive, the direction of the product points in the same direction as $\vec{\mathrm{A}}$; if *c* is negative, the direction of the product points in the opposite direction as $\vec{\mathrm{A}}$.

Glossary

component (of a 2-d vector)
: a piece of a vector that points in either the vertical or the horizontal direction; every 2-d vector can be expressed as a sum of two vertical and horizontal vector components

commutative
: refers to the interchangeability of order in a function; vector addition is commutative because the order in which vectors are added together does not affect the final sum

direction (of a vector)
: the orientation of a vector in space

head (of a vector)
: the end point of a vector; the location of the tip of the vector’s arrowhead; also referred to as the

"tip"

head-to-tail method
a method of adding vectors in which the tail of each vector is placed at the head of the previous vector

magnitude (of a vector)
the length or size of a vector; magnitude is a scalar quantity

resultant
the sum of two or more vectors

resultant vector
the vector sum of two or more vectors

scalar
a quantity with magnitude but no direction

tail
the start point of a vector; opposite to the head or tip of the arrow

Conceptual Questions

1: Which of the following is a vector: a person's height, the altitude on Mt. Everest, the age of the Earth, the boiling point of water, the cost of this book, the Earth's population, the acceleration of gravity?

2: Give a specific example of a vector, stating its magnitude, units, and direction.

3: What do vectors and scalars have in common? How do they differ?

4: Two campers in a national park hike from their cabin to the same spot on a lake, each taking a different path, as illustrated below. The total distance traveled along Path 1 is 7.5 km, and that along Path 2 is 8.2 km. What is the final displacement of each camper?

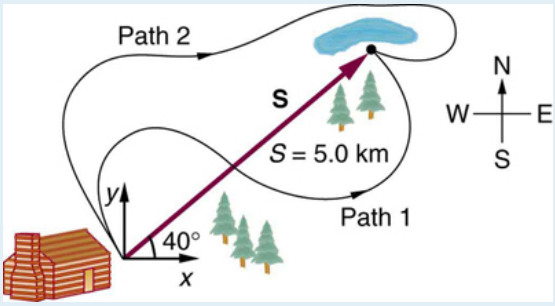


Figure 19.

5: If an airplane pilot is told to fly 123 km in a straight line to get from San Francisco to Sacramento, explain why he could end up anywhere on the circle shown in Figure 20. What other information would he need to get to Sacramento?

Figure 20.

6: Suppose you take two steps A and B (that is, two nonzero displacements). Under what circumstances can you end up at your starting point? More generally, under what circumstances can two nonzero vectors add to give zero? Is the maximum distance you can end up from the starting point A+B the sum of the lengths of the two steps?

7: Explain why it is not possible to add a scalar to a vector.

8: If you take two steps of different sizes, can you end up at your starting point? More generally, can two vectors with different magnitudes ever add to zero? Can three or more?

Problems & Exercises

Use graphical methods to solve these problems. You may assume data taken from graphs is accurate to three digits.

1: Find the following for path A in Figure 21: (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

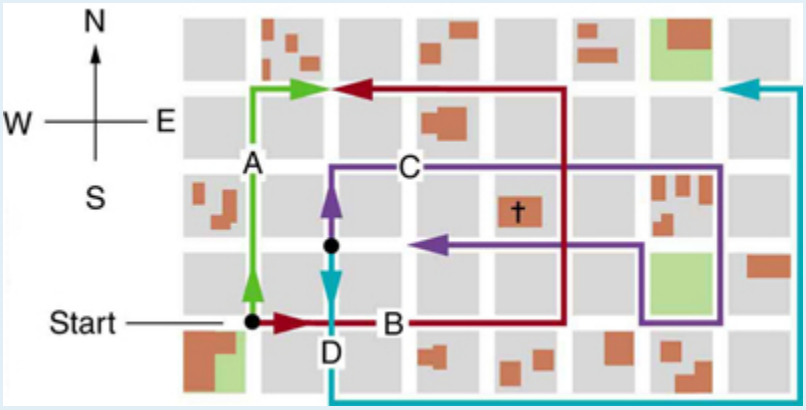


Figure 21. *The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.*

2: Find the following for path B in Figure 21: (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

3: Find the north and east components of the displacement for the hikers shown in Figure 19.

4: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $\overrightarrow{\mathrm{A}}$ and $\vec{\mathrm{B}}$, as in Figure 22, then this problem asks you to find their sum $\vec{\mathrm{R}} = \vec{\mathrm{A}} + \vec{\mathrm{B}}$.)

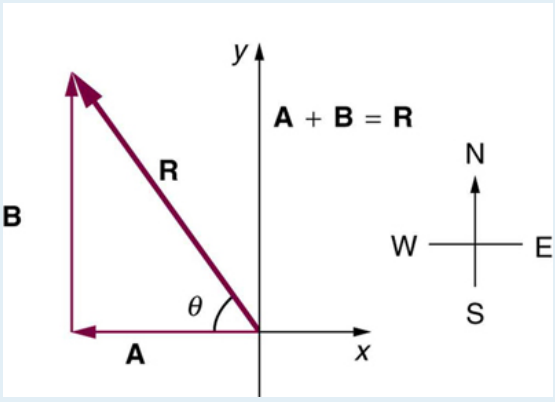


Figure 22. *The two displacements A and B add to give a total displacement R having magnitude R and direction θ.*

5: Suppose you first walk 12.0 m in a direction 20° west of north and then 20.0 m in a direction 40.0° south of west. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$, as in Figure 23, then this problem finds their sum $\vec{\mathrm{R}} = \vec{\mathrm{A}} + \vec{\mathrm{B}}$.)

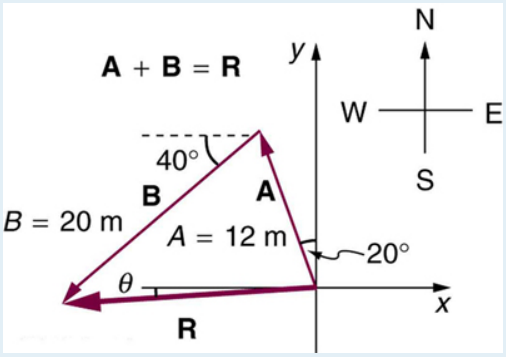


Figure 23.

6: Repeat the problem above, but reverse the order of the two legs of the walk; show that you get the same final result. That is, you first walk leg $\vec{\mathrm{B}}$, which is 20.0 m in a direction exactly 40° south of west, and then leg $\vec{\mathrm{A}}$, which is 12.0 m in a direction exactly 20° west of north. (This problem shows that $\vec{\mathrm{A}} + \vec{\mathrm{B}} = \vec{\mathrm{B}} + \vec{\mathrm{A}}$.)

7: (a) Repeat the problem two problems prior, but for the second leg you walk 20.0 m in a direction 40.0° north of east (which is equivalent to subtracting $\vec{\mathrm{B}}$ from $\vec{\mathrm{A}}$ —that is, to finding $\vec{\mathrm{R}}' = \vec{\mathrm{A}} - \vec{\mathrm{B}}$). (b) Repeat the problem two problems prior, but now you first walk 20.0 m in a direction 40.0° south of west and then 12.0 m in a direction 20.0° east of south (which is equivalent to subtracting $\vec{\mathrm{A}}$ from $\vec{\mathrm{B}}$—that is, to finding $\vec{\mathrm{R}}'' = \vec{\mathrm{B}} - \vec{\mathrm{A}} = -\vec{\mathrm{R}}'$). Show that this is the case.

8: Show that the *order* of addition of three vectors does not affect their sum. Show this property by choosing any three vectors $\vec{\mathrm{A}}$, $\vec{\mathrm{B}}$, and $\vec{\mathrm{C}}$, all having different lengths and directions. Find the sum $\vec{\mathrm{A}} + \vec{\mathrm{B}} + \vec{\mathrm{C}}$ then find their sum when added in a different order and show the result is the same. (There are five other orders in which $\vec{\mathrm{A}}$, $\vec{\mathrm{B}}$, and $\vec{\mathrm{C}}$ can be added; choose only one.)

9: Show that the sum of the vectors discussed in Example 2 gives the result shown in Figure 17.

10: Find the magnitudes of velocities $\boldsymbol{v_A}$ and $\boldsymbol{v_B}$ in Figure 24

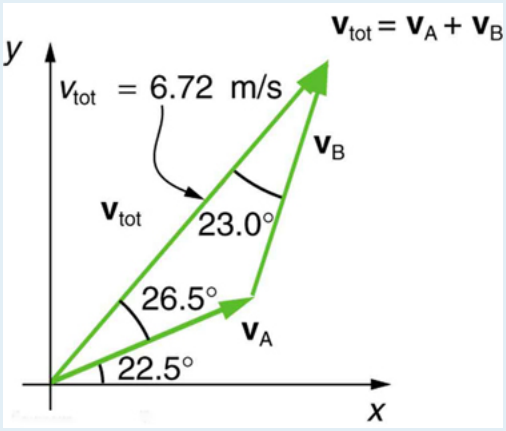


Figure 24. *The two velocities* $\boldsymbol{v_A}$ *and* $\boldsymbol{v_B}$ *add to give a total* $\boldsymbol{v_{tot}}$.

11: Find the components of $\boldsymbol{v}_{\mathbf{tot}}$ along the *x*– and *y*-axes in Figure 24.

12: Find the components of $\boldsymbol{v}_{\mathbf{tot}}$ along a set of perpendicular axes rotated 30° counterclockwise relative to those in Figure 24.

Solutions

Problems & Exercises

1: (a) 480 m (b) 379 m, 18.4^{o} east of north

3: north component 3.21 km, east component 3.83 k,

5: 19.5 m, 4.65^{o} south of west

7: (a) 26.6 m, 65.1^{o} north of east (b) 26.6 m, 65.1^{o} south of west

9: 52.9 m, 90.1^{o} with respect to the *x*-axis.

11: *x*-component 4.41 m/s, *y*-component 5.07 m/s

3.3 Vector Addition and Subtraction: Analytical Methods

Summary

- Understand the rules of vector addition and subtraction using analytical methods.
- Apply analytical methods to determine vertical and horizontal component vectors.
- Apply analytical methods to determine the magnitude and direction of a resultant vector.

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like $\vec{\mathrm{A}}$ in Figure 1, we may wish to find which two perpendicular vectors, A_x and A_y, add to produce it. Notice these perpendicular component vectors will not have vector arrows in this text.

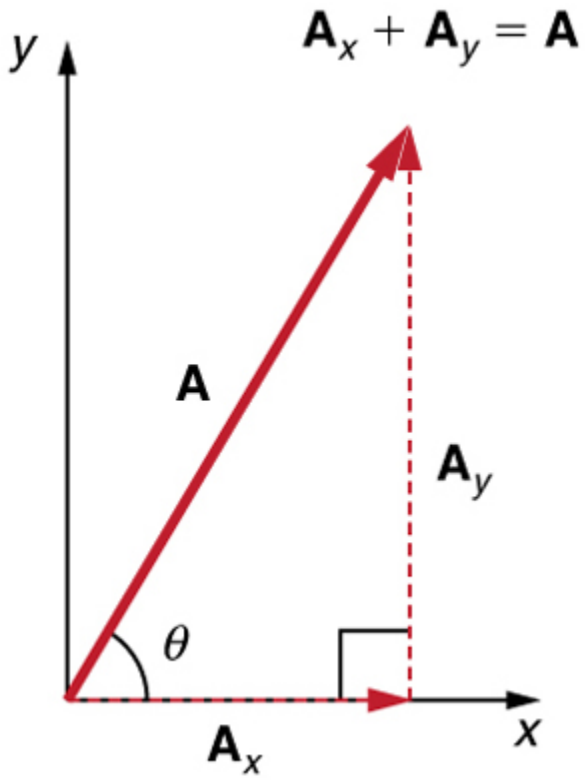


Figure 1. *The vector A, with its tail at the origin of an x, y-coordinate system, is shown together with its x- and y-components, A_x and A_y. These vectors form a right triangle. The analytical relationships among these vectors are summarized below.*

A_x and A_y are defined to be the components of $\vec{\mathrm{A}}$ along the x- and y-axes. The three vectors $\vec{\mathrm{A}}$, A_x, and A_y form a right triangle:

$$\mathrm{A}_x + \mathrm{A}_y = \vec{\mathrm{A}}.$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $\mathrm{A}_x = 3\ \mathrm{m}$ east, $\mathrm{A}_y = 4\ \mathrm{m}$ north, and $\vec{\mathrm{A}} = 5\ \mathrm{m}$ north-east, then it is true that the vectors $\mathrm{A}_x + \mathrm{A}_y = \vec{\mathrm{A}}$. However, it is *not* true that the sum of the magnitudes of the vectors is also equal. That is,

$$3\ \mathrm{m} + 4\ \mathrm{m} \neq 5\ \mathrm{m}$$

Thus,

$$\mathrm{A}_x + \mathrm{A}_y \neq \vec{\mathrm{A}}$$

If the vector $\vec{\mathrm{A}}$ is known, then its magnitude *A* and its angle *θ* (its direction) are known. To find A_x and A_y, its x- and y-components, we use the following relationships for a right triangle.

$$\mathrm{A}_x = A\cos\theta$$

and

$$\mathrm{A}_y = A \sin\theta.$$

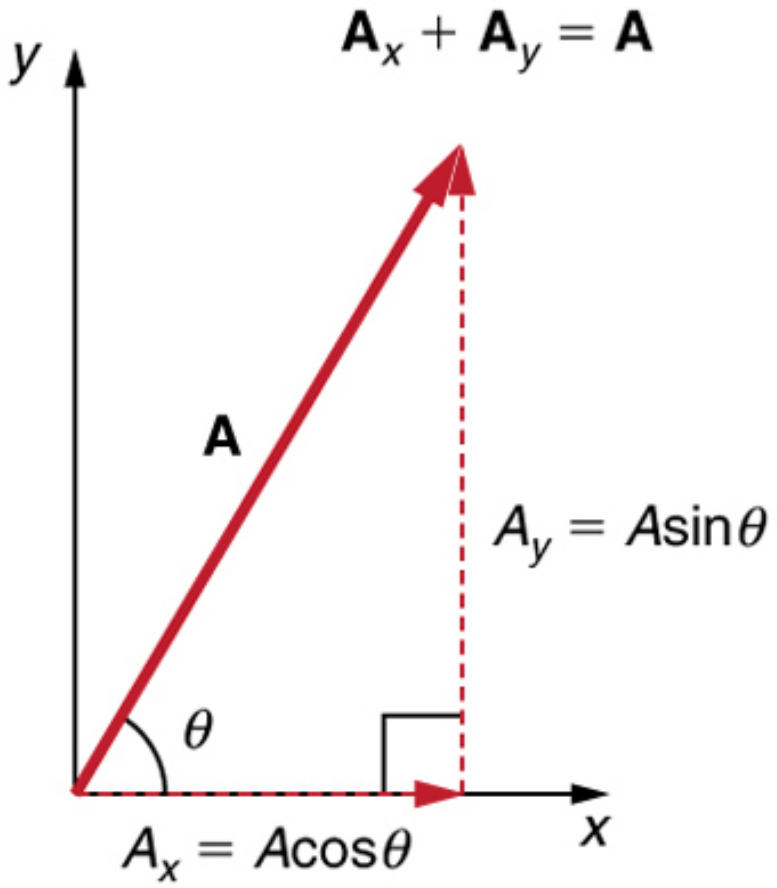


Figure 2. *The magnitudes of the vector components A_x and A_y can be related to the resultant vector A and the angle θ with trigonometric identities. Here we see that $A_x = A \cos\theta$ and $A_y = A \sin\theta$.*

Suppose, for example, that $\vec{\mathrm{A}}$ is the vector representing the total displacement of the person walking in a city considered in Chapter 3.1 Kinematics in Two Dimensions: An Introduction and Chapter 3.2 Vector Addition and Subtraction: Graphical Methods.

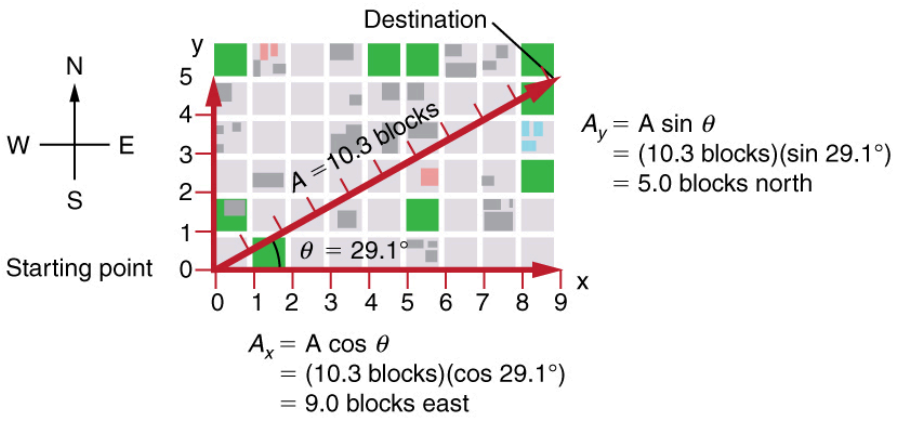


Figure 3. *We can use the relationships $A_x = A \cos \theta$ and $A_y = A \sin\theta$ to determine the magnitude of the horizontal and vertical component vectors in this example.*

Then A = 10.3 blocks and θ = 29.1°, so that

$$\mathrm{A}_x = \mathrm{A}\ \cos\theta = (10.3\ \text{blocks})(\cos\ 29.1^o) = 9.0\ \text{blocks}$$
$$\mathrm{A}_y = \mathrm{A}\ \sin\theta = (10.3\ \text{blocks})(\sin\ 29.1^o) = 5.0\ \text{blocks}.$$

Calculating a Resultant Vector

If the perpendicular components A_x and A_y of a vector $\vec{\mathrm{A}}$ are known, then $\vec{\mathrm{A}}$ can also be found analytically. To find the magnitude A and direction θ of a vector from its perpendicular components A_x and A_y, we use the following relationships:

$$A = \sqrt{{\mathrm{A}_x}^2 + {\mathrm{A}_y}^2}$$
$$\theta = \tan^{-1}(\mathrm{A}_y/\mathrm{A}_x).$$

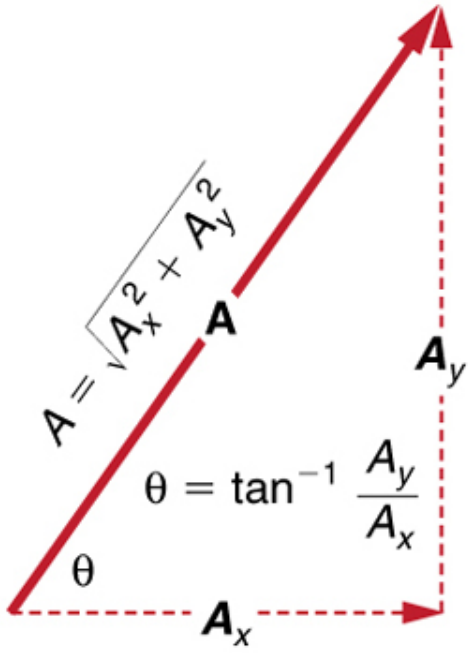


Figure 4. *The magnitude and direction of the resultant vector can be determined once the horizontal and vertical components A_x and A_y have been determined.*

Note that the equation $A = \sqrt{{A_x}^2 + {A_y}^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if A_x and A_y are 9 and 5 blocks, respectively, then $A = \sqrt{9^2 + 5^2} = 10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = \tan^{-1}(5/9) = 29.1^0$, as before.

DETERMINING VECTORS AND VECTOR COMPONENTS WITH ANALYTICAL METHODS

Equations $A_x = A\cos\theta$ and $A_y = A\sin\theta$ are used to find the perpendicular components of a vector—that is, to go from A and θ to A_x and A_y. Equations $A = \sqrt{{A_x}^2 + {A_y}^2}$ and \theta = \tan^{-1} \left(\text{A}_y / \text{A}_x \right) are used to find a vector from its perpendicular components—that is, to go from A_x and A_y to A and θ. Both processes are crucial to analytical methods of vector addition and subtraction.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider Figure 5, in which the vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ are added to produce the resultant $\vec{\mathrm{R}}$.

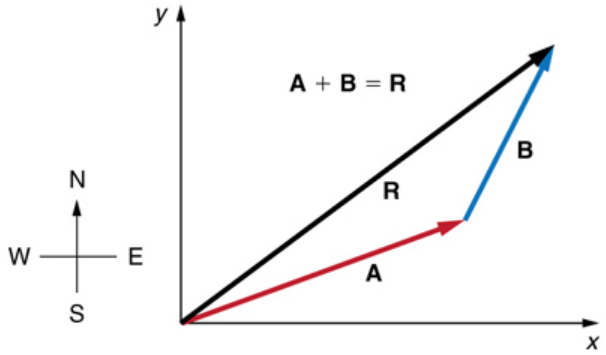


Figure 5. *Vectors A and B are two legs of a walk, and R is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of R.*

If $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ represent two legs of a walk (two displacements), then $\vec{\mathrm{R}}$ is the total displacement. The person taking the walk ends up at the tip of $\vec{\mathrm{R}}$. There are many ways to arrive at the same point. In particular, the person could have walked first in the x-direction and then in the y-direction. Those paths are the x- and y-components of the resultant, R_x and R_y. If we know R_x and R_y, we can find R and θ using the equations $\mathrm{A} = \sqrt{{\mathrm{A}_x}^2 + {\mathrm{A}_y}^2}$ and $\theta = \tan^{-1}(\mathrm{A}_y/\mathrm{A}_x)$. When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

Step 1. *Identify the x- and y-axes that will be used in the problem. Then, find the components of each vector to be added along the chosen perpendicular axes.* Use the equations $\mathrm{A}_x = A\cos\theta$ and $\mathrm{A}_y = A\sin\theta$ to find the components. In Figure 6, these components are A_x, A_y, B_x, and B_y. The angles that vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ make with the *x*-axis are θ_A and θ_B, respectively.

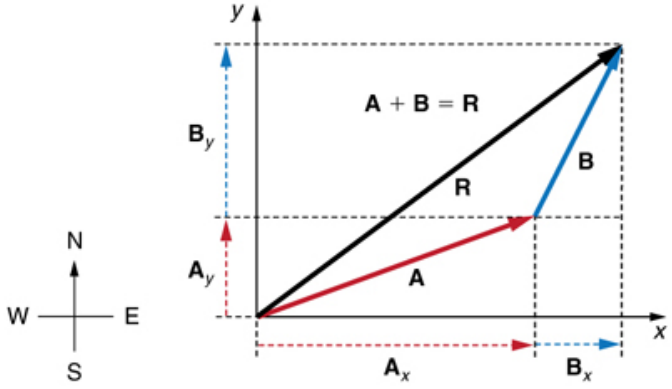


Figure 6. *To add vectors A and B, first determine the horizontal and vertical components of each vector. These are the dotted vectors A_x, A_y, B_x and B_y shown in the image.*

Step 2. *Find the components of the resultant along each axis by adding the components of the individual vectors along that axis.* That is, as shown in Figure 7,

$$\mathrm{R}_x = \mathrm{A}_x + \mathrm{B}_x$$

and

$$\mathrm{R}_y = \mathrm{A}_y + \mathrm{B}_y.$$

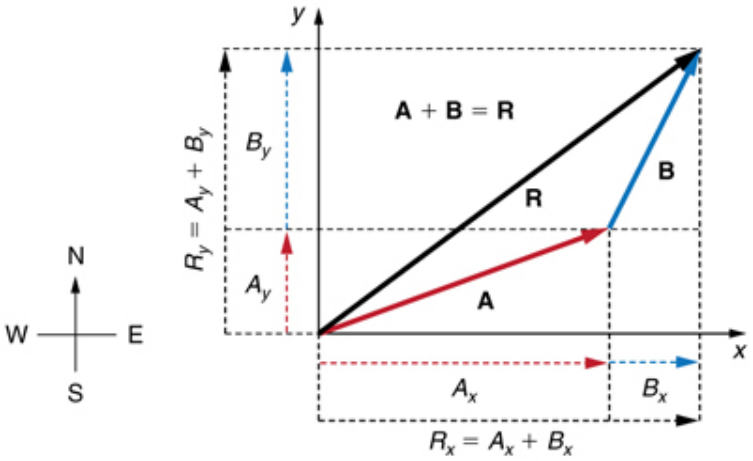


Figure 7. *The magnitude of the vectors A_x and B_x add to give the magnitude R_x of the resultant vector in the horizontal direction. Similarly, the magnitudes of the vectors A_y and B_y add to give the magnitude R_y of the resultant vector in the vertical direction.*

Components along the same axis, say the *x*-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the *y*-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of $\vec{\mathrm{R}}$ are known, its magnitude and direction can be found.

Step 3. *To get the magnitude R of the resultant, use the Pythagorean theorem:*

$$\mathrm{R} = \sqrt{{\mathrm{R}_x}^2 + {\mathrm{R}_y}^2}.$$

Step 4. *To get the direction of the resultant:*

$$\theta = \tan^{-1}(\mathrm{R}_y/\mathrm{R}_x).$$

The following example illustrates this technique for adding vectors using perpendicular components.

Example 1: Adding Vectors Using Analytical Methods

Add the vector $\vec{\mathrm{A}}$ to the vector $\vec{\mathrm{B}}$ shown in Figure 8, using perpendicular components along the *x*– and *y*-axes. The *x*– and *y*-axes are along the east–west and north–south directions, respectively. Vector $\vec{\mathrm{A}}$ represents the first leg of a walk in which a person walks 53.0 m in a direction 20.0° north of east. Vector $\vec{\mathrm{B}}$ represents the second leg, a displacement of 34.0 m in a direction 63.0° north of east.

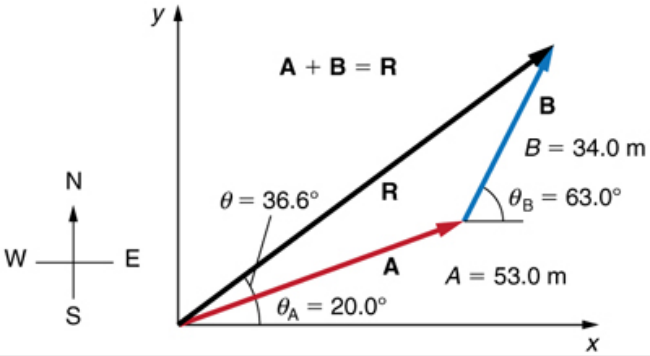


Figure 8. *Vector A has magnitude 53.0 m and direction 20.0^0 north of the x-axis. Vector B has magnitude 34.0 m and direction 63.0^0 north of the x-axis. You can use analytical methods to determine the magnitude and direction of R.*

Strategy

The components of $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ along the *x*– and *y*-axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

Solution

Following the method outlined above, we first find the components of $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$along the *x*– and *y*-axes. Note that A = 53.0 m, θ_A = 20.0°, B = 34.0 m, and θ_B = 63.0°. We find the *x*-components by using $\mathrm{A}_x = A\cos\theta$, which gives

$$\begin{aligned}\mathrm{A}_x &= A\cos\theta_A = (53.0\text{ m})(\cos 20.0^o)\\ &= (53.0\text{ m})(0.940) = 49.8\text{ m}\end{aligned}$$

and

$$\begin{aligned}\mathrm{B}_x &= B\cos\theta_B = (34.0\text{ m})(\cos 63.0^o)\\ &= (34.0\text{ m})(0.454) = 15.4\text{ m}\end{aligned}$$

Similarly, the *y*-components are found using $\mathrm{A}_y = A\sin\theta$:

$$\begin{aligned}\mathrm{A}_y &= A\sin\theta_A = (53.0\text{ m})(\sin 20.0^o)\\ &= (53.0\text{ m})(0.342) = 18.1\text{ m}\end{aligned}$$

and

$$\begin{aligned} \mathrm{B}_y &= B\cos\sin_B = (34.0\ \mathrm{m})(\sin 63.0^o) \\ &= (34.0\ \mathrm{m})(0.891) = 30.3\ \mathrm{m} \end{aligned}$$

The *x*– and *y*-components of the resultant are thus

$$\mathrm{R}_x = \mathrm{A}_x + \mathrm{B}_x = 49.8\ \mathrm{m} + 15.4\ \mathrm{m} = 65.2\ \mathrm{m}$$

and

$$\mathrm{R}_y = \mathrm{A}_y + \mathrm{B}_y = 18.1\ \mathrm{m} + 30.3\ \mathrm{m} = 48.4\ \mathrm{m}.$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

$$\mathrm{R} = \sqrt{\mathrm{R}_x^2 + \mathrm{R}_y^2} = \sqrt{(65.2)^2 + (48.4)^2\ \mathrm{m}}$$

so that

$$R = 81.2\ \mathrm{m}.$$

Finally, we find the direction of the resultant:

$$\theta = \tan^{-1}(\mathrm{R}_y/\mathrm{R}_x) = +\tan^{-1}(48.4/65.2).$$

Thus,

$$\theta = \tan^{-1}(0.742) = 36.6^o.$$

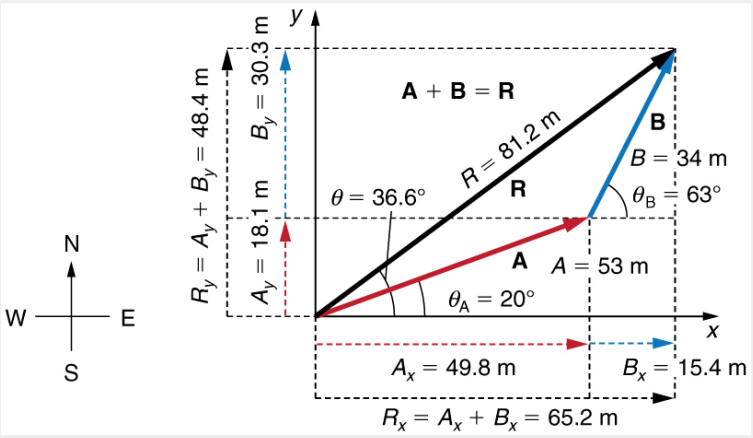


Figure 9. *Using analytical methods, we see that the magnitude of R is 81.2 m and its direction is 36.6^0 north of east.*

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector. That is, $\vec{\mathrm{A}} - \vec{\mathrm{B}} \equiv \vec{\mathrm{A}} + (-\vec{\mathrm{B}})$. Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition*. The components of $-\vec{\mathrm{B}}$ are the negatives of the components of $\vec{\mathrm{B}}$. The *x*– and *y*-components of the resultant $\vec{\mathrm{A}} - \vec{\mathrm{B}} = \vec{\mathrm{R}}$ are thus

$$\mathrm{R}_x = \mathrm{A}_x + (-\mathrm{B}_x)$$

and

$$R_y = A_y + (-B_y)$$

and the rest of the method outlined above is identical to that for addition. (See Figure 10.)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, Chapter 3.4 Projectile Motion, is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.

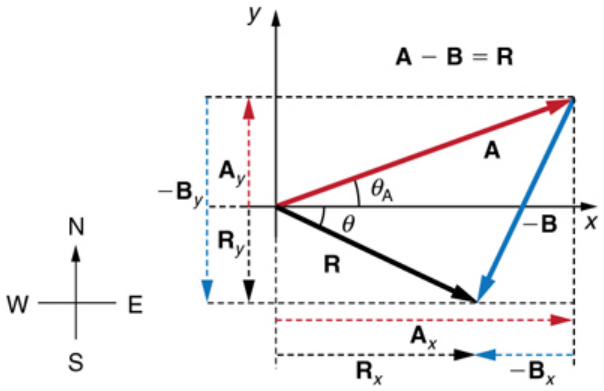


Figure 10. *The subtraction of the two vectors shown in Figure 5. The components of -B are the negatives of the components of B. The method of subtraction is the same as that for addition.*

PHET EXPLORATIONS: VECTOR ADDITION

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.

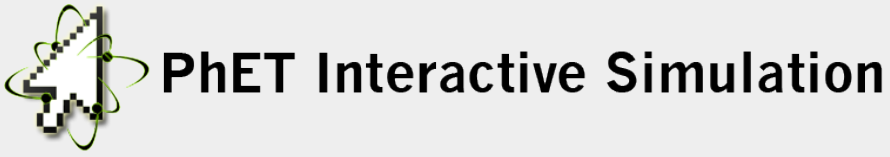


Figure 11. *Vector Addition*

Summary

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ using the analytical method are as follows:

  Step 1: Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations

$$\begin{aligned} \mathrm{A}_x &= A\cos\theta \\ \mathrm{B}_x &= B\cos\theta \end{aligned}$$

  and

$$\begin{aligned} \mathrm{A}_y &= A\sin\theta \\ \mathrm{B}_y &= B\sin\theta. \end{aligned}$$

  Step 2: Add the horizontal and vertical components of each vector to determine the components R_x and R_y of the resultant vector, $\vec{\mathrm{R}}$:

$$\mathrm{R}_x = \mathrm{A}_x + \mathrm{B}_x$$

  and

$$\mathrm{R}_y = \mathrm{A}_y + \mathrm{B}_y.$$

  Step 3: Use the Pythagorean theorem to determine the magnitude, *R*, of the resultant vector $\vec{\mathrm{R}}$:

$$R = \sqrt{{\mathrm{R}_x}^2 + {\mathrm{R}_y}^2}.$$

  Step 4: Use a trigonometric identity to determine the direction, θ, of *R*:

$$\theta = \tan^{-1}(\mathrm{R}_y/\mathrm{R}_x).$$

Glossary

analytical method
the method of determining the magnitude and direction of a resultant vector using the Pythagorean theorem and trigonometric identities

Conceptual Questions

1: Suppose you add two vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$. What relative direction between them produces the resultant with

the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

2: Give an example of a nonzero vector that has a component of zero.

3: Explain why a vector cannot have a component greater than its own magnitude.

4: If the vectors $\vec{A}$ and $\vec{B}$ are perpendicular, what is the component of $\vec{A}$ along the direction of $\vec{B}$? What is the component of $\vec{B}$ along the direction of $\vec{A}$?

Problems & Exercises

1: Find the following for path C in Figure 12: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

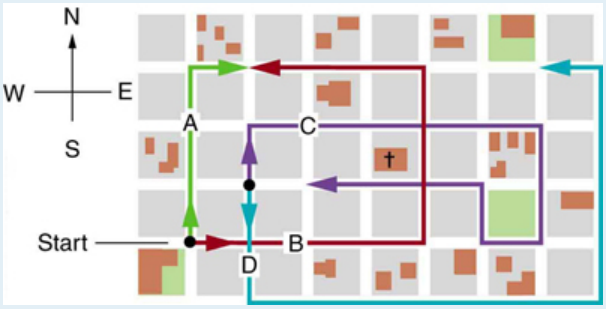


Figure 12. *The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.*

2: Find the following for path D in Figure 12: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

3: Find the north and east components of the displacement from San Francisco to Sacramento shown in Figure 13.

Figure 13.

4: Solve the following problem using analytical techniques: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $vec\mathrm{A}$ and $\vec{\mathrm{B}}$, as in Figure 14, then this problem asks you to find their sum $\vec{\mathrm{R}} = \vec{\mathrm{A}} + \vec{\mathrm{B}}$.)

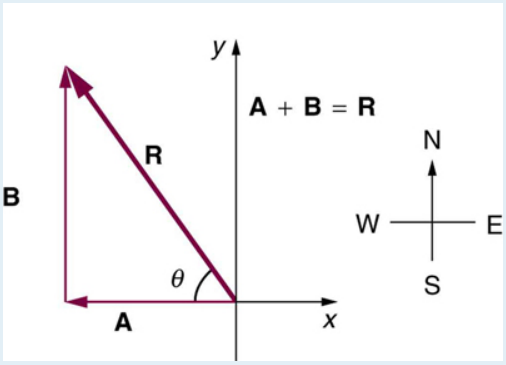


Figure 14. *The two displacements A and B add to give a total displacement R having magnitude R and direction θ.*

Note that you can also solve this graphically. Discuss why the analytical technique for solving this problem is potentially more accurate than the graphical technique.

5: Repeat Exercise 4 using analytical techniques, but reverse the order of the two legs of the walk and show that you get the same final result. (This problem shows that adding them in reverse order gives the same result—that is, $\vec{\mathrm{B}} + \vec{\mathrm{A}} = \vec{\mathrm{A}} + \vec{\mathrm{B}}$.) Discuss how taking another path to reach the same point might help to overcome an obstacle blocking you other path.

6: You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (This determination is equivalent to find the components of the displacement along the east and north directions.) (b) Show that you still arrive at the same point if the east and north legs are reversed in order.

7: Do Exercise 4 again using analytical techniques and change the second leg of the walk to 25.0 m straight south. (This is equivalent to subtracting $\vec{B}$ from $\vec{A}$ —that is, finding $\vec{R}' = \vec{A} - \vec{B}$) (b) Repeat again, but now you first walk 25.0 m north and then 18.0 m east. (This is equivalent to subtract $\vec{A}$ from $\vec{B}$ —that is, to find $\vec{A} = \vec{B} + \vec{C}$. Is that consistent with your result?)

8: A new landowner has a triangular piece of flat land she wishes to fence. Starting at the west corner, she measures the first side to be 80.0 m long and the next to be 105 m. These sides are represented as displacement vectors $\vec{A}$ from $\vec{B}$ in Figure 15. She then correctly calculates the length and orientation of the third side $\vec{C}$. What is her result?

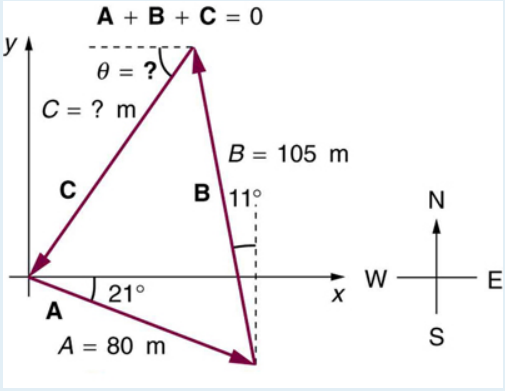


Figure 15.

9: You fly 32.0 km in a straight line in still air in the direction 35.0° south of west. (a) Find the distances you would have to fly straight south and then straight west to arrive at the same point. (This determination is equivalent to finding the components of the displacement along the south and west directions.) (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. These are the components of the displacement along a different set of axes—one rotated 45°.

10: A farmer wants to fence off his four-sided plot of flat land. He measures the first three sides, shown as $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Figure 16, and then correctly calculates the length and orientation of the fourth side $\vec{D}$. What is his result?

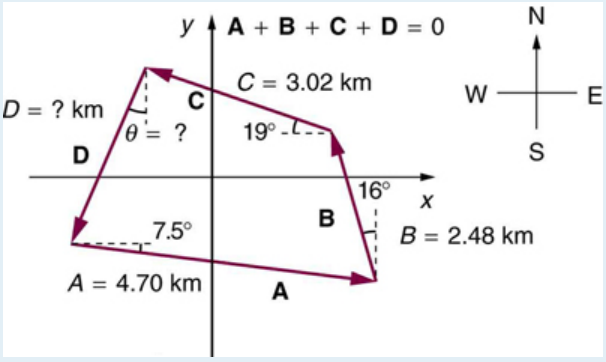


Figure 16.

11: In an attempt to escape his island, Gilligan builds a raft and sets to sea. The wind shifts a great deal during

the day, and he is blown along the following straight lines: 2.50 km 45.0° north of west; then 4.70 km 60.0° south of east; then 1.30 km 25.0° south of west; then 5.10 km straight east; then 1.70 km 5.00° east of north; then 7.20 km 55.0° south of west; and finally 2.80 km 10.0° north of east. What is his final position relative to the island?

12: Suppose a pilot flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east as shown in Figure 17. Find her total distance ***R*** from the starting point and the direction θ of the straight-line path to the final position. Discuss qualitatively how this flight would be altered by a wind from the north and how the effect of the wind would depend on both wind speed and the speed of the plane relative to the air mass.

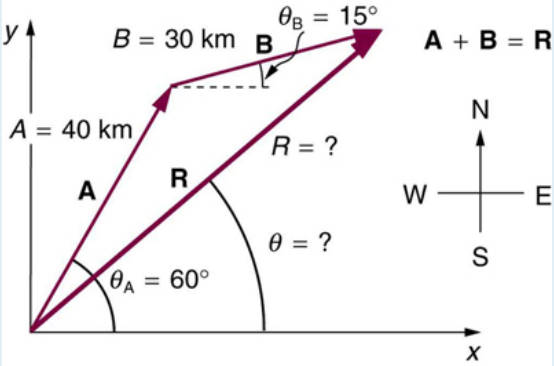


Figure 17.

Solutions

Problems & Exercises

1: (a) 1.56 km (b) 120 m east

3: North-component 87.0 km, east-component 87.0 km

5: 30.8 m, 35.8 degrees west of north

7: (a) 30.8 m, 54.2^{o} south of west (b) 30.8 m, 54.2^{o} north of east

9: (a) 18.4 km south, then 26.2 km west (b) 31.5 km at 45^{o} south of west, then 5.56 km at 45^{o} west of north

11: 7.34 km, 63.5^{o} south of east

3.4 Projectile Motion

Summary

- Identify and explain the properties of a projectile, such as acceleration due to gravity, range, maximum height, and trajectory.
- Determine the location and velocity of a projectile at different points in its trajectory.
- Apply the principle of independence of motion to solve projectile motion problems.

Projectile motion is the **motion** of an object thrown or projected into the air, subject to only the acceleration of gravity. The object is called a **projectile**, and its path is called its **trajectory**. The motion of falling objects, as covered in Chapter 2.6 Problem-Solving Basics for One-Dimensional Kinematics, is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this section, we consider two-dimensional projectile motion, such as that of a football or other object for which **air resistance** *is negligible*.

The most important fact to remember here is that *motions along perpendicular axes are independent* and thus can be analyzed separately. This fact was discussed in Chapter 3.1 Kinematics in Two Dimensions: An Introduction, where vertical and horizontal motions were seen to be independent. The key to analyzing two-dimensional projectile motion is to break it into two motions, one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible, because acceleration due to gravity is vertical—thus, there will be no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the *x*-axis and the vertical axis the *y*-axis. Figure 1 illustrates the notation for displacement, where $\vec{s}$ is defined to be the total displacement and *x* and *y* are its components along the horizontal and vertical axes, respectively. The magnitudes of these vectors are *s*, *x*, and *y*. (Note that in the last section we used the notation $\vec{\mathrm{A}}$ to represent a vector with components $\mathrm{A_x}$ and $\mathrm{A_y}$. If we continued this format, we would call displacement $\vec{\mathrm{s}}$ with components $\mathrm{s_x}$ and $\mathrm{s_y}$. However, to simplify the notation, we will simply represent the component vectors as *x* and *y*.)

Of course, to describe motion we must deal with velocity and acceleration, as well as with displacement. We must find their components along the *x*– and *y*-axes, too. We will assume all forces except gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple: $\mathrm{a}_y = -g = -9.80\ \mathrm{m/s^2}$. (Note that this definition assumes that the upwards direction is defined as the positive direction. If you arrange the coordinate system instead such that the downwards direction is positive, then acceleration due to gravity takes a positive value.) Because gravity is vertical, $\mathrm{a}_x = 0$. Both accelerations are constant, so the kinematic equations can be used.

REVIEW OF KINEMATIC EQUATIONS (CONSTANT α)

$$x = x_0 + \bar{v}t$$

$$\bar{v} = \frac{\vec{v}_0 + \vec{v}}{2}$$

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$x = x_0 + v_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}^2 = {\vec{v}_0}^2 + 2\vec{a}(x - x_0)$$

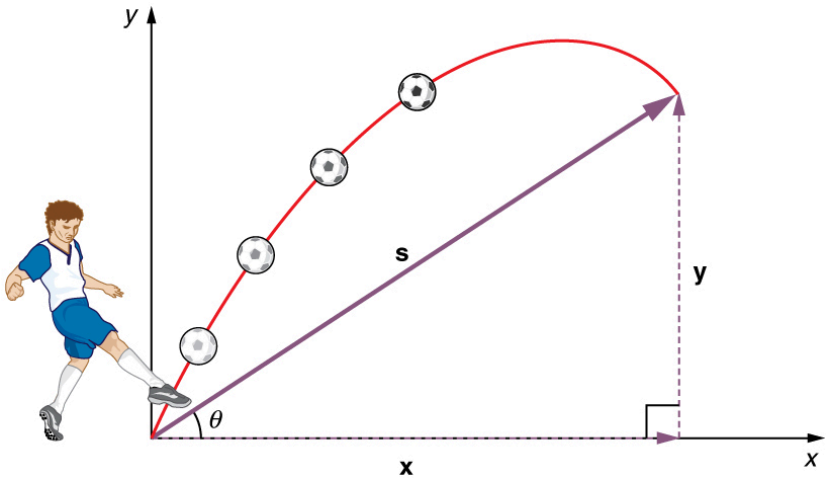


Figure 1. *The total displacement s of a soccer ball at a point along its path. The vector s has components x and y along the horizontal and vertical axes. Its magnitude is s, and it makes an angle θ with the horizontal.*

Given these assumptions, the following steps are then used to analyze projectile motion:

Step 1. *Resolve or break the motion into horizontal and vertical components along the x- and y-axes.* These axes are perpendicular, so $A_x = A\cos\theta$ and $A_y = A\sin\theta$ are used. The magnitude of the components of displacement $\vec{\mathbf{s}}$ along these axes are *x* and *y*. The magnitudes of the components of the velocity $\vec{v}$ are $\vec{v}_x = \vec{v}\cos\ \theta$ and $\vec{v}_y = \vec{v}\sin\ \theta$, where *v* is the magnitude of the velocity and $\boldsymbol{\theta}$ is its direction, as shown in Figure 2. Initial values are denoted with a subscript 0, as usual.

Step 2. *Treat the motion as two independent one-dimensional motions, one horizontal and the other vertical.* The kinematic equations for horizontal and vertical motion take the following forms:

HORIZONTAL MOTION (CONSTANT $a_x = 0$)

$$x = x_0 + v_0 t$$

$$v_x = v_{0x} \text{ (Velocity is constant)}$$

VERTICAL MOTION (POSITIVE IS UP, $a_y = -g = -9.80\ \text{m/s}^2$)

$$y = y_0 + \frac{1}{2}(v_0 + v_y)t$$

$$v_y = v_{0y} - gt$$

$$y = y_0 + v_{0y}t + \frac{1}{2}gt^2$$

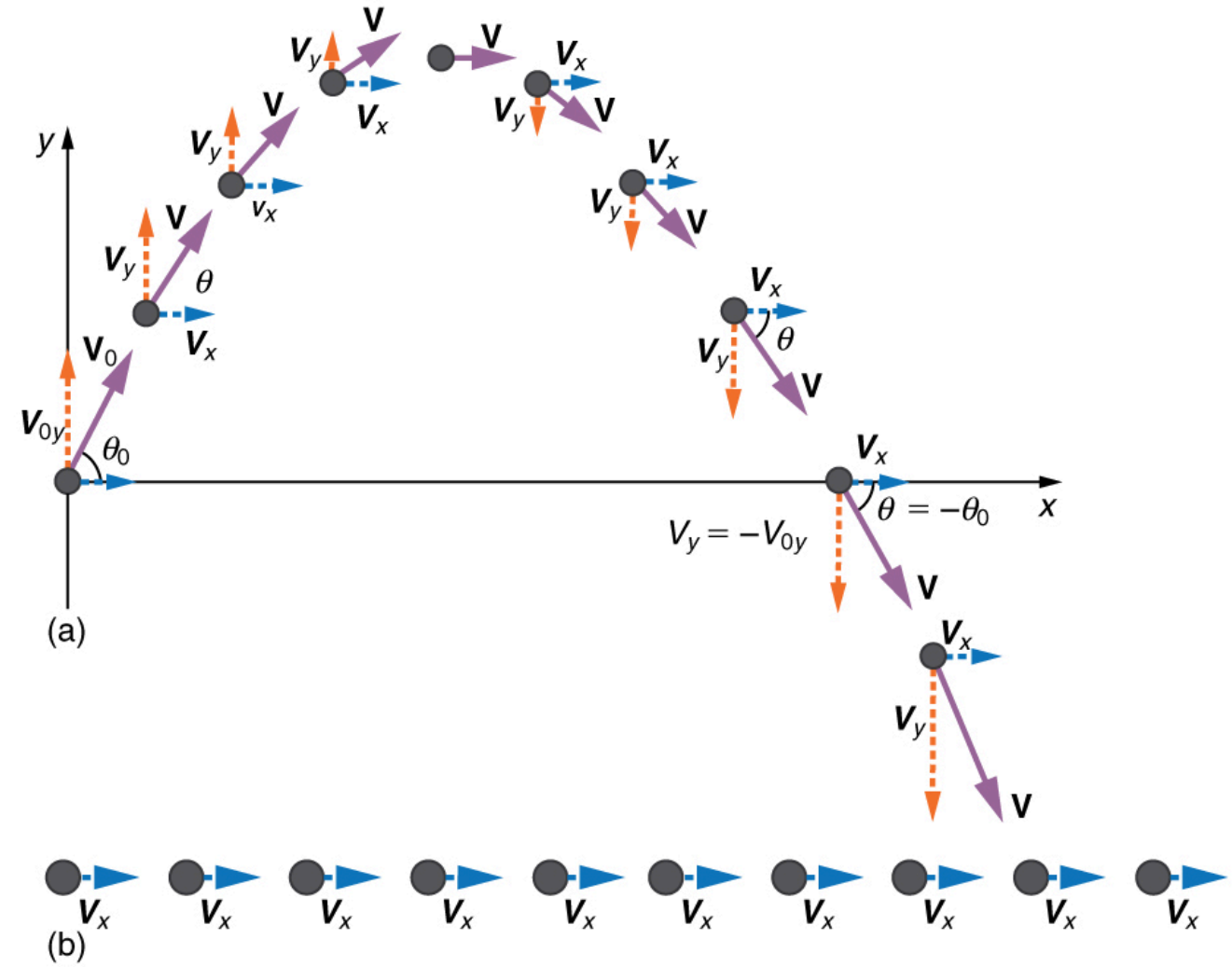
$$v^2 = {v_{0y}}^2 + 2g(y - y_0)$$

Step 3. *Solve for the unknowns in the two separate motions—one horizontal and one vertical.* Note that the only common variable between the motions is time *t*. The problem solving procedures here are the same as for one-dimensional **kinematics** and are illustrated in the solved examples below.

Step 4. *Recombine the two motions to find the total displacement* $\vec{\text{s}}$ *and velocity* $\vec{\text{v}}$. Because the *x* – and *y* -motions are perpendicular, we determine these vectors by using the techniques outlined in the Chapter 3.3 Vector Addition and Subtraction: Analytical Methods and employing $A = \sqrt{\text{A}_x^2 + \text{A}_y^2}$ and $\theta = \tan^{-1}(\text{A}_y / \text{A}_x)$ in the following form, where θ_d is the direction of the displacement $\vec{\text{s}}$ and θ is the direction of the velocity $\vec{\text{v}}$:

$$s = \sqrt{\text{x}^2 + \text{y}^2}$$
$$\theta_d = \tan^{-1}(\text{y}/\text{x})$$

$$v = \sqrt{\mathrm{v}_x^2 + \mathrm{v}_y^2}$$
$$\theta = \tan^{-1}(\mathrm{v}_y / \mathrm{v}_x).$$


y
x
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(a)

$\mathbf{V}_x$
(b)

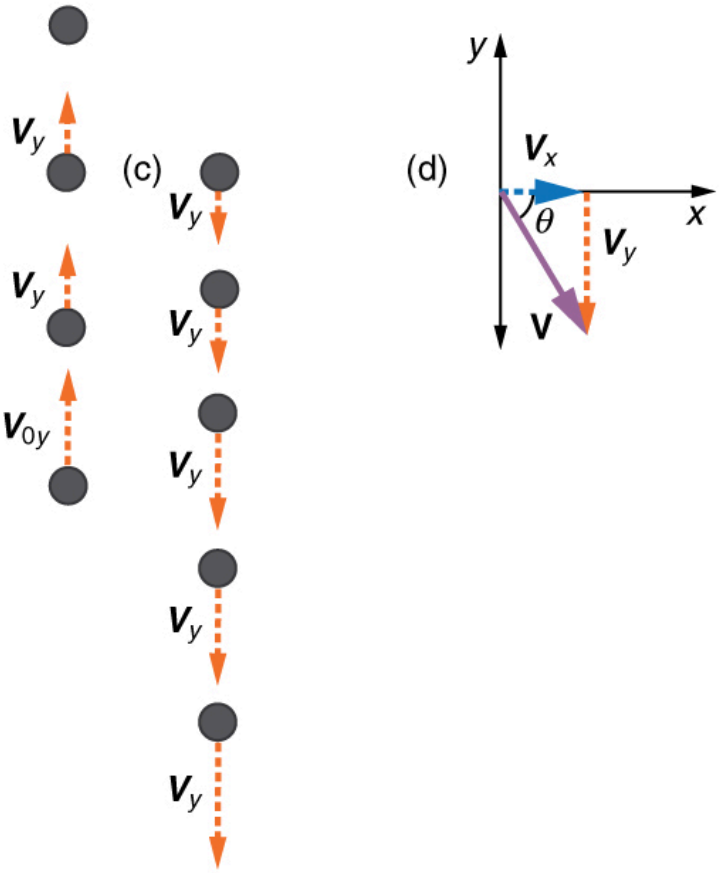

$\mathbf{V}_y$
$\mathbf{V}_{0y}$
(c)
(d)
y
x
θ
$\mathbf{V}$
$\mathbf{V}_x$

Figure 2. *(a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. (b) The horizontal motion is simple, because $a_x = 0$ and v_x is thus constant. (c) The velocity in the vertical direction begins to decrease as the object rises; at its highest point, the vertical velocity is zero. As the object falls towards the Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity. (d) The x – and y -motions are recombined to give the total velocity at any given point on the trajectory.*

Example 1: A Fireworks Projectile Explodes High and Away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of 75.0° above the horizontal, as illustrated in Figure 3. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passed between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes?

Strategy

Because air resistance is negligible for the unexploded shell, the analysis method outlined above can be used. The motion can be broken into horizontal and vertical motions in which $\vec{a}_x = 0$ and $\vec{a}_y = -g$. We can then define x_0 and y_0 to be zero and solve for the desired quantities.

Solution for (a)

By “height” we mean the altitude or vertical position y above the starting point. The highest point in any trajectory, called the apex, is reached when $v_y = 0$. Since we know the initial and final velocities as well as the initial position, we use the following equation to find y:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$

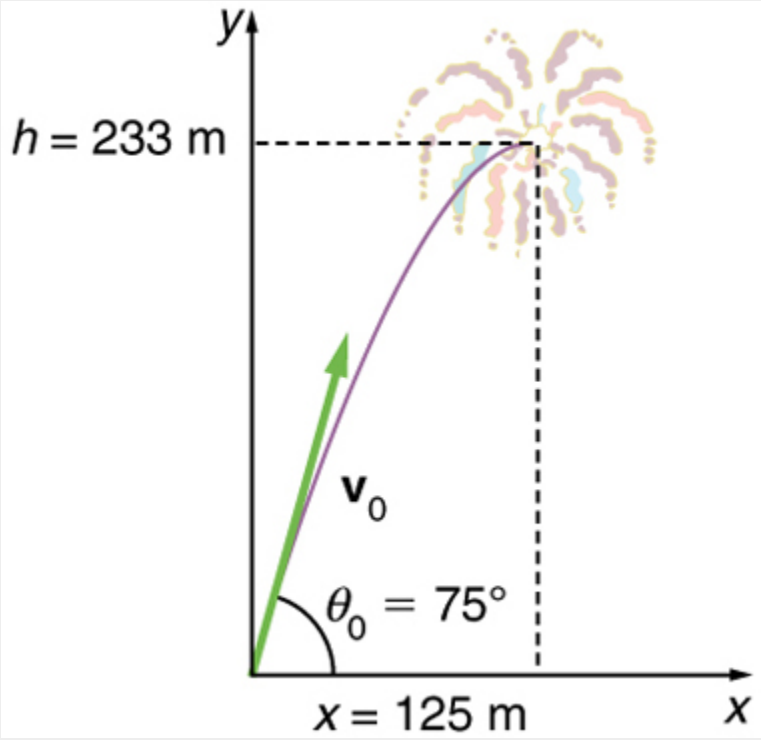


Figure 3. *The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.*

Because y_0 and v_y are both zero, the equation simplifies to

$$0 = v_{0y}^2 - 2gy.$$

Solving for y gives

$$y = \frac{v_{0y}^2}{2g}.$$

Now we must find $\vec{v}_{0y}$, the component of the initial velocity in the y-direction. It is given by $\vec{v}_{0y} = \vec{v}_0 \sin \theta$, where $\vec{v}_0$ is the initial velocity of 70.0 m/s, and $\theta_0 = 75.0°$ is the initial angle. Thus,

$$v_{0y} = v_0 \sin\theta_0 = (70.0 \text{ m/s})(\sin 75^o) = 67.6 \text{ m/s}.$$

and y is

$$y = \frac{(67.6 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)},$$

so that

$$y = 233 \text{ m}.$$

Discussion for (a)

Note that because up is positive, the initial velocity is positive, as is the maximum height, but the acceleration due to gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6 m/s initial vertical component of velocity will reach a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, and so the initial velocity would have to be somewhat larger than that given to reach the same height.

Solution for (b)

As in many physics problems, there is more than one way to solve for the time to the highest point. In this case, the easiest method is to use $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$. Because y_0 is zero, this equation reduces to simply

$$y = \frac{1}{2}(v_{0y} + v_y)t.$$

Note that the final vertical velocity, $\vec{v}_y$, at the highest point is zero. Thus,

$$t = \frac{2y}{(v_{0y} + v_y)} = \frac{2(233\ \text{m})}{(67.6\ \text{m/s})}.$$

$$= 6.90\ \text{s}$$

Discussion for (b)

This time is also reasonable for large fireworks. When you are able to see the launch of fireworks, you will notice several seconds pass before the shell explodes. (Another way of finding the time is by using $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$, and solving the quadratic equation for *t*.)

Solution for (c)

Because air resistance is negligible, $\vec{a}_x = 0$ and the horizontal velocity is constant, as discussed above. The horizontal displacement is horizontal velocity multiplied by time as given by $x = x_0 + v_x t$, where x_0 is equal to zero:

$$x = v_x t,$$

where $\vec{v}_x$ is the *x*-component of the velocity, which is given by $\vec{v}_x = \vec{v}_0 \cos\ \theta_0$. Now,

$$v_x = v_0 \cos\theta_0 = (70.0\ \text{m/s})(\cos 75.0^o) = 18.1\ \text{m/s}.$$

The time *t* for both motions is the same, and so *x* is

$$x = (18.1\ \text{m/s})(6.90\ \text{s}) = 125\ \text{m}.$$

Discussion for (c)

The horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. Once the shell explodes, air resistance has a major effect, and many fragments will land directly below.

In solving part (a) of the preceding example, the expression we found for y is valid for any projectile motion where air resistance is negligible. Call the maximum height $y = h$; then,

$$h = \frac{v_{0y}^2}{2g}.$$

This equation defines the *maximum height of a projectile* and depends only on the vertical component of the initial velocity.

DEFINING A COORDINATE SYSTEM

It is important to set up a coordinate system when analyzing projectile motion. One part of defining the coordinate system is to define an origin for the ***x*** and ***y*** positions. Often, it is convenient to choose the initial position of the object as the origin such that $x_0 = 0$ and $y_0 = 0$. It is also important to define the positive and negative directions in the x and y directions. Typically, we define the positive vertical direction as upwards, and the positive horizontal direction is usually the direction of the object's motion. When this is the case, the vertical acceleration, g, takes a negative value (since it is directed downwards towards the Earth). However, it is occasionally useful to define the coordinates differently. For example, if you are analyzing the motion of a ball thrown downwards from the top of a cliff, it may make sense to define the positive direction downwards since the motion of the ball is solely in the downwards direction. If this is the case, g takes a positive value.

Example 2: Calculating Projectile Motion: Hot Rock Projectile

Kilauea in Hawaii is the world's most continuously active volcano. Very active volcanoes characteristically eject red-hot rocks and lava rather than smoke and ash. Suppose a large rock is ejected from the volcano with a speed of 25.0 m/s and at an angle 35.0° above the horizontal, as shown in Figure 4. The rock strikes the side of the volcano at an altitude 20.0 m lower than its starting point. (a) Calculate the time it takes the rock to follow this path. (b) What are the magnitude and direction of the rock's velocity at impact?

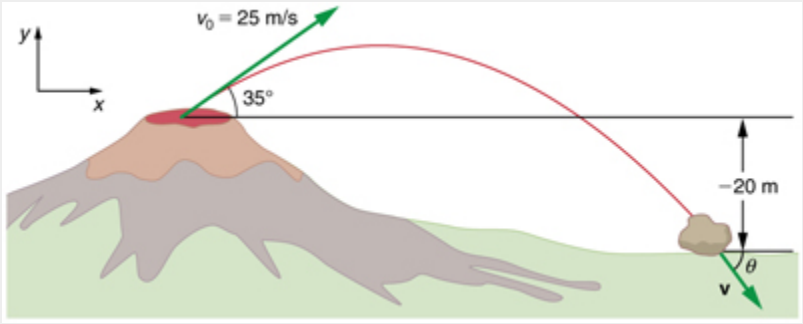


Figure 4. *The trajectory of a rock ejected from the Kilauea volcano.*

Strategy

Again, resolving this two-dimensional motion into two independent one-dimensional motions will allow us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. We will solve for ***t*** first. While the rock is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, the vertical and horizontal results will be recombined to obtain *v* and θ_v at the final time *t* determined in the first part of the example.

Solution for (a)

While the rock is in the air, it rises and then falls to a final position 20.0 m lower than its starting altitude. We can find the time for this by using

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2.$$

If we take the initial position y_0 to be zero, then the final position is *y* = -20.0 m. Now the initial vertical velocity is the vertical component of the initial velocity, found from $\vec{v}_{0y} = \vec{v}_0 \sin\ \theta_0 = (25.0\ \text{m/s})(\sin\ 35.0^o) = 14.3\ \text{m/s}$. Substituting known values yields

$$-20.0\ \text{m} = (14.3\ \text{m/s})t - 4.90\ \ \text{m/s}^2t^2.$$

Rearranging terms gives a quadratic equation in *t*:

$$4.90\ \text{m/s}^2t^2 - 14.3\ \text{m/s}t - 20.0\ \text{m} = 0.$$

This expression is a quadratic equation of the form $at^2 + bt + c = 0$, where the constants are *a* = 4.90, *b* = -14.3, and *c*=-20.0. Its solutions are given by the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation yields two solutions: *t* = 3.96 and *t* = -1.03. (It is left as an exercise for the reader to verify these solutions.) The time is *t* = 3.96 s or -1.03 s. The negative value of time implies an event before the start of motion, and so we discard it. Thus,

$$t = 3.96\ \text{s}.$$

Discussion for (a)

The time for projectile motion is completely determined by the vertical motion. So any projectile that has an initial vertical velocity of 14.3 m/s and lands 20.0 m below its starting altitude will spend 3.96 s in the air.

Solution for (b)

From the information now in hand, we can find the final horizontal and vertical velocities $\vec{v}_x$ and $\vec{v}_y$ and combine them to find the total velocity $\vec{v}$ and the angle θ_0 it makes with the horizontal. Of course, $\vec{v}_x$ is constant so we can solve for it at any horizontal location. In this case, we chose the starting point since we know both the initial velocity and initial angle. Therefore:

$$\vec{v}_x = \vec{v}_0 \cos\theta_0 = (25.0\ \text{m/s})(\cos 35^o) = 20.5\ \text{m/s}.$$

The final vertical velocity is given by the following equation:

$$v_y = v_{0y} - gt,$$

where v_{0y} was found in part (a) to be 14.3 m/s. Thus,

$$\vec{v}_y = 14.3 \text{ m/s} - (9.80 \text{ m/s}^2)(3.96 \text{ s})$$

so that

$$\vec{v}_y = -24.5 \text{ m/s}.$$

To find the magnitude of the final velocity ***v*** we combine its perpendicular components, using the following equation:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20.5 \text{ m/s})^2 + (-24.5 \text{ m/s})^2},$$

which gives

$$v = 31.9 \text{ m/s}.$$

The direction θ_v is found from the equation:

$$\theta_v = \tan^{-1}(\vec{v}_y/\vec{v}_x)$$

so that

$$\theta_v = \tan^{-1}(-24.5/20.5) = \tan^{-1}(-1.19).$$

Thus,

$$\theta_v = -50.1^o.$$

Discussion for (b)

The negative angle means that the velocity is 50.1° below the horizontal. This result is consistent with the fact that the final vertical velocity is negative and hence downward—as you would expect because the final altitude is 20.0 m lower than the initial altitude. (See Figure 4.)

One of the most important things illustrated by projectile motion is that vertical and horizontal motions are independent of each other. Galileo was the first person to fully comprehend this characteristic. He used it to predict the range of a projectile. On level ground, we define **range** to be the horizontal distance *R* traveled by a projectile. Galileo and many others were interested in the range of projectiles primarily for military purposes—such as aiming cannons. However, investigating the range of projectiles can shed light on other interesting phenomena, such as the orbits of satellites around the Earth. Let us consider projectile range further.

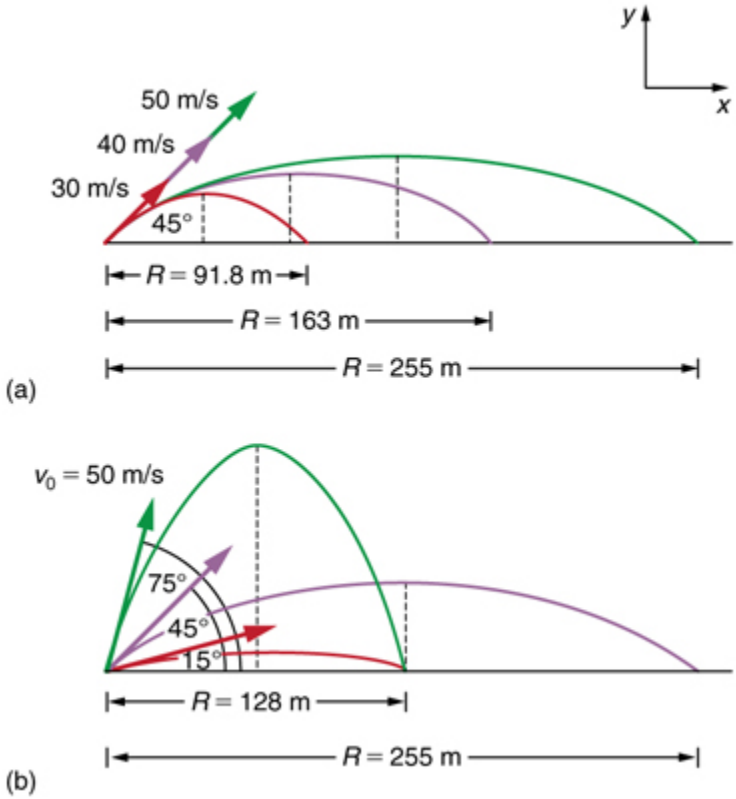


Figure 5. *Trajectories of projectiles on level ground. (a) The greater the initial speed v_0, the greater the range for a given initial angle. (b) The effect of initial angle $\theta_0$on the range of a projectile with a given initial speed. Note that the range is the same for 15° and 75°, although the maximum heights of those paths are different.*

How does the initial velocity of a projectile affect its range? Obviously, the greater the initial speed v_0, the greater the range, as shown in Figure 5(a). The initial angle θ_0 also has a dramatic effect on the range, as illustrated in Figure 5(b). For a fixed initial speed, such as might be produced by a cannon, the maximum range is obtained with θ_0= 45°. This is true only for conditions neglecting air resistance. If air resistance is considered, the maximum angle is approximately 38°. Interestingly, for every initial angle except 45° there are two angles that give the same range—the sum of those angles is 90°. The range also depends on the value of the acceleration of gravity g. The lunar astronaut Alan Shepherd was able to drive a golf ball a great distance on the Moon because gravity is weaker there. The range R of a projectile on *level ground* for which air resistance is negligible is given by

$$R = \frac{v_0^2 \sin 2\theta_0}{g},$$

where $\vec{v}_0$ is the initial speed and θ_0 is the initial angle relative to the horizontal. The proof of this equation is left as an end-of-chapter problem (hints are given), but it does fit the major features of projectile range as described.

When we speak of the range of a projectile on level ground, we assume that R is very small compared with the circumference of the Earth. If, however, the range is large, the Earth curves away below the

projectile and acceleration of gravity changes direction along the path. The range is larger than predicted by the range equation given above because the projectile has farther to fall than it would on level ground. (See Figure 6.) If the initial speed is great enough, the projectile goes into orbit. This possibility was recognized centuries before it could be accomplished. When an object is in orbit, the Earth curves away from underneath the object at the same rate as it falls. The object thus falls continuously but never hits the surface. These and other aspects of orbital motion, such as the rotation of the Earth, will be covered analytically and in greater depth later in this text.

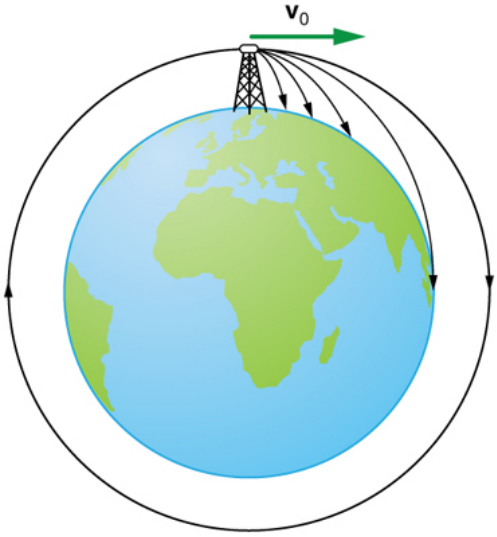


Figure 6. *Projectile to satellite. In each case shown here, a projectile is launched from a very high tower to avoid air resistance. With increasing initial speed, the range increases and becomes longer than it would be on level ground because the Earth curves away underneath its path. With a large enough initial speed, orbit is achieved.*

PHET EXPLORATIONS: PROJECTILE MOTION

Blast a Buick out of a cannon! Learn about projectile motion by firing various objects. Set the angle, initial speed, and mass. Add air resistance. Make a game out of this simulation by trying to hit a target.

PhET Interactive Simulation

Figure 7. *Projectile Motion*

Summary

- Projectile motion is the motion of an object through the air that is subject only to the acceleration of gravity.
- To solve projectile motion problems, perform the following steps:

    1. Determine a coordinate system. Then, resolve the position and/or velocity of the object in the horizontal and vertical components. The components of position $\vec{\mathrm{s}}$ are given by the quantities *x* and *y*, and the components of the velocity $\vec{\mathrm{v}}$ are given by $\vec{v}_x = \vec{v}\cos\ \theta$ and $\vec{v}_y = \vec{v}\sin\ \theta$
    2. , where ***v*** is the magnitude of the velocity and θ is its direction.
    3. Analyze the motion of the projectile in the horizontal direction using the following equations:
    $$(a_x = 0)$$
    $$x = x_0 + v_0 t$$
    $$v_x = v_{0x}\ (\text{Velocity is constant})$$
    4. Analyze the motion of the projectile in the vertical direction using the following equations:
    $$(\text{POSITIVE IS UP, } a_y = -g = -9.80\ \text{m/s}^2)$$
    $$y = y_0 + \frac{1}{2}(v_0 + v_y)t$$
    $$v_y = v_{0y} - gt$$
    $$y = y_0 + v_{0y}t + \frac{1}{2}gt^2$$
    $$v^2 = {v_{0y}}^2 + 2g(y - y_0)$$
    5. Recombine the horizontal and vertical components of location and/or velocity using the following equations:
    $$s = \sqrt{x^2 + y^2}$$
    $$\theta = \tan^{-1}(y/x)$$
    $$v = \sqrt{v_x^2 + v_y^2}$$
    $$\theta_v = \tan^{-1}(v_y/v_x).$$

- The maximum height *h* of a projectile launched with initial vertical velocity $\vec{v}_{0y}$ is given by
$$h = \frac{v_{0y}^2}{2g},$$
- The maximum horizontal distance traveled by a projectile is called the range. The range *R* of a projectile on level ground launched at an angle θ_0 above the horizontal with initial speed v_0 is given by

$$\mathrm{R} = \frac{v_0^2 \sin 2\theta_0}{g}.$$

Glossary

air resistance
: a frictional force that slows the motion of objects as they travel through the air; when solving basic physics problems, air resistance is assumed to be zero

kinematics
: the study of motion without regard to mass or force

motion
: displacement of an object as a function of time

projectile
: an object that travels through the air and experiences only acceleration due to gravity

projectile motion
: the motion of an object that is subject only to the acceleration of gravity

range
: the maximum horizontal distance that a projectile travels

trajectory
: the path of a projectile through the air

Conceptual Questions

1: Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither 0° nor 90°): (a) Is the velocity ever zero? (b) When is the velocity a minimum? A maximum? (c) Can the velocity ever be the same as the initial velocity at a time other than at $t = 0$? (d) Can the speed ever be the same as the initial speed at a time other than at $t = 0$?

2: Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither 0° nor 90°): (a) Is the acceleration ever zero? (b) Is the acceleration ever in the same direction as a component of velocity? (c) Is the acceleration ever opposite in direction to a component of velocity?

3: For a fixed initial speed, the range of a projectile is determined by the angle at which it is fired. For all but the maximum, there are two angles that give the same range. Considering factors that might affect the ability of an archer to hit a target, such as wind, explain why the smaller angle (closer to the horizontal) is preferable. When would it be necessary for the archer to use the larger angle? Why does the punter in a football game use the higher trajectory?

4: During a lecture demonstration, a professor places two coins on the edge of a table. She then flicks one of the coins horizontally off the table, simultaneously nudging the other over the edge. Describe the subsequent motion of the two coins, in particular discussing whether they hit the floor at the same time.

Problems & Exercises

1: A projectile is launched at ground level with an initial speed of 50.0 m/s at an angle of 30.0° above the horizontal. It strikes a target above the ground 3.00 seconds later. What are the *x* and *y* distances from where the projectile was launched to where it lands?

2: A ball is kicked with an initial velocity of 16 m/s in the horizontal direction and 12 m/s in the vertical direction. (a) At what speed does the ball hit the ground? (b) For how long does the ball remain in the air? (c)What maximum height is attained by the ball?

3: A ball is thrown horizontally from the top of a 60.0-m building and lands 100.0 m from the base of the building. Ignore air resistance. (a) How long is the ball in the air? (b) What must have been the initial horizontal component of the velocity? (c) What is the vertical component of the velocity just before the ball hits the ground? (d) What is the velocity (including both the horizontal and vertical components) of the ball just before it hits the ground?

4: (a) A daredevil is attempting to jump his motorcycle over a line of buses parked end to end by driving up 32° ramp at a speed of 40.0 m/s (144 km/h). How many buses can he clear if the top of the takeoff ramp is at the same height as the bus tops and the buses are 20.0 m long? (b) Discuss what your answer implies about the margin of error in this act—that is, consider how much greater the range is than the horizontal distance he must travel to miss the end of the last bus. (Neglect air resistance.)

5: An archer shoots an arrow at a 75.0 m distant target; the bull’s-eye of the target is at same height as the release height of the arrow. (a) At what angle must the arrow be released to hit the bull’s-eye if its initial speed is 35.0 m/s? In this part of the problem, explicitly show how you follow the steps involved in solving projectile motion problems. (b) There is a large tree halfway between the archer and the target with an overhanging horizontal branch 3.50 m above the release height of the arrow. Will the arrow go over or under the branch?

6: A rugby player passes the ball 7.00 m across the field, where it is caught at the same height as it left his hand. (a) At what angle was the ball thrown if its initial speed was 12.0 m/s, assuming that the smaller of the two possible angles was used? (b) What other angle gives the same range, and why would it not be used? (c) How long did this pass take?

7: Verify the ranges for the projectiles in Figure 5(a) for $\theta = 45°$ and the given initial velocities.

8: Verify the ranges shown for the projectiles in Figure 5(b) for an initial velocity of 50 m/s at the given initial angles.

9: The cannon on a battleship can fire a shell a maximum distance of 32.0 km. (a) Calculate the initial velocity of the shell. (b) What maximum height does it reach? (At its highest, the shell is above 60% of the atmosphere—but air resistance is not really negligible as assumed to make this problem easier.) (c) The ocean is not flat, because the Earth is curved. Assume that the radius of the Earth is 6.37×10^3 km. How many meters lower will its surface be 32.0 km from the ship along a horizontal line parallel to the surface at the ship? Does your answer imply that error introduced by the assumption of a flat Earth in projectile motion is significant here?

10: An arrow is shot from a height of 1.5 m toward a cliff of height H. It is shot with a velocity of 30 m/s at an angle of 60° above the horizontal. It lands on the top edge of the cliff 4.0 s later. (a) What is the height of

the cliff? (b) What is the maximum height reached by the arrow along its trajectory? (c) What is the arrow's impact speed just before hitting the cliff?

11: In the standing broad jump, one squats and then pushes off with the legs to see how far one can jump. Suppose the extension of the legs from the crouch position is 0.600 m and the acceleration achieved from this position is 1.25 times the acceleration due to gravity, g. How far can they jump? State your assumptions. (Increased range can be achieved by swinging the arms in the direction of the jump.)

12: The world long jump record is 8.95 m (Mike Powell, USA, 1991). Treated as a projectile, what is the maximum range obtainable by a person if he has a take-off speed of 9.5 m/s? State your assumptions.

13: Serving at a speed of 170 km/h, a tennis player hits the ball at a height of 2.5 m and an angle θ below the horizontal. The service line is 11.9 m from the net, which is 0.91 m high. What is the angle θ such that the ball just crosses the net? Will the ball land in the service box, whose out line is 6.40 m from the net?

14: A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. (a) If the ball is thrown at an angle of 25° relative to the ground and is caught at the same height as it is released, what is its initial speed relative to the ground? (b) How long does it take to get to the receiver? (c) What is its maximum height above its point of release?

15: Gun sights are adjusted to aim high to compensate for the effect of gravity, effectively making the gun accurate only for a specific range. (a) If a gun is sighted to hit targets that are at the same height as the gun and 100.0 m away, how low will the bullet hit if aimed directly at a target 150.0 m away? The muzzle velocity of the bullet is 275 m/s. (b) Discuss qualitatively how a larger muzzle velocity would affect this problem and what would be the effect of air resistance.

16: An eagle is flying horizontally at a speed of 3.00 m/s when the fish in her talons wiggles loose and falls into the lake 5.00 m below. Calculate the velocity of the fish relative to the water when it hits the water.

17: An owl is carrying a mouse to the chicks in its nest. Its position at that time is 4.00 m west and 12.0 m above the center of the 30.0 cm diameter nest. The owl is flying east at 3.50 m/s at an angle 30.0° below the horizontal when it accidentally drops the mouse. Is the owl lucky enough to have the mouse hit the nest? To answer this question, calculate the horizontal position of the mouse when it has fallen 12.0 m.

18: Suppose a soccer player kicks the ball from a distance 30 m toward the goal. Find the initial speed of the ball if it just passes over the goal, 2.4 m above the ground, given the initial direction to be 40° above the horizontal.

19: Can a goalkeeper at her/ his goal kick a soccer ball into the opponent's goal without the ball touching the ground? The distance will be about 95 m. A goalkeeper can give the ball a speed of 30 m/s.

20: The free throw line in basketball is 4.57 m (15 ft) from the basket, which is 3.05 m (10 ft) above the floor. A player standing on the free throw line throws the ball with an initial speed of 7.15 m/s, releasing it at a height of 2.44 m (8 ft) above the floor. At what angle above the horizontal must the ball be thrown to exactly hit the basket? Note that most players will use a large initial angle rather than a flat shot because it allows for a larger margin of error. Explicitly show how you follow the steps involved in solving projectile motion problems.

21: In 2007, Michael Carter (U.S.) set a world record in the shot put with a throw of 24.77 m. What was the

initial speed of the shot if he released it at a height of 2.10 m and threw it at an angle of 38.0° above the horizontal? (Although the maximum distance for a projectile on level ground is achieved at 45° when air resistance is neglected, the actual angle to achieve maximum range is smaller; thus, 38° will give a longer range than 45° in the shot put.)

22: A basketball player is running at 5.00 m/s directly toward the basket when he jumps into the air to dunk the ball. He maintains his horizontal velocity. (a) What vertical velocity does he need to rise 0.750 m above the floor? (b) How far from the basket (measured in the horizontal direction) must he start his jump to reach his maximum height at the same time as he reaches the basket?

23: A football player punts the ball at a 45.0° angle. Without an effect from the wind, the ball would travel 60.0 m horizontally. (a) What is the initial speed of the ball? (b) When the ball is near its maximum height it experiences a brief gust of wind that reduces its horizontal velocity by 1.50 m/s. What distance does the ball travel horizontally?

24: Prove that the trajectory of a projectile is parabolic, having the form $y = ax + bx^2$. To obtain this expression, solve the equation $x = v_{0x}t$ for t and substitute it into the expression for $y = v_{0y}t - (1/2)gt^2$ (These equations describe the x and y positions of a projectile that starts at the origin.) You should obtain an equation of the form $y = ax + bx^2$ where a and b are constants.

25: Derive $\mathrm{R} = \dfrac{v_0^2 \sin 2\theta_0}{g}$ for the range of a projectile on level ground by finding the time t at which y becomes zero and substituting this value of t into the expression for $x - x_0$, noting that $\mathrm{R} = x - x_0$.

26: Unreasonable Results (a) Find the maximum range of a super cannon that has a muzzle velocity of 4.0 km/s. (b) What is unreasonable about the range you found? (c) Is the premise unreasonable or is the available equation inapplicable? Explain your answer. (d) If such a muzzle velocity could be obtained, discuss the effects of air resistance, thinning air with altitude, and the curvature of the Earth on the range of the super cannon.

27: Construct Your Own Problem Consider a ball tossed over a fence. Construct a problem in which you calculate the ball's needed initial velocity to just clear the fence. Among the things to determine are; the height of the fence, the distance to the fence from the point of release of the ball, and the height at which the ball is released. You should also consider whether it is possible to choose the initial speed for the ball and just calculate the angle at which it is thrown. Also examine the possibility of multiple solutions given the distances and heights you have chosen.

Solutions

Problems & Exercises

1: x = 1.30 m x 10^2, y = 30.9 m

3: (a) 3.50 s (b) 28.6 m/s (c) 34.3 m/s (d) 44.7 m/s, 50.2^{o} below horizontal

5: (a) 18.4^{o} (b) The arrow will go over the branch.

7: $\mathrm{R} = \dfrac{v_0^2}{\sin 2\theta_0 g}$. For 45^{o}, $\mathrm{R} = \dfrac{v_0^2}{g}$. R = 91.8 m for v_0 = 30 m/s; R = 163 m for v_0 = 40 m/s; R = 255 m for v_0 = 50 m/s.

9: (a) 560 m/s (b) 8.00 x 10^{3} m (c) 80.0 m. This error is not significant because it is only 1% of the answer in part (b).

11: 1.50 m, assuming launch angle of 45^{o}

13: $\theta = 6.1^{0}$ yes, the ball lands at 5.3 m from the net

15: (a) -0.486 m (b) The larger the muzzle velocity, the smaller the deviation in the vertical direction, because the time of flight would be smaller. Air resistance would have the effect of decreasing the time of flight, therefore increasing the vertical deviation.

17: 4.23 m. No, the owl is not lucky; he misses the nest.

19: No, the maximum range (neglecting air resistance) is about 92 m.

21: 15.0 m/s

23: (a) 24.2 m/s (b) The ball travels a total of 57.4 m with the brief gust of wind.

25: $y - y_0 = 0 = v_{0y}t - \dfrac{1}{2}gt^2 = (v_0 \sin\theta)t - \dfrac{1}{2}gt^2,$

so that $t = \dfrac{2(v_0 \sin\theta)}{g}$

$x - x_0 = v_{0x}t = (v_0 \cos\theta)t = R$, and substituting for t gives:

$$R = v_0 \cos\theta(\frac{2v_0 \sin\theta}{g}) = (\frac{2v_0^2 \sin\theta\cos\theta}{g})$$

since $2\sin\theta\cos\theta = \sin 2\theta$, the range is: $\mathrm{R} = \dfrac{v_0^2 \sin 2\theta}{g}$.

3.5 Optional: Addition of Velocities

Summary

- Apply principles of vector addition to determine relative velocity.
- Explain the significance of the observer in the measurement of velocity.

Relative Velocity

If a person rows a boat across a rapidly flowing river and tries to head directly for the other shore, the boat instead moves *diagonally* relative to the shore, as in Figure 1. The boat does not move in the direction in which it is pointed. The reason, of course, is that the river carries the boat downstream. Similarly, if a small airplane flies overhead in a strong crosswind, you can sometimes see that the plane is not moving in the direction in which it is pointed, as illustrated in Figure 2. The plane is moving straight ahead relative to the air, but the movement of the air mass relative to the ground carries it sideways.

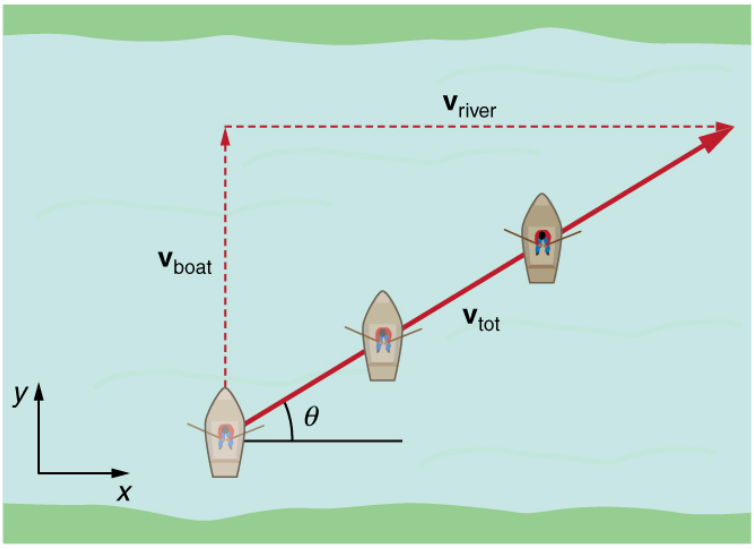


Figure 1. *A boat trying to head straight across a river will actually move diagonally relative to the shore as shown. Its total velocity (solid arrow) relative to the shore is the sum of its velocity relative to the river plus the velocity of the river relative to the shore.*

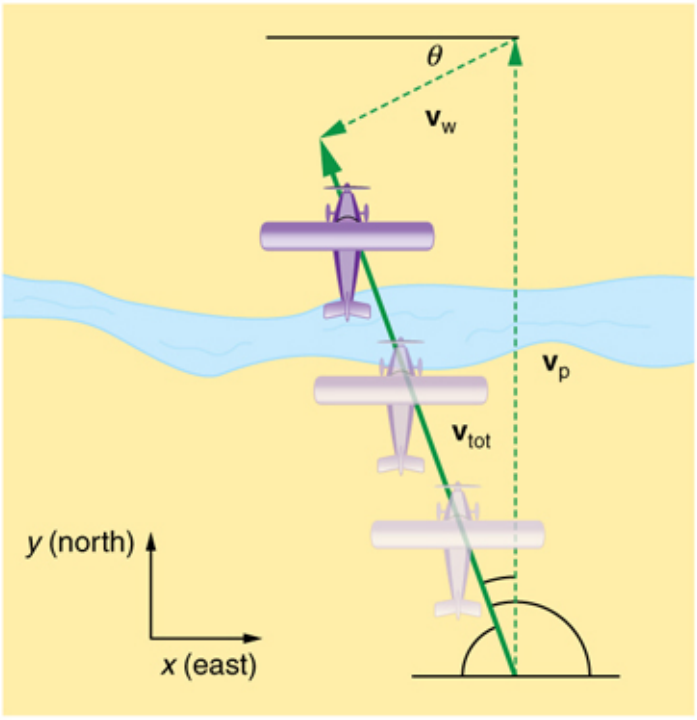


Figure 2. *An airplane heading straight north is instead carried to the west and slowed down by wind. The plane does not move relative to the ground in the direction it points; rather, it moves in the direction of its total velocity (solid arrow).*

In each of these situations, an object has a **velocity** relative to a medium (such as a river) and that medium has a velocity relative to an observer on solid ground. The velocity of the object *relative to the observer* is the sum of these velocity vectors, as indicated in Figure 1 and Figure 2. These situations are only two of many in which it is useful to add velocities. In this module, we first re-examine how to add velocities and then consider certain aspects of what relative velocity means.

How do we add velocities? Velocity is a vector (it has both magnitude and direction); the rules of **vector addition** apply to the addition of velocities, just as they do for any other vectors. In one-dimensional motion, the addition of velocities is simple—they add like ordinary numbers. For example, if a field hockey player is moving at 5 m/s straight toward the goal and drives the ball in the same direction with a velocity of 30 m/s relative to her body, then the velocity of the ball is 35 m/s relative to the stationary, profusely sweating goalkeeper standing in front of the goal.

In two-dimensional motion, either graphical or analytical techniques can be used to add velocities. We will concentrate on analytical techniques. The following equations give the relationships between the magnitude and direction of velocity (v and θ) and its components (v_x and v_y) along the x– and y-axes of an appropriately chosen coordinate system:

$$v_x = v\cos\theta$$
$$v_y = v\sin\theta$$
$$v = \sqrt{v_x^2 + v_y^2}$$
$$\theta = \tan^{-1}(v_y/v_x).$$

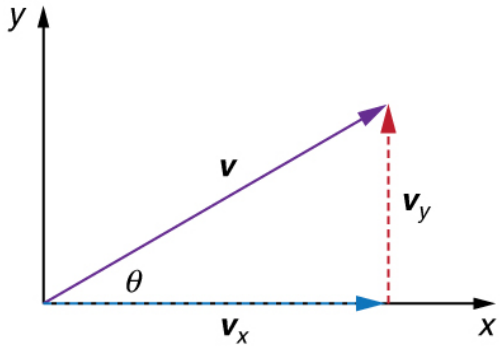


Figure 3. *The velocity, v, of an object traveling at an angle θ to the horizontal axis is the sum of component vectors v_x and v_y.*

These equations are valid for any vectors and are adapted specifically for velocity. The first two equations are used to find the components of a velocity when its magnitude and direction are known. The last two are used to find the magnitude and direction of velocity when its components are known.

TAKE-HOME EXPERIMENT: RELATIVE VELOCITY OF A BOAT

Fill a bathtub half-full of water. Take a toy boat or some other object that floats in water. Unplug the drain so water starts to drain. Try pushing the boat from one side of the tub to the other and perpendicular to the flow of water. Which way do you need to push the boat so that it ends up immediately opposite? Compare the directions of the flow of water, heading of the boat, and actual velocity of the boat.

Example 1: Adding Velocities: A Boat on a River

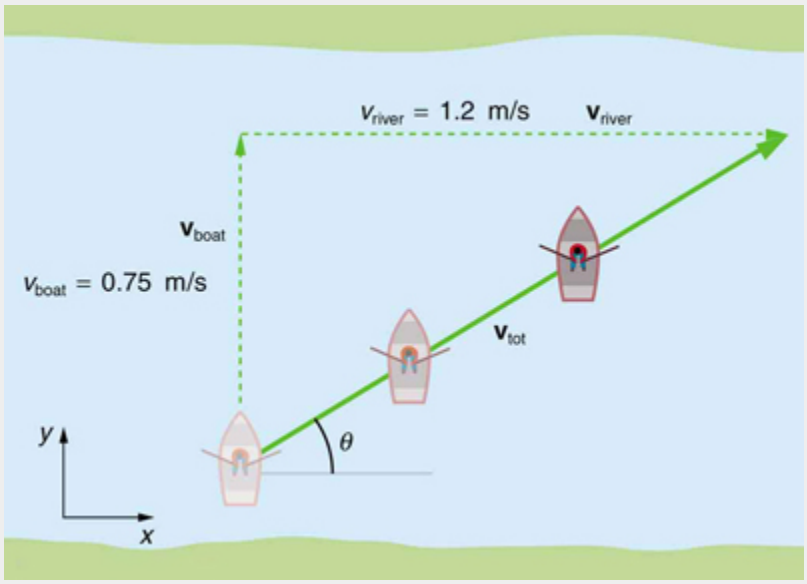


Figure 4. *A boat attempts to travel straight across a river at a speed 0.75 m/s. The current in the river, however, flows at a speed of 1.20 m/s to the right. What is the total displacement of the boat relative to the shore?*

Refer to Figure 4, which shows a boat trying to go straight across the river. Let us calculate the magnitude and direction of the boat's velocity relative to an observer on the shore, v_{tot}. The velocity of the boat, v_{boat}, is 0.75 m/s in the *y*-direction relative to the river and the velocity of the river, v_{river}, is 1.20 m/s to the right.

Strategy

We start by choosing a coordinate system with its x-axis parallel to the velocity of the river, as shown in Figure 4. Because the boat is directed straight toward the other shore, its velocity relative to the water is parallel to the *y*-axis and perpendicular to the velocity of the river. Thus, we can add the two velocities by using the equations $v_{tot} = \sqrt{v_x^2 + v_y^2}$ and $\theta = \tan^{-1}(v_y / v_x)$ directly.

Solution

The magnitude of the total velocity is

$$v_{tot} = \sqrt{v_x^2 + v_y^2},$$

where

$$v_x = v_{river} = 1.20 \text{ m/s}$$

and

$$v_y = v_{boat} = 0.750 \text{ m/s}.$$

Thus,

$$v_{tot} = \sqrt{(1.20 \text{ m/s})^2 + (0.750 \text{ m/s})^2}$$

yielding

$$v_{tot} = 1.42 \text{ m/s}.$$

The direction of the total velocity, θ, is given by:

$$\theta = \tan^{-1}(v_y/v_x) = \tan^{-1}(0.750/1.20).$$

This equation gives

$$\theta = 32.0^0.$$

Discussion

Both the magnitude *v* and the direction θ of the total velocity are consistent with Figure 4. Note that because the velocity of the river is large compared with the velocity of the boat, it is swept rapidly downstream. This result is evidenced by the small angle (only 32.0°) the total velocity has relative to the riverbank.

Example 2: Calculating Velocity: Wind Velocity Causes an Airplane to Drift

Calculate the wind velocity for the situation shown in Figure 5. The plane is known to be moving at 45.0 m/s due north relative to the air mass, while its velocity relative to the ground (its total velocity) is 38.0 m/s in a direction 20.0° west of north.

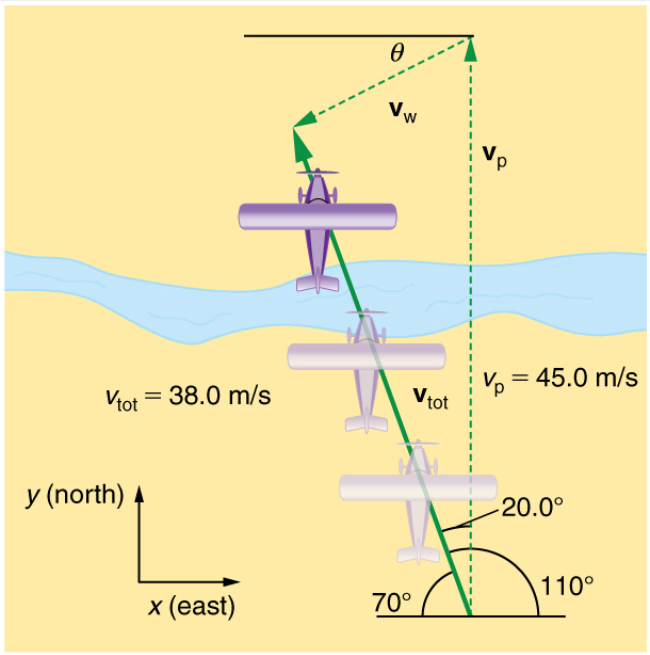


Figure 5. *An airplane is known to be heading north at 45.0 m/s, though its velocity relative to the ground is 38.0 m/s at an angle west of north. What is the speed and direction of the wind?*

Strategy

In this problem, somewhat different from the previous example, we know the total velocity v_{tot} and that it is the sum of two other velocities, v_w (the wind) and v_p (the plane relative to the air mass). The quantity v_p is known, and we are asked to find v_w. None of the velocities are perpendicular, but it is possible to find their components along a common set of perpendicular axes. If we can find the components of v_w, then we can combine them to solve for its magnitude and direction. As shown in Figure 5, we choose a coordinate system with its x-axis due east and its y-axis due north (parallel to v_p).

Solution

Because v_{tot} is the vector sum of the v_w and v_p, its x– and y-components are the sums of the x– and y-components of the wind and plane velocities. Note that the plane only has vertical component of velocity so $v_{px} = 0$ and $v_{py} = v_p$. That is,

$$v_{totx} = v_{wx}$$

and

$$v_{toty} = v_{wy} + v_p.$$

We can use the first of these two equations to find v_{wx}:

$$v_{wy} = v_{totx} = v_{tot} \cos 110^0.$$

Because v_{tot} = 38.0 m/s and cos 110° = -0.342 we have

$$v_{wy} = (38.0 \text{ m/s})(-0.342) = -13 \text{ m/s}.$$

The minus sign indicates motion west which is consistent with the diagram.

Now, to find v_{wy} we note that

$$v_{toty} = v_{wy} + v_p$$

Here $v_{toty} = v_{tot} \sin 110°$; thus,

$$v_{wy} = (38.0 \text{ m/s})(0.940) - 45.0 \text{ m/s} = -9.29 \text{ m/s}.$$

This minus sign indicates motion south which is consistent with the diagram.

Now that the perpendicular components of the wind velocity v_{wx} and v_{wy} are known, we can find the magnitude and direction of v_w. First, the magnitude is

$$\begin{aligned} v_w &= \sqrt{v_{wx}^2 + v_{wy}^2} \\ &= (-13.0 \text{ m/s})^2 + (-9.29 \text{ m/s})^2 \end{aligned}$$

so that

$$v_w = 16.0 \text{m/s}.$$

The direction is:

$$\theta = \tan^{-1}(v_{wy}/v_{wx}) = \tan^{-1}(-9.29/-13.0)$$

giving

$$\theta = 35.6^0.$$

Discussion

The wind's speed and direction are consistent with the significant effect the wind has on the total velocity of the plane, as seen in Figure 5. Because the plane is fighting a strong combination of crosswind and head-wind, it ends up with a total velocity significantly less than its velocity relative to the air mass as well as heading in a different direction.

Note that in both of the last two examples, we were able to make the mathematics easier by choosing a coordinate system with one axis parallel to one of the velocities. We will repeatedly find that choosing an appropriate coordinate system makes problem solving easier. For example, in projectile motion we always use a coordinate system with one axis parallel to gravity.

Relative Velocities and Classical Relativity

When adding velocities, we have been careful to specify that the *velocity is relative to some reference frame*. These velocities are called **relative velocities**. For example, the velocity of an airplane relative to an air mass is different from its velocity relative to the ground. Both are quite different from the velocity of an airplane relative to its passengers (which should be close to zero). Relative velocities are one aspect of **relativity**, which is defined to be the study of how different observers moving relative to each other measure the same phenomenon.

Nearly everyone has heard of relativity and immediately associates it with Albert Einstein (1879–1955), the greatest physicist of the 20th century. Einstein revolutionized our view of nature with his *modern*

theory of relativity, which we shall study in later chapters. The relative velocities in this section are actually aspects of classical relativity, first discussed correctly by Galileo and Isaac Newton. **Classical relativity** is limited to situations where speeds are less than about 1% of the speed of light—that is, less than 3,000 km/s. Most things we encounter in daily life move slower than this speed.

Let us consider an example of what two different observers see in a situation analyzed long ago by Galileo. Suppose a sailor at the top of a mast on a moving ship drops his binoculars. Where will it hit the deck? Will it hit at the base of the mast, or will it hit behind the mast because the ship is moving forward? The answer is that if air resistance is negligible, the binoculars will hit at the base of the mast at a point directly below its point of release. Now let us consider what two different observers see when the binoculars drop. One observer is on the ship and the other on shore. The binoculars have no horizontal velocity relative to the observer on the ship, and so he sees them fall straight down the mast. (See Figure 6.) To the observer on shore, the binoculars and the ship have the *same* horizontal velocity, so both move the same distance forward while the binoculars are falling. This observer sees the curved path shown in Figure 6. Although the paths look different to the different observers, each sees the same result—the binoculars hit at the base of the mast and not behind it. To get the correct description, it is crucial to correctly specify the velocities relative to the observer.

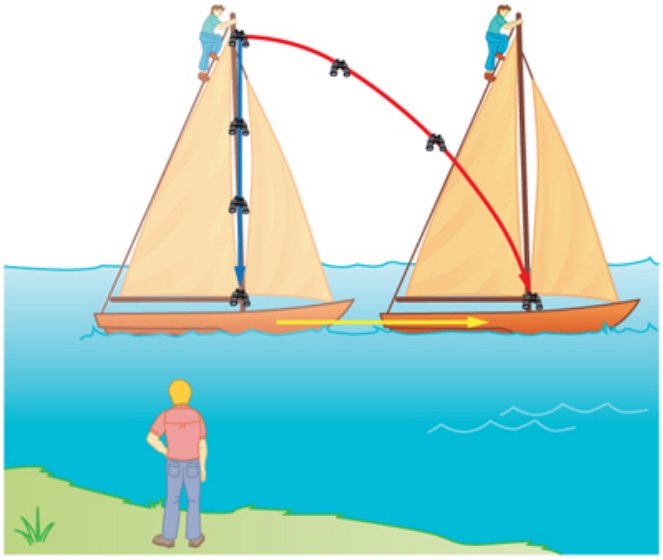


Figure 6. *Classical relativity. The same motion as viewed by two different observers. An observer on the moving ship sees the binoculars dropped from the top of its mast fall straight down. An observer on shore sees the binoculars take the curved path, moving forward with the ship. Both observers see the binoculars strike the deck at the base of the mast. The initial horizontal velocity is different relative to the two observers. (The ship is shown moving rather fast to emphasize the effect.)*

Example 3: Calculating Relative Velocity: An Airline Passenger Drops a Coin

An airline passenger drops a coin while the plane is moving at 260 m/s. What is the velocity of the coin when it strikes the floor 1.50 m below its point of release: (a) Measured relative to the plane? (b) Measured relative to the Earth?

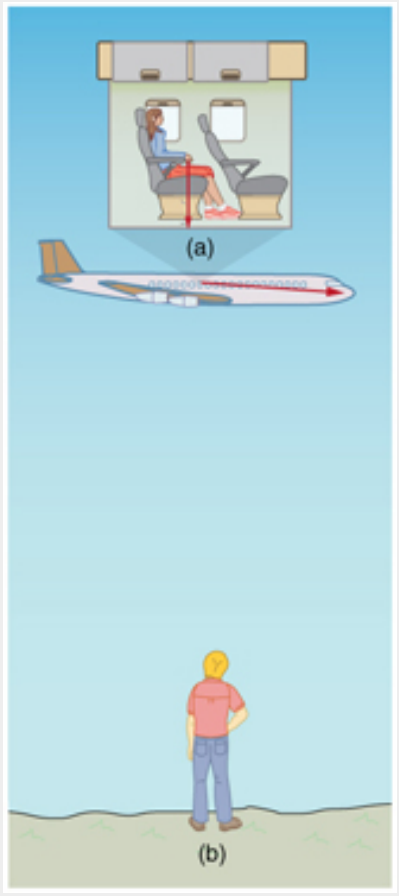


Figure 7. *The motion of a coin dropped inside an airplane as viewed by two different observers. (a) An observer in the plane sees the coin fall straight down. (b) An observer on the ground sees the coin move almost horizontally.*

Strategy

Both problems can be solved with the techniques for falling objects and projectiles. In part (a), the initial velocity of the coin is zero relative to the plane, so the motion is that of a falling object (one-dimensional). In part (b), the initial velocity is 260 m/s horizontal relative to the Earth and gravity is vertical, so this motion is a projectile motion. In both parts, it is best to use a coordinate system with vertical and horizontal axes.

Solution for (a)

Using the given information, we note that the initial velocity and position are zero, and the final position is 1.50 m. The final velocity can be found using the equation:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$

Substituting known values into the equation, we get

$$v_y^2 = 0^2 - 2(9.80 \text{ m/s}^2)(-1.50 \text{ m} - 0\text{m}) = 29.4 \text{ m}^2/\text{s}^2$$

yielding

$$v_y = -5.42 \text{ m/s}.$$

We know that the square root of 29.4 has two roots: 5.42 and -5.42. We choose the negative root because we know that the velocity is directed downwards, and we have defined the positive direction to be upwards. There is no initial horizontal velocity relative to the plane and no horizontal acceleration, and so the motion is straight down relative to the plane.

Solution for (b)

Because the initial vertical velocity is zero relative to the ground and vertical motion is independent of horizontal motion, the final vertical velocity for the coin relative to the ground is v_y = -5.42 m/s, the same as found in part (a). In contrast to part (a), there now is a horizontal component of the velocity. However, since there is no horizontal acceleration, the initial and final horizontal velocities are the same and v_x = 260 m/s. The *x*– and *y*-components of velocity can be combined to find the magnitude of the final velocity:

$$v = \sqrt{{v_x}^2 + {v_y}^2}.$$

Thus,

$$v = \sqrt{(260 \text{ m/s})^2 + (-5.42 \text{ m/s})^2}$$

yielding

$$v = 260.06 \text{ m/s}.$$

The direction is given by:

$$\theta = \tan^{-1}(v_y/v_x) = \tan^{-1}(-5.42/260)$$

so that

$$\theta = \tan^{-1}(-0.0208) = -1.19^0.$$

Discussion

In part (a), the final velocity relative to the plane is the same as it would be if the coin were dropped from rest on the Earth and fell 1.50 m. This result fits our experience; objects in a plane fall the same way when the plane is flying horizontally as when it is at rest on the ground. This result is also true in moving cars. In part (b), an observer on the ground sees a much different motion for the coin. The plane is moving so fast horizontally to begin with that its final velocity is barely greater than the initial velocity. Once again, we see that in two dimensions, vectors do not add like ordinary numbers—the final velocity v in part (b) is *not* (260 – 5.42) m/s; rather, it is 260.06 m/s. The velocity's magnitude had to be calculated to five digits to see any difference from that of the airplane. The motions as seen by different observers (one in the plane and one on the ground) in this example are analogous to those

discussed for the binoculars dropped from the mast of a moving ship, except that the velocity of the plane is much larger, so that the two observers see *very* different paths. (See Figure 7.) In addition, both observers see the coin fall 1.50 m vertically, but the one on the ground also sees it move forward 144 m (this calculation is left for the reader). Thus, one observer sees a vertical path, the other a nearly horizontal path.

MAKING CONNECTIONS: RELATIVITY AND EINSTEIN

Because Einstein was able to clearly define how measurements are made (some involve light) and because the speed of light is the same for all observers, the outcomes are spectacularly unexpected. Time varies with observer, energy is stored as increased mass, and more surprises await.

PHET EXPLORATIONS: MOTION IN 2D

Try the new "Ladybug Motion 2D" simulation for the latest updated version. Learn about position, velocity, and acceleration vectors. Move the ball with the mouse or let the simulation move the ball in four types of motion (2 types of linear, simple harmonic, circle).

Figure 8. *Motion in 2D*

Summary

- Velocities in two dimensions are added using the same analytical vector techniques, which are rewritten as

$$\begin{aligned} v_x &= v\cos\theta \\ v_y &= v\sin\theta \\ v &= \sqrt{v_x^2 + v_y^2} \\ \theta &= \tan^{-1}(v_y/v_x). \end{aligned}$$

- Relative velocity is the velocity of an object as observed from a particular reference frame, and it varies dramatically with reference frame.
- **Relativity** is the study of how different observers measure the same phenomenon, particularly when the observers move relative to one another. **Classical relativity** is limited to situations where speed is less than about 1% of the speed of light (3000 km/s).

Glossary

classical relativity
the study of relative velocities in situations where speeds are less than about 1% of the speed of light—that is, less than 3000 km/s

relative velocity
the velocity of an object as observed from a particular reference frame

relativity
the study of how different observers moving relative to each other measure the same phenomenon

velocity
speed in a given direction

vector addition
the rules that apply to adding vectors together

Conceptual Questions

1: What frame or frames of reference do you instinctively use when driving a car? When flying in a commercial jet airplane?

2: A basketball player dribbling down the court usually keeps his eyes fixed on the players around him. He is moving fast. Why doesn't he need to keep his eyes on the ball?

3: If someone is riding in the back of a pickup truck and throws a softball straight backward, is it possible for the ball to fall straight down as viewed by a person standing at the side of the road? Under what condition would this occur? How would the motion of the ball appear to the person who threw it?

4: The hat of a jogger running at constant velocity falls off the back of his head. Draw a sketch showing the path of the hat in the jogger's frame of reference. Draw its path as viewed by a stationary observer.

5: A clod of dirt falls from the bed of a moving truck. It strikes the ground directly below the end of the truck. What is the direction of its velocity relative to the truck just before it hits? Is this the same as the direction of its velocity relative to ground just before it hits? Explain your answers.

Problems & Exercises

1: Bryan Allen pedaled a human-powered aircraft across the English Channel from the cliffs of Dover to Cap Gris-Nez on June 12, 1979. (a) He flew for 169 min at an average velocity of 3.53 m/s in a direction 45°

south of east. What was his total displacement? (b) Allen encountered a headwind averaging 2.00 m/s almost precisely in the opposite direction of his motion relative to the Earth. What was his average velocity relative to the air? (c) What was his total displacement relative to the air mass?

2: A seagull flies at a velocity of 9.00 m/s straight into the wind. (a) If it takes the bird 20.0 min to travel 6.00 km relative to the Earth, what is the velocity of the wind? (b) If the bird turns around and flies with the wind, how long will he take to return 6.00 km? (c) Discuss how the wind affects the total round-trip time compared to what it would be with no wind.

3: Near the end of a marathon race, the first two runners are separated by a distance of 45.0 m. The front runner has a velocity of 3.50 m/s, and the second a velocity of 4.20 m/s. (a) What is the velocity of the second runner relative to the first? (b) If the front runner is 250 m from the finish line, who will win the race, assuming they run at constant velocity? (c) What distance ahead will the winner be when she crosses the finish line?

4: Verify that the coin dropped by the airline passenger in the Example 3 travels 144 m horizontally while falling 1.50 m in the frame of reference of the Earth.

5: A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. The ball is thrown at an angle of 25.0° relative to the ground and is caught at the same height as it is released. What is the initial velocity of the ball *relative to the quarterback* ?

6: A ship sets sail from Rotterdam, The Netherlands, heading due north at 7.00 m/s relative to the water. The local ocean current is 1.50 m/s in a direction 40.0° north of east. What is the velocity of the ship relative to the Earth?

7: (a) A jet airplane flying from Darwin, Australia, has an air speed of 260 m/s in a direction 5.0° south of west. It is in the jet stream, which is blowing at 35.0 m/s in a direction 15° south of east. What is the velocity of the airplane relative to the Earth? (b) Discuss whether your answers are consistent with your expectations for the effect of the wind on the plane's path.

8: (a) In what direction would the ship in Exercise 6 have to travel in order to have a velocity straight north relative to the Earth, assuming its speed relative to the water remains 7.00 m/s? (b) What would its speed be relative to the Earth?

9: (a) Another airplane is flying in a jet stream that is blowing at 45.0 m/s in a direction 20° south of east (as in Exercise 7). Its direction of motion relative to the Earth is 45.0° south of west, while its direction of travel relative to the air is 5.00° south of west. What is the airplane's speed relative to the air mass? (b) What is the airplane's speed relative to the Earth?

10: A sandal is dropped from the top of a 15.0-m-high mast on a ship moving at 1.75 m/s due south. Calculate the velocity of the sandal when it hits the deck of the ship: (a) relative to the ship and (b) relative to a stationary observer on shore. (c) Discuss how the answers give a consistent result for the position at which the sandal hits the deck.

11: The velocity of the wind relative to the water is crucial to sailboats. Suppose a sailboat is in an ocean current that has a velocity of 2.20 m/s in a direction 30.0° east of north relative to the Earth. It encounters a wind that has a velocity of 4.50 m/s in a direction of 50.0° south of west relative to the Earth. What is the velocity of the wind relative to the water?

12: The great astronomer Edwin Hubble discovered that all distant galaxies are receding from our Milky Way Galaxy with velocities proportional to their distances. It appears to an observer on the Earth that we are at the center of an expanding universe. Figure 9 illustrates this for five galaxies lying along a straight line, with the Milky Way Galaxy at the center. Using the data from the figure, calculate the velocities: (a) relative to galaxy 2 and (b) relative to galaxy 5. The results mean that observers on all galaxies will see themselves at the center of the expanding universe, and they would likely be aware of relative velocities, concluding that it is not possible to locate the center of expansion with the given information.

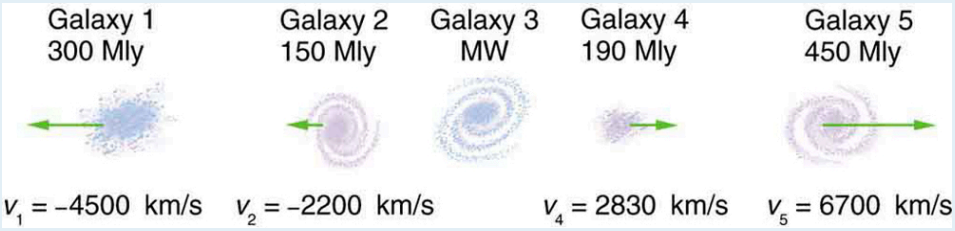


Figure 9. *Five galaxies on a straight line, showing their distances and velocities relative to the Milky Way (MW) Galaxy. The distances are in millions of light years (Mly), where a light year is the distance light travels in one year. The velocities are nearly proportional to the distances. The sizes of the galaxies are greatly exaggerated; an average galaxy is about 0.1 Mly across.*

13: (a) Use the distance and velocity data in Figure 9 to find the rate of expansion as a function of distance.

(b) If you extrapolate back in time, how long ago would all of the galaxies have been at approximately the same position? The two parts of this problem give you some idea of how the Hubble constant for universal expansion and the time back to the Big Bang are determined, respectively.

14: An athlete crosses a 25-m-wide river by swimming perpendicular to the water current at a speed of 0.5 m/s relative to the water. He reaches the opposite side at a distance 40 m downstream from his starting point. How fast is the water in the river flowing with respect to the ground? What is the speed of the swimmer with respect to a friend at rest on the ground?

15: A ship sailing in the Gulf Stream is heading 25.0° west of north at a speed of 4.00 m/s relative to the water. Its velocity relative to the Earth is 4.80 m/s 5.00° west of north. What is the velocity of the Gulf Stream? (The velocity obtained is typical for the Gulf Stream a few hundred kilometers off the east coast of the United States.)

16: An ice hockey player is moving at 8.00 m/s when he hits the puck toward the goal. The speed of the puck relative to the player is 29.0 m/s. The line between the center of the goal and the player makes a 90.0° angle relative to his path as shown in Figure 10. What angle must the puck's velocity make relative to the player (in his frame of reference) to hit the center of the goal?

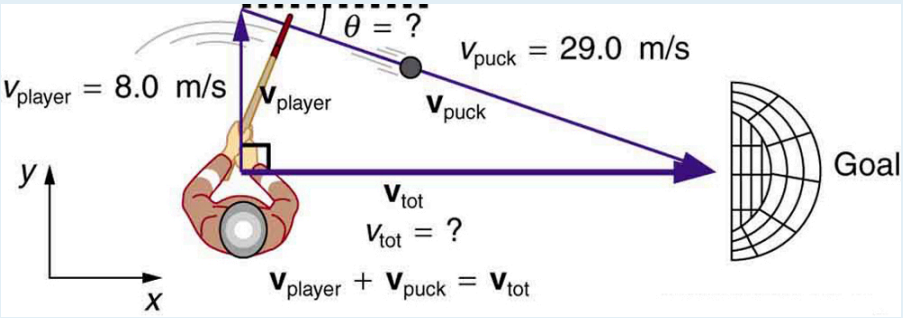


Figure 10. *An ice hockey player moving across the rink must shoot backward to give the puck a velocity toward the goal.*

15: Unreasonable Results Suppose you wish to shoot supplies straight up to astronauts in an orbit 36,000 km above the surface of the Earth. (a) At what velocity must the supplies be launched? (b) What is unreasonable about this velocity? (c) Is there a problem with the relative velocity between the supplies and the astronauts when the supplies reach their maximum height? (d) Is the premise unreasonable or is the available equation inapplicable? Explain your answer.

16: Unreasonable Results A commercial airplane has an air speed of 280 m/s due east and flies with a strong tailwind. It travels 3000 km in a direction 5° south of east in 1.50 h. (a) What was the velocity of the plane relative to the ground? (b) Calculate the magnitude and direction of the tailwind's velocity. (c) What is unreasonable about both of these velocities? (d) Which premise is unreasonable?

17: Construct Your Own Problem Consider an airplane headed for a runway in a cross wind. Construct a problem in which you calculate the angle the airplane must fly relative to the air mass in order to have a velocity parallel to the runway. Among the things to consider are the direction of the runway, the wind speed and direction (its velocity) and the speed of the plane relative to the air mass. Also calculate the speed of the airplane relative to the ground. Discuss any last minute maneuvers the pilot might have to perform in order for the plane to land with its wheels pointing straight down the runway.

Solutions

Problems & Exercises:

1: (a) 35.8 km south of east (b) 5.53 m/s south of east (c) 56.1 km south of east

3: (a) 0.70 m/s faster (b) Second runner wins (c) 4.17 m

5: 17.0 m/s, 22.1^{o}

7: (a) 230 m/s, 8.0^{o} south of west (b) The wind should make the plane travel slower and more to the south, which is what was calculated

9: (a) 63.5 m/s (b) 29.6 m/s

11: 6.68 m/s, 53.3^{o} south of west

13: (a) 14.9 km/s/Mly (b) 20.2 billion years

15: 1.72 m/s, 42.3^{o} north of east

Chapter 4 Dynamics: Force and Newton's Laws of Motion

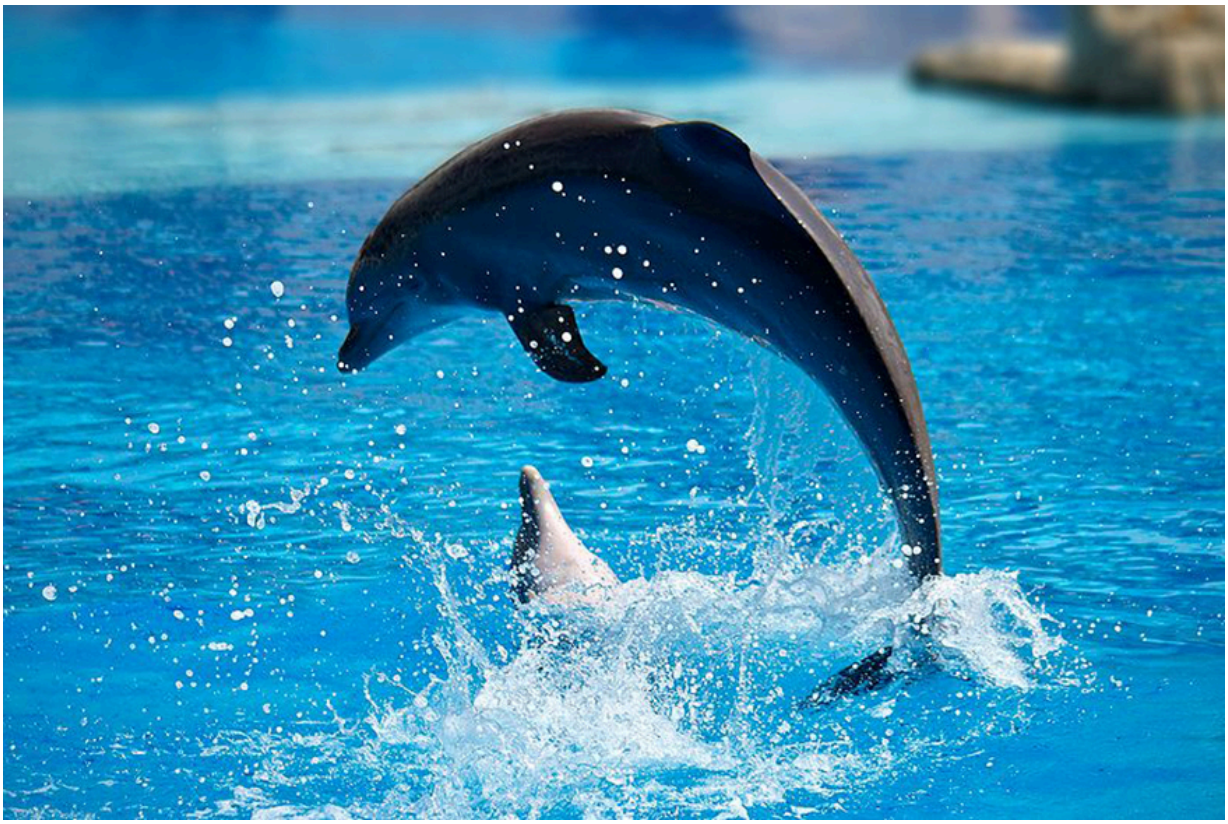
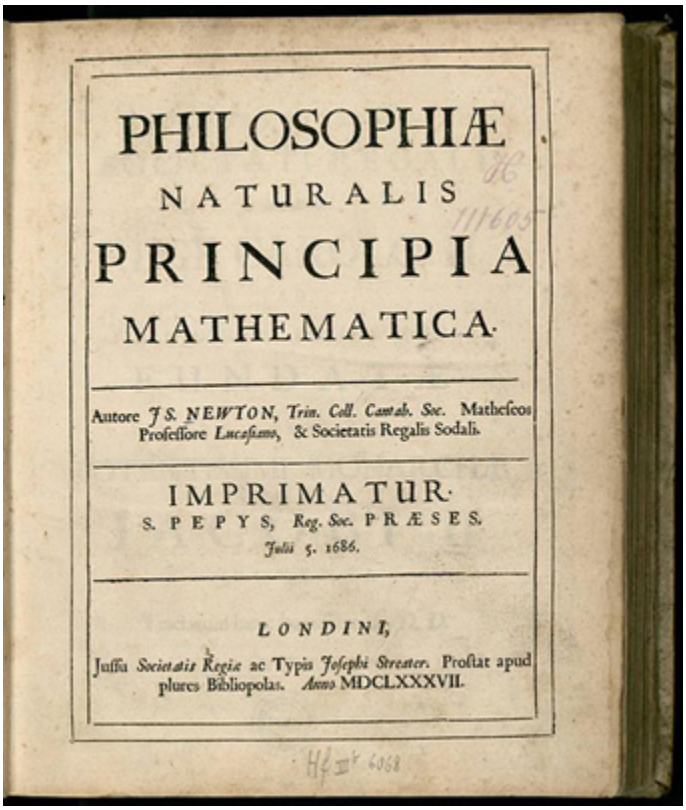


Figure 1. *Newton's laws of motion describe the motion of the dolphin's path. (credit: Jin Jang)*

Motion draws our attention. Motion itself can be beautiful, causing us to marvel at the forces needed to achieve spectacular motion, such as that of a dolphin jumping out of the water, or a pole vaulter, or the flight of a bird, or the orbit of a satellite. The study of motion is kinematics, but kinematics only *describes* the way objects move—their velocity and their acceleration. **Dynamics** considers the forces that affect the motion of moving objects and systems. Newton's laws of motion are the foundation of dynamics. These laws provide an example of the breadth and simplicity of principles under which nature functions. They are also universal laws in that they apply to similar situations on Earth as well as in space.

Isaac Newton's (1642–1727) laws of motion were just one part of the monumental work that has made him legendary. The development of Newton's laws marks the transition from the Renaissance into the modern era. This transition was characterized by a revolutionary change in the way people thought about the physical universe. For many centuries natural philosophers had debated the nature of the universe based largely on certain rules of logic with great weight given to the thoughts of earlier classical

philosophers such as Aristotle (384–322 BC). Among the many great thinkers who contributed to this change were Newton and Galileo.


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu Societatis Regiæ ac Typis Josephi Streater. Proſtat apud plures Bibliopolas. Anno MDCLXXXVII.

Figure 2. *Isaac Newton's monumental work, Philosophiae Naturalis Principia Mathematica, was published in 1687. It proposed scientific laws that are still used today to describe the motion of objects. (credit: Service commun de la documentation de l'Université de Strasbourg)*

Galileo was instrumental in establishing *observation* as the absolute determinant of truth, rather than "logical" argument. Galileo's use of the telescope was his most notable achievement in demonstrating the importance of observation. He discovered moons orbiting Jupiter and made other observations that were inconsistent with certain ancient ideas and religious dogma. For this reason, and because of the manner in which he dealt with those in authority, Galileo was tried by the Inquisition and punished. He spent the final years of his life under a form of house arrest. Because others before Galileo had also made discoveries by *observing* the nature of the universe, and because repeated observations verified those of Galileo, his work could not be suppressed or denied. After his death, his work was verified by others, and his ideas were eventually accepted by the church and scientific communities.

Galileo also contributed to the formation of what is now called Newton's first law of motion. Newton made use of the work of his predecessors, which enabled him to develop laws of motion, discover the law of gravity, invent calculus, and make great contributions to the theories of light and color. It is amazing that many of these developments were made with Newton working alone, without the benefit of the usual interactions that take place among scientists today.

It was not until the advent of modern physics early in the 20th century that it was discovered that Newton's laws of motion produce a good approximation to motion only when the objects are moving

at speeds much, much less than the speed of light and when those objects are larger than the size of most molecules (about 10^{-9} m in diameter). These constraints define the realm of classical mechanics, as discussed in Chapter 1 Introduction to the Nature of Science and Physics. At the beginning of the 20^{th} century, Albert Einstein (1879–1955) developed the theory of relativity and, along with many other scientists, developed quantum theory. This theory does not have the constraints present in classical physics. All of the situations we consider in this chapter are in the realm of classical physics.

MAKING CONNECTIONS: PAST AND PRESENT PHILOSOPHY

The importance of observation and the concept of *cause and effect* were not always so entrenched in human thinking. This realization was a part of the evolution of modern physics from natural philosophy. The achievements of Galileo, Newton, Einstein, and others were key milestones in the history of scientific thought. Most of the scientific theories that are described in this book descended from the work of these scientists.

4.1 Development of Force Concept

Summary

- Define force.

Dynamics is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. Our intuitive definition of **force**—that is, a push or a pull—is a good place to start. We know that a push or pull has both magnitude and direction (therefore, it is a vector quantity) and can vary considerably in each regard. For example, a cannon exerts a strong force on a cannonball that is launched into the air. In contrast, Earth exerts only a tiny downward pull on a flea. Our everyday experiences also give us a good idea of how multiple forces add. If two people push in different directions on a third person, as illustrated in Figure 1, we might expect the total force to be in the direction shown. Since force is a vector, it adds just like other vectors, as illustrated in Figure 2(a) for two ice skaters. Forces, like other vectors, are represented by arrows and can be added using the familiar head-to-tail method or by trigonometric methods. These ideas were developed in Chapter 3 Two-Dimensional Kinematics.

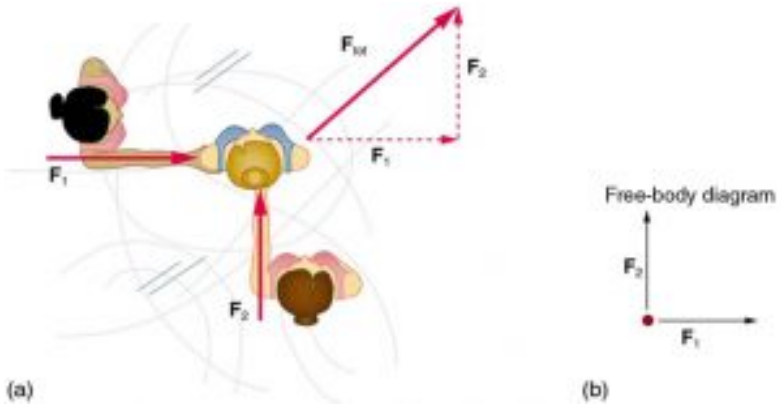


Figure 1. *Part (a) shows an overhead view of two ice skaters pushing on a third. Forces are vectors and add like other vectors, so the total force on the third skater is in the direction shown. In part (b), we see a free-body diagram representing the forces acting on the third skater.*

Figure 1(b) is our first example of a **free-body diagram**, which is a technique used to illustrate all the **external forces** acting on a body. The body is represented by a single isolated point (or free body), and only those forces acting *on* the body from the outside (external forces) are shown. (These forces are the only ones shown, because only external forces acting on the body affect its motion. We can ignore any internal forces within the body.) Free-body diagrams are very useful in analyzing forces acting on a system and are employed extensively in the study and application of Newton's laws of motion.

A more quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard distance. One possibility is to stretch a spring a certain fixed distance, as illustrated in Figure 2, and use the force it exerts to pull itself back to its relaxed shape—called a *restoring force*—as a standard. The magnitude of all other forces can be stated as multiples of this standard unit of force. Many other possibilities exist for standard forces and some alternative definitions of force will be given later in this chapter.

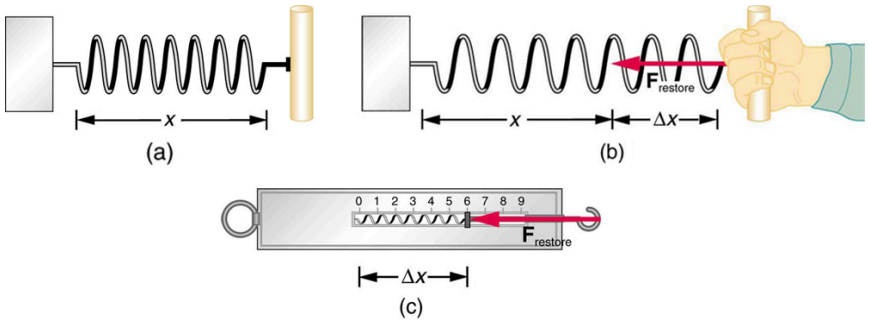


Figure 2. *The force exerted by a stretched spring can be used as a standard unit of force. (a) This spring has a length* ***x*** *when undistorted. (b) When stretched a distance Δx, the spring exerts a restoring force,* F_{restore}*, which is reproducible. (c) A spring scale is one device that uses a spring to measure force. The force* F_{restore} *is exerted on whatever is attached to the hook. Here* F_{restore} *has a magnitude of 6 units in the force standard being employed.*

TAKE-HOME EXPERIMENT: FORCE STANDARDS

To investigate force standards and cause and effect, get two identical rubber bands. Hang one rubber band vertically on a hook. Find a small household item that could be attached to the rubber band using a paper clip, and use this item as a weight to investigate the stretch of the rubber band. Measure the amount of stretch produced in the rubber band with one, two, and four of these (identical) items suspended from the rubber band. What is the relationship between the number of items and the amount of stretch? How large a stretch would you expect for the same number of items suspended from two rubber bands? What happens to the amount of stretch of the rubber band (with the weights attached) if the weights are also pushed to the side with a pencil?

Section Summary

- **Dynamics** is the study of how forces affect the motion of objects.
- **Force** is a push or pull that can be defined in terms of various standards, and it is a vector having both magnitude and direction.
- **External forces** are any outside forces that act on a body. A **free-body diagram** is a drawing of all external forces acting on a body.

Glossary

dynamics
: the study of how forces affect the motion of objects and systems

external force
: a force acting on an object or system that originates outside of the object or system

free-body diagram
: a sketch showing all of the external forces acting on an object or system; the system is represented by a dot, and the forces are represented by vectors extending outward from the dot

force
: a push or pull on an object with a specific magnitude and direction; can be represented by vectors; can be expressed as a multiple of a standard force

Conceptual Questions

1: Propose a force standard different from the example of a stretched spring discussed in the text. Your standard must be capable of producing the same force repeatedly.

2: What properties do forces have that allow us to classify them as vectors?

4.2 Newton's First Law of Motion: Inertia

Summary

- Define mass and inertia.
- Understand Newton's first law of motion.

Experience suggests that an object at rest will remain at rest if left alone, and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. What **Newton's first law of motion** states, however, is the following:

NEWTON'S FIRST LAW OF MOTION

A body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force.

Note the repeated use of the verb "remains." We can think of this law as preserving the status quo of motion.

Rather than contradicting our experience, **Newton's first law of motion** states that there must be a *cause* (which is a net external force) *for there to be any change in velocity (either a change in magnitude or direction)*. We will define *net external force* in the next section. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappeared, would the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the *cause* of the slowing (consistent with Newton's first law). The object would not slow down at all if friction were completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Newton's first law is completely general and can be applied to anything from an object sliding on a table to a satellite in orbit to blood pumped from the heart. Experiments have thoroughly verified that any change in velocity (speed or direction) must be caused by an external force. The idea of *generally applicable or universal laws* is important not only here—it is a basic feature of all laws of physics. Identifying these laws is like recognizing patterns in nature from which further patterns can be discovered. The genius of Galileo, who first developed the idea for the first law, and Newton, who clarified it, was to ask the fundamental question, "What is the cause?" Thinking in terms of cause and effect is a worldview fundamentally different from the typical ancient Greek approach when questions such as "Why does a tiger have stripes?" would have been answered in Aristotelian fashion, "That is the nature of the beast." True perhaps, but not a useful insight.

Mass

The property of a body to remain at rest or to remain in motion with constant velocity is called **inertia**. Newton's first law is often called the **law of inertia**. As we know from experience, some objects have more inertia than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its **mass**. Roughly speaking, mass is a measure of the amount of "stuff" (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of various types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is very difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this manner. Operationally, the masses of objects are determined by comparison with the standard kilogram.

Check Your Understanding

1: Which has more mass: a kilogram of cotton balls or a kilogram of gold?

Section Summary

- **Newton's first law of motion** states that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force. This is also known as the **law of inertia**.
- **Inertia** is the tendency of an object to remain at rest or remain in motion. Inertia is related to an object's mass.
- **Mass** is the quantity of matter in a substance.

Glossary

inertia
: the tendency of an object to remain at rest or remain in motion

law of inertia
: see Newton's first law of motion

mass
: the quantity of matter in a substance; measured in kilograms

Newton's first law of motion
: a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force; also known as the law of inertia

Conceptual Questions

1: How are inertia and mass related?

2: What is the relationship between weight and mass? Which is an intrinsic, unchanging property of a body?

Solutions

Check Your Understanding

1: They are equal. A kilogram of one substance is equal in mass to a kilogram of another substance. The quantities that might differ between them are volume and density.

4.3 Newton's Second Law of Motion: Concept of a System

Summary

- Define net force, external force, and system.
- Understand Newton's second law of motion.
- Apply Newton's second law to determine the weight of an object.

Newton's second law of motion is closely related to Newton's first law of motion. It mathematically states the cause and effect relationship between force and changes in motion. Newton's second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an **acceleration**. Newton's first law says that a net external force causes a change in motion; thus, we see that a *net external force causes acceleration*.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an **external force** acts from outside the **system** of interest. For example, in Figure 1(a) the system of interest is the wagon plus the child in it. The two forces exerted by the other children are external forces. An internal force acts between elements of the system. Again looking at Figure 1(a), the force the child in the wagon exerts to hang onto the wagon is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton's first law. (The internal forces actually cancel, as we shall see in the next section.) *You must define the boundaries of the system before you can determine which forces are external*. Sometimes the system is obvious, whereas other times identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton's laws. This concept will be revisited many times on our journey through physics.

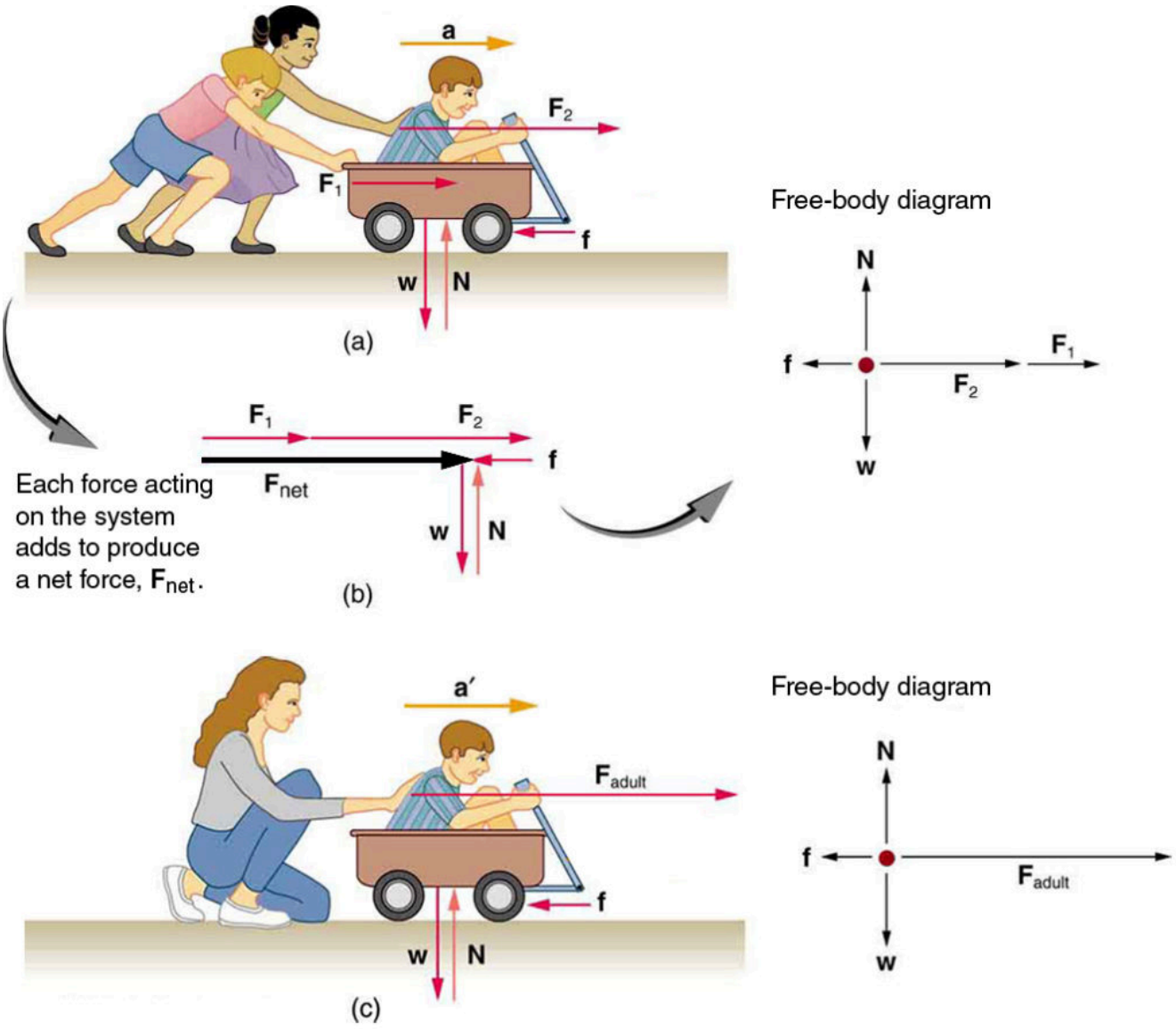


Figure 1. *Different forces exerted on the same mass produce different accelerations. (a) Two children push a wagon with a child in it. Arrows representing all external forces are shown. The system of interest is the wagon and its rider. The weight* ***w*** *of the system and the support of the ground* ***N*** *are also shown for completeness and are assumed to cancel. The vector* ***f*** *represents the friction acting on the wagon, and it acts to the left, opposing the motion of the wagon. (b) All of the external forces acting on the system add together to produce a net force,* $\boldsymbol{F_{net}}$*. The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, we draw the vectors collinearly. (c) A larger net external force produces a larger acceleration (****a'>a****) when an adult pushes the child.*

Now, it seems reasonable that acceleration should be directly proportional to and in the same direction as the net (total) external force acting on a system. This assumption has been verified experimentally and is illustrated in Figure 1. In part (a), a smaller force causes a smaller acceleration than the larger force illustrated in part (c). For completeness, the vertical forces are also shown; they are assumed to cancel since there is no acceleration in the vertical direction. The vertical forces are the weight *w* and the support of the ground *N*, and the horizontal force *f* represents the force of friction. These will be discussed in more detail in later sections. For now, we will define friction as a force that opposes the

motion past each other of objects that are touching. Figure 1(b) shows how vectors representing the external forces add together to produce a net force, F_{net}.

To obtain an equation for Newton's second law, we first write the relationship of acceleration and net external force as the proportionality

$$\vec{a} \propto \vec{F}_{\text{net}},$$

where the symbol $\propto$ means "proportional to," and F_{net} is the net external force. (The net external force is the vector sum of all external forces and can be determined graphically, using the head-to-tail method, or analytically, using components. The techniques are the same as for the addition of other vectors, and are covered in Chapter 3 Two-Dimensional Kinematics.) This proportionality states what we have said in words—*acceleration is directly proportional to the net external force*. Once the system of interest is chosen, it is important to identify the external forces and ignore the internal ones. It is a tremendous simplification not to have to consider the numerous internal forces acting between objects within the system, such as muscular forces within the child's body, let alone the myriad of forces between atoms in the objects, but by doing so, we can easily solve some very complex problems with only minimal error due to our simplification

Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. And indeed, as illustrated in Figure 2, the same net external force applied to a car produces a much smaller acceleration than when applied to a basketball. The proportionality is written as

$$a \propto \frac{1}{m}$$

where ***m*** is the mass of the system. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is exactly linearly proportional to the net external force.

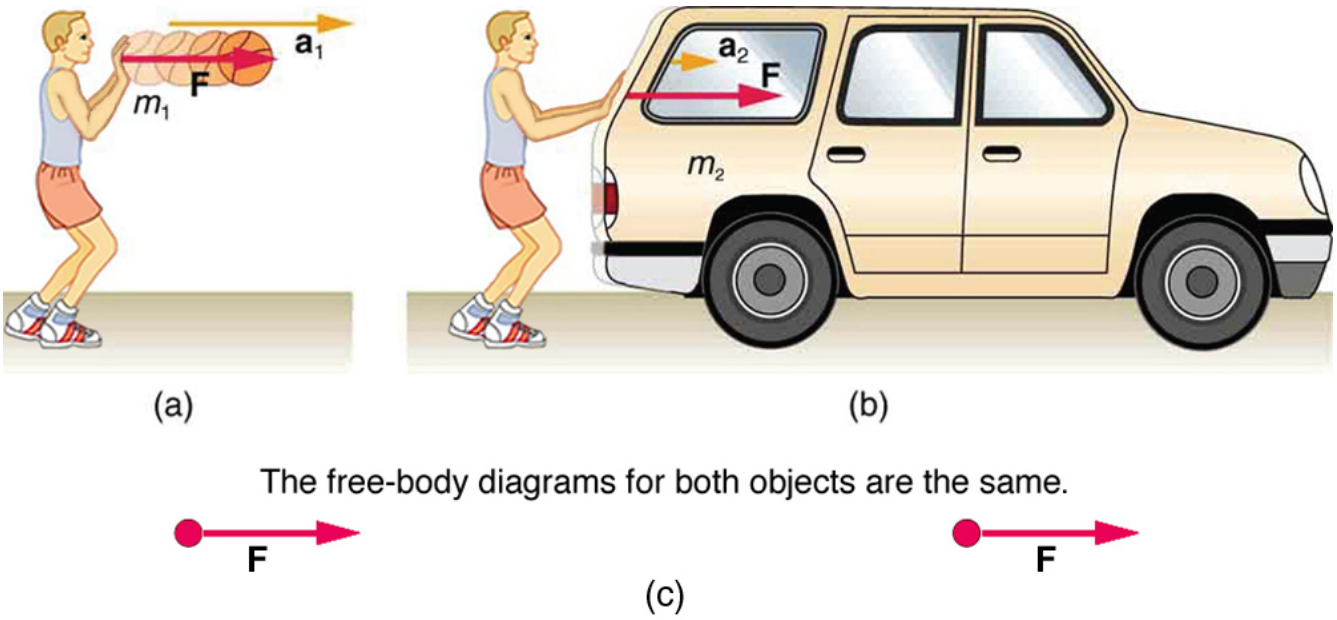


Figure 2. *The same force exerted on systems of different masses produces different accelerations. (a) A basketball player pushes on a basketball to make a pass. (The effect of gravity on the ball is ignored.) (b) The same player exerts an identical force on a stalled SUV and produces a far smaller acceleration (even if friction is negligible). (c) The free-body diagrams are identical, permitting direct comparison of the two situations. A series of patterns for the free-body diagram will emerge as you do more problems.*

It has been found that the acceleration of an object depends *only* on the net external force and the mass of the object. Combining the two proportionalities just given yields Newton's second law of motion.

NEWTON'S SECOND LAW OF MOTION

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.

In equation form, Newton's second law of motion is

$$\vec{\mathrm{a}} = \frac{\vec{\mathrm{F}}_{\mathrm{net}}}{m}.$$

This is often written in the more familiar form

$$\vec{\mathrm{F}}_{\mathrm{net}} = m\vec{\mathrm{a}}.$$

When only the magnitude of force and acceleration are considered, this equation is simply

$$F_{\mathrm{net}} = ma.$$

Although these last two equations are really the same, the first gives more insight into what Newton's second law means. The law is a *cause and effect relationship* among three quantities that is not simply based on their definitions. The validity of the second law is completely based on experimental verification.

Units of Force

$\vec{\mathrm{F}}_{\mathrm{net}} = m\vec{\mathrm{a}}$ is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the newton (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of 1 m/s^2. That is, from $\vec{\mathrm{F}}_{\mathrm{net}} = m\vec{\mathrm{a}}$ we can see that,

$$1\ \mathrm{N} = 1\ \mathrm{kg} \cdot \mathrm{m/s}^2$$

While almost the entire world uses the newton for the unit of force, in the United States the most familiar unit of force is the pound (lb), where 1 N = 0.225 lb.

Weight and the Gravitational Force

When an object is dropped, it accelerates toward the centre of Earth. Newton's second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its weight *w*. Weight can be denoted as a vector $\vec{\mathrm{w}}$ because it has a direction; *down* is, by definition, the direction of gravity, and hence weight is a downward force. The magnitude of weight is denoted as *w*. Galileo was instrumental in showing that, in the absence of air resistance, all objects fall with the same acceleration *g*. Using Galileo's result and Newton's second law, we can derive an equation for weight.

Consider an object with mass *m* falling downward toward Earth. It experiences only the downward force of gravity, which has magnitude *w*. Newton's second law states that the magnitude of the net external force on an object is $F_{\mathrm{net}} = ma$.

Since the object experiences only the downward force of gravity, $F_{\mathrm{net}} = w$. We know that the acceleration of an object due to gravity is *g*, or $a = g$. Substituting these into Newton's second law gives

WEIGHT

This is the equation for *weight*—the gravitational force on a mass *m*:

$$\mathrm{w} = \mathrm{m}\ \mathrm{g}$$

Since $g = 9.80\ \mathrm{m/s}^2$ on Earth, the weight of a 1.0 kg object on Earth is 9.8 N, as we see:

$$w = \mathrm{mg} = (1.0\ \mathrm{kg})\ (9.80\ \ \mathrm{m/s}^2) = 9.8\ \ \mathrm{N}$$

Recall that ***g*** can take a positive or negative value, depending on the positive direction in the coordinate system. Be sure to take this into consideration when solving problems with weight.

When the net external force on an object is its weight, we say that it is in **free-fall**. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

The acceleration due to gravity g varies slightly over the surface of Earth, so that the weight of an object depends on location and is not an intrinsic property of the object. Weight varies dramatically if one leaves Earth's surface. On the Moon, for example, the acceleration due to gravity is only 1.67 m/s^2. A 1.0-kg mass thus has a weight of 9.8 N on Earth and only about 1.7 N on the Moon.

The broadest definition of weight in this sense is that *the weight of an object is the gravitational force on it from the nearest large body*, such as Earth, the Moon, the Sun, and so on. This is the most common and useful definition of weight in physics. It differs dramatically, however, from the definition of weight used by NASA and the popular media in relation to space travel and exploration. When they speak of "weightlessness" and "microgravity," they are really referring to the phenomenon we call "free-fall" in physics. We shall use the above definition of weight, and we will make careful distinctions between free-fall and actual weightlessness.

It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much "stuff") and does not vary in classical physics, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object. Furthermore, the terms *mass* and *weight* are used interchangeably in everyday language; for example, our medical records often show our "weight" in kilograms, but never in the correct units of newtons.

COMMON MISCONCEPTIONS: MASS VS. WEIGHT

Mass and weight are often used interchangeably in everyday language. However, in science, these terms are distinctly different from one another. Mass is a measure of how much matter is in an object. The typical measure of mass is the kilogram (or the "slug" in English units). Weight, on the other hand, is a measure of the force of gravity acting on an object. Weight is equal to the mass of an object (m) multiplied by the acceleration due to gravity (g). Like any other force, weight is measured in terms of newtons (or pounds in English units).

Assuming the mass of an object is kept intact, it will remain the same, regardless of its location. However, because weight depends on the acceleration due to gravity, the weight of an object *can change* when the object enters into a region with stronger or weaker gravity. For example, the acceleration due to gravity on the Moon is 1.67 m/s^2 (which is much less than the acceleration due to gravity on Earth, 9.80 m/s^2). If you measured your weight on Earth and then measured your weight on the Moon, you would find that you "weigh" much less, even though you do not look any skinnier. This is because the force of gravity is weaker on the Moon. In fact, when people say that they are "losing weight," they really mean that they are losing "mass" (which in turn causes them to weigh less).

TAKE-HOME EXPERIMENT: MASS AND WEIGHT

What do bathroom scales measure? When you stand on a bathroom scale, what happens to the scale? It depresses slightly. The scale contains springs that compress in proportion to your weight—similar to rubber bands expanding when pulled. The springs provide a measure of your weight (for an object which is not accelerating). This is a force in newtons (or pounds). In most countries, the measurement is divided by 9.80 to give a reading in mass units of kilograms. The scale measures weight but is calibrated to provide information about mass. While standing on a bathroom scale, push down on a table next to you. What happens to the reading? Why? Would your scale measure the same "mass" on Earth as on the Moon?

Example 1: What Acceleration Can a Person Produce when Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb) parallel to the ground. The mass of the mower is 24 kg. What is its acceleration**?**

Figure 3. *The net force on a lawn mower is 51 N to the right. At what rate does the lawn mower accelerate to the right?*

Strategy

Since F_{net} and m are given, the acceleration can be calculated directly from Newton's second law as stated in $F_{\text{net}} = ma$.

Solution

The magnitude of the acceleration a is $a = \frac{F_{net}}{m}$. Entering known values gives

$$a = \frac{51\ \text{N}}{24\ \text{kg}}$$

Substituting the units kg m/s^2 for N (because that is what a newton is) yields

$$a = \frac{51 \text{ kg} \cdot \text{m/s}^2}{24 \text{ kg}} = 2.1 \text{ m/s}^2.$$

Discussion

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moves forward), and the vertical forces must cancel if there is to be no acceleration in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long because the person's top speed would soon be reached.

Example 2: What Rocket Thrust Accelerates This Sled?

Prior to manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets. Calculate the magnitude of force exerted by each rocket, called its thrust *T*, for the four-rocket propulsion system shown in Figure 4. The sled's initial acceleration is 49 m/s^2, the mass of the system is 2100 kg, and the force of friction opposing the motion is known to be 650 N.

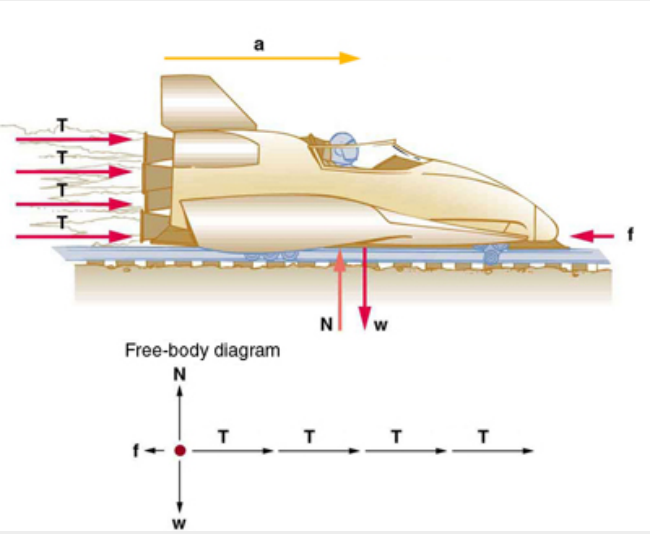


Figure 4. *A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust T. As in other situations where there is only horizontal acceleration, the vertical forces cancel. The ground exerts an upward force N on the system that is equal in magnitude and opposite in direction to its weight, w. The system here is the sled, its rockets, and rider, so none of the forces between these objects are considered. The arrow representing friction (f) is drawn larger than scale.*

Strategy

Although there are forces acting vertically and horizontally, we assume the vertical forces cancel since there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem. Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in the figure.

Solution

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines. Since we have defined the direction of the force and acceleration as acting "to the right," we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

$$F_{\text{net}} = ma,$$

where F_{net} is the net force along the horizontal direction. We can see from Figure 4 that the engine thrusts add, while friction opposes the thrust. In equation form, the net external force is

$$F_{\text{net}} = 4T - f.$$

Substituting this into Newton's second law gives

$$F_{\text{net}} = ma = 4T - f.$$

Using a little algebra, we solve for the total thrust 4*T*:

$$4T = ma + f.$$

Substituting known values yields

$$4T = ma + f = (2100 \text{ kg})(49 \text{ m/s}^2) + 650 \text{ N}.$$

So the total thrust is

$$4T = 1.0 \times 10^5 \text{ N},$$

and the individual thrusts are

$$T = \frac{1.0 \times 10^5 \text{ N}}{4} = 2.6 \times 10^4 \text{ N}.$$

Discussion

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance and the setup designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45 *g*'s. (Recall that *g*, the acceleration due to gravity, is 9.80 m/s^2.When we say that an acceleration is 45 *g*'s, it is 45 × 9.80 m/s^2, which is approximately 440 m/s^2.) While living subjects are not used any more, land speeds of 10,000 km/h have been obtained with rocket sleds. In this example, as in the preceding one, the system of interest is obvious. We will see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton's second law of motion is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature. The next section introduces the third and final law of motion.

Section Summary

- Acceleration, $\vec{a}$, is defined as a change in velocity, meaning a change in its magnitude or direction, or both.
- An external force is one acting on a system from outside the system, as opposed to internal forces, which act between components within the system.
- Newton's second law of motion states that the acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.
- In equation form, Newton's second law of motion is $\text{a} = \dfrac{F_{\text{net}}}{m}$.
- This is often written in the more familiar form: $F_{\text{net}} = ma$.
- The weight *w* of an object is defined as the force of gravity acting on an object of mass *m*. The object experiences an acceleration due to gravity *g*:

  $$w = mg.$$

- If the only force acting on an object is due to gravity, the object is in free fall.
- Friction is a force that opposes the motion past each other of objects that are touching.

Glossary

acceleration
the rate at which an object's velocity changes over a period of time

free-fall
a situation in which the only force acting on an object is the force due to gravity

friction
a force past each other of objects that are touching; examples include rough surfaces and air resistance

net external force
the vector sum of all external forces acting on an object or system; causes a mass to accelerate

Newton's second law of motion
the net external force F_{net} on an object with mass m is proportional to and in the same direction as the acceleration of the object, a, and inversely proportional to the mass; defined mathematically as $\mathrm{a} = \dfrac{F_{\text{net}}}{m}$

system
defined by the boundaries of an object or collection of objects being observed; all forces originating from outside of the system are considered external forces

weight
the force w due to gravity acting on an object of mass m; defined mathematically as: $w = mg$, where g is the magnitude and direction of the acceleration due to gravity

Conceptual Questions

1: Which statement is correct? (a) Net force causes motion. (b) Net force causes change in motion. Explain your answer and give an example.

2: Why can we neglect forces such as those holding a body together when we apply Newton's second law of motion?

3: Explain how the choice of the "system of interest" affects which forces must be considered when applying Newton's second law of motion.

4: Describe a situation in which the net external force on a system is not zero, yet its speed remains constant.

5: A system can have a nonzero velocity while the net external force on it *is* zero. Describe such a situation.

6: A rock is thrown straight up. What is the net external force acting on the rock when it is at the top of its trajectory?

7: (a) Give an example of different net external forces acting on the same system to produce different accelerations. (b) Give an example of the same net external force acting on systems of different masses, producing different accelerations. (c) What law accurately describes both effects? State it in words and as an equation.

8: If the acceleration of a system is zero, are no external forces acting on it? What about internal forces? Explain your answers.

9: If a constant, nonzero force is applied to an object, what can you say about the velocity and acceleration of the object?

10: The gravitational force on the basketball in Figure 2 is ignored. When gravity *is* taken into account, what is the direction of the net external force on the basketball—above horizontal, below horizontal, or still horizontal?

Problems & Exercises

You may assume data taken from illustrations is accurate to three digits.

1: A 63.0-kg sprinter starts a race with an acceleration of 4.20 m/s^2.What is the net external force on him?

2: If the sprinter from the previous problem accelerates at that rate for 20 m, and then maintains that velocity for the remainder of the 100-m dash, what will be his time for the race?

3: A cleaner pushes a 4.50-kg laundry cart in such a way that the net external force on it is 60.0 N. Calculate the magnitude of its acceleration.

4: Since astronauts in orbit are apparently weightless, a clever method of measuring their masses is needed to monitor their mass gains or losses to adjust diets. One way to do this is to exert a known force on an astronaut and measure the acceleration produced. Suppose a net external force of 50.0 N is exerted and the astronaut's acceleration is measured to be 0.893 m/s^2. (a) Calculate her mass. (b) By exerting a force on the astronaut, the vehicle in which they orbit experiences an equal and opposite force. Discuss how this would affect the measurement of the astronaut's acceleration. Propose a method in which recoil of the vehicle is avoided.

5: In Figure 3, the net external force on the 24-kg mower is stated to be 51 N. a) If the force of friction opposing the motion is 24 N, what force *F* (in newtons) is the person exerting on the mower? b) Suppose the mower is moving at 1.5 m/s when the force *F* is removed. How far will the mower go before stopping?

6: The same rocket sled drawn in Figure 5 is decelerated at a rate of 196 m/s^2. What force is necessary to produce this deceleration? Assume that the rockets are off. The mass of the system is 2100 kg.

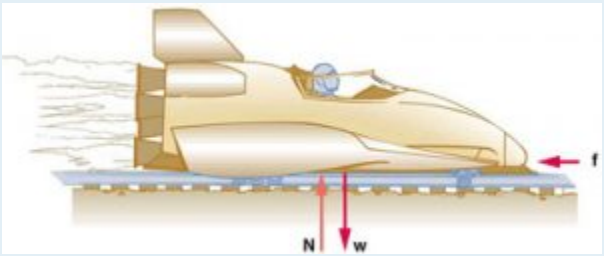


Figure 5.

7: (a) If the rocket sled shown in Figure 6 starts with only one rocket burning, what is the magnitude of its acceleration? Assume that the mass of the system is 2100 kg, the thrust T is 2.4×10^4 N, and the force of friction opposing the motion is known to be 650 N. (b) Why is the acceleration not one-fourth of what it is with all rockets burning?

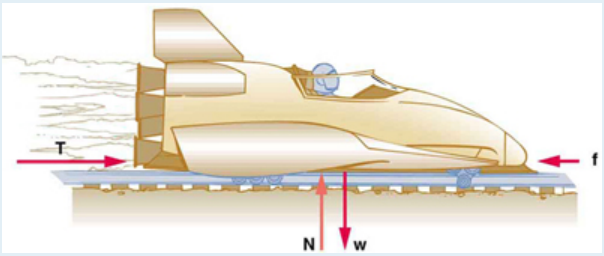


Figure 6.

8: What is the deceleration of the rocket sled if it comes to rest in 1.1 s from a speed of 1000 km/h? (Such deceleration caused one test subject to black out and have temporary blindness.)

9: Suppose two children push horizontally, but in exactly opposite directions, on a third child in a wagon. The first child exerts a force of 75.0 N, the second a force of 90.0 N, friction is 12.0 N, and the mass of the third child plus wagon is 23.0 kg. (a) What is the system of interest if the acceleration of the child in the wagon is to be calculated? (b) Draw a free-body diagram, including all forces acting on the system. (c) Calculate the acceleration. (d) What would the acceleration be if friction were 15.0 N?

10: A powerful motorcycle can produce an acceleration of 3.50 m/s^2 while traveling at 90.0 km/h. At that speed the forces resisting motion, including friction and air resistance, total 400 N. (Air resistance is analogous to air friction. It always opposes the motion of an object.) What is the magnitude of the force the motorcycle exerts backward on the ground to produce its acceleration if the mass of the motorcycle with rider is 245 kg?

11: The rocket sled shown in Figure 7 accelerates at a rate of 49.0 m/s^2. Its passenger has a mass of 75.0 kg. (a) Calculate the horizontal component of the force the seat exerts against his body. Compare this with his weight by using a ratio. (b) Calculate the direction and magnitude of the total force the seat exerts against his body.

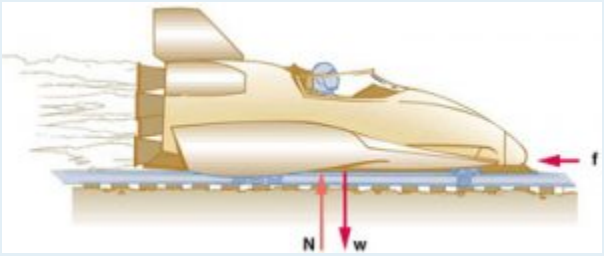


Figure 7.

12: Repeat the previous problem for the situation in which the rocket sled decelerates at a rate of 201 m/s^2. In this problem, the forces are exerted by the seat and restraining belts.

13: The weight of an astronaut plus his space suit on the Moon is only 250 N. How much do they weigh on Earth? What is the mass on the Moon? On Earth?

14: Suppose the mass of a fully loaded module in which astronauts take off from the Moon is 10,000 kg. The thrust of its engines is 30,000 N. (a) Calculate its the magnitude of acceleration in a vertical takeoff from the Moon. (b) Could it lift off from Earth? If not, why not? If it could, calculate the magnitude of its acceleration.

Solutions

Problems & Exercises

1: 265 N

3: 13.3 m/s

5 a) 75 N b) net force = 24 N backwards due to friction

Acceleration = 24N/24 kg = 1.0 m/s^2 backwards so displacement = 1.1 meters

7: (a) 12 m/s^2 (b) The acceleration is not one-fourth of what it was with all rockets burning because the frictional force is still as large as it was with all rockets burning.

9: (a) The system is the child in the wagon plus the wagon.

(b)

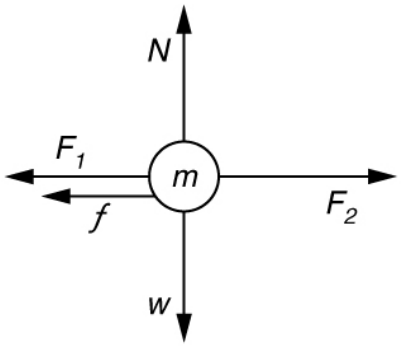


Figure 8.

(c) $\vec{a} = 0.130\ \text{m/s}^2$ in the direction of the second child's push. (d) $\vec{a} = 0.00\ \text{m/s}^2$

11: (a) 3.68 x 10^3 N. This force is 5.00 times greater than his weight. (b) 3750 N; 11.3^o above horizontal

13: 1.5 x 10^3 N, 150 kg, 150 kg

4.4 Newton's Third Law of Motion: Symmetry in Forces

Summary

- Understand Newton's third law of motion.
- Apply Newton's third law to define systems and solve problems of motion.

There is a passage in the musical *Man of la Mancha* that relates to Newton's third law of motion. Sancho, in describing a fight with his wife to Don Quixote, says, "Of course I hit her back, Your Grace, but she's a lot harder than me and you know what they say, 'Whether the stone hits the pitcher or the pitcher hits the stone, it's going to be bad for the pitcher.'" This is exactly what happens whenever one body exerts a force on another—the first also experiences a force (equal in magnitude and opposite in direction). Numerous common experiences, such as stubbing a toe or throwing a ball, confirm this. It is precisely stated in **Newton's third law of motion**.

NEWTON'S THIRD LAW OF MOTION

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts.

This law represents a certain *symmetry in nature*: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as "action-reaction," where the force exerted is the action and the force experienced as a consequence is the reaction. Newton's third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton's third law at work by taking a look at how people move about. Consider a swimmer pushing off from the side of a pool, as illustrated in Figure 1. She pushes against the pool wall with her feet and accelerates in the direction *opposite* to that of her push. The wall has exerted an equal and opposite force back on the swimmer. You might think that two equal and opposite forces would cancel, but they do not *because they act on different systems*. In this case, there are two systems that we could investigate: the swimmer or the wall. If we select the swimmer to be the system of interest, as in the figure, then $F_{\text{wall on feet}}$ is an external force on this system and affects its motion. The swimmer moves in the direction of $F_{\text{wall on feet}}$. In contrast, the force $F_{\text{feet on wall}}$ acts on the wall and not on our system of interest. Thus $F_{\text{feet on wall}}$ does not directly affect the motion of the system and does not cancel

$F_{\text{wall on feet}}$. Note that the swimmer pushes in the direction opposite to that in which she wishes to move. The reaction to her push is thus in the desired direction.

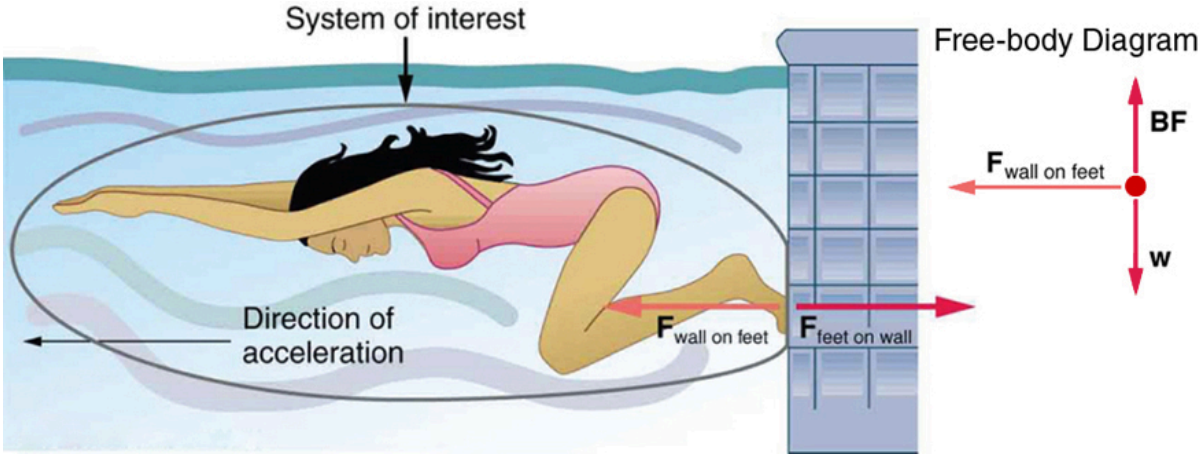


Figure 1. *When the swimmer exerts a force* $F_{\text{feet on wall}}$ *on the wall, she accelerates in the direction opposite to that of her push. This means the net external force on her is in the direction opposite to* $F_{\text{feet on wall}}$. *This opposition occurs because, in accordance with Newton's third law of motion, the wall exerts a force* $F_{\text{wall on feet}}$ *on her, equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Note that* $F_{\text{feet on wall}}$ *does not act on this system (the swimmer) and, thus, does not cancel* $F_{\text{wall on feet}}$. *Thus the free-body diagram shows only* $F_{\text{wall on feet}}$, *w, the gravitational force, and BF, the buoyant force of the water supporting the swimmer's weight. The vertical forces w and BF cancel since there is no vertical motion.*

Other examples of Newton's third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward. Helicopters create lift by pushing air down, thereby experiencing an upward reaction force. Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need. For example, the wings of a bird force air downward and backward in order to get lift and move forward. An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski. In a situation similar to Sancho's, professional cage fighters experience reaction forces when they punch, sometimes breaking their hand by hitting an opponent's body.

In another example, rockets move forward by expelling gas backward at high velocity. Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle—Newton's third law of motion. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket, as shown in Figure 2. This reaction force is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases.

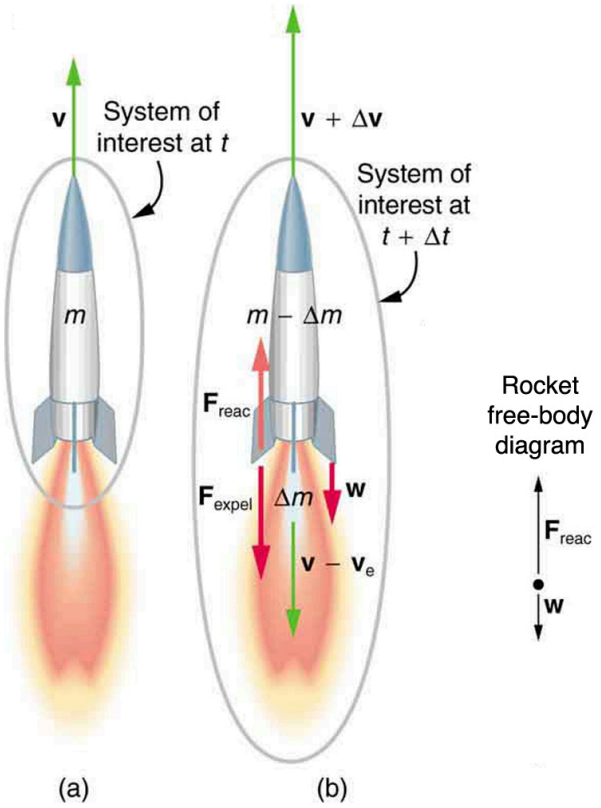


Figure 2. *(a) This rocket has a mass m and an upward velocity* ***v****. The net external force on the system is−mg, if air resistance is neglected. (b) A time Δt later the system has two main parts, the ejected gas and the remainder of the rocket. The reaction force on the rocket is what overcomes the gravitational force and accelerates it upward.*

MAKING CONNECTIONS: TAKE-HOME EXPERIMENT—PROPULSION OF A BALLOON

Hold a balloon and fill it with air. Then, let the balloon go. In which direction does the air come out of the balloon and in which direction does the balloon get propelled? If you fill the balloon with water and then let the balloon go, does the balloon's direction change? Explain your answer.

Example 1: Getting Up To Speed: Choosing the Correct System

A physics professor pushes a cart of demonstration equipment to a lecture hall, as seen in Figure 3. Her mass is 65.0 kg, the cart's is 12.0 kg, and the equipment's is 7.0 kg. Calculate the acceleration

produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.

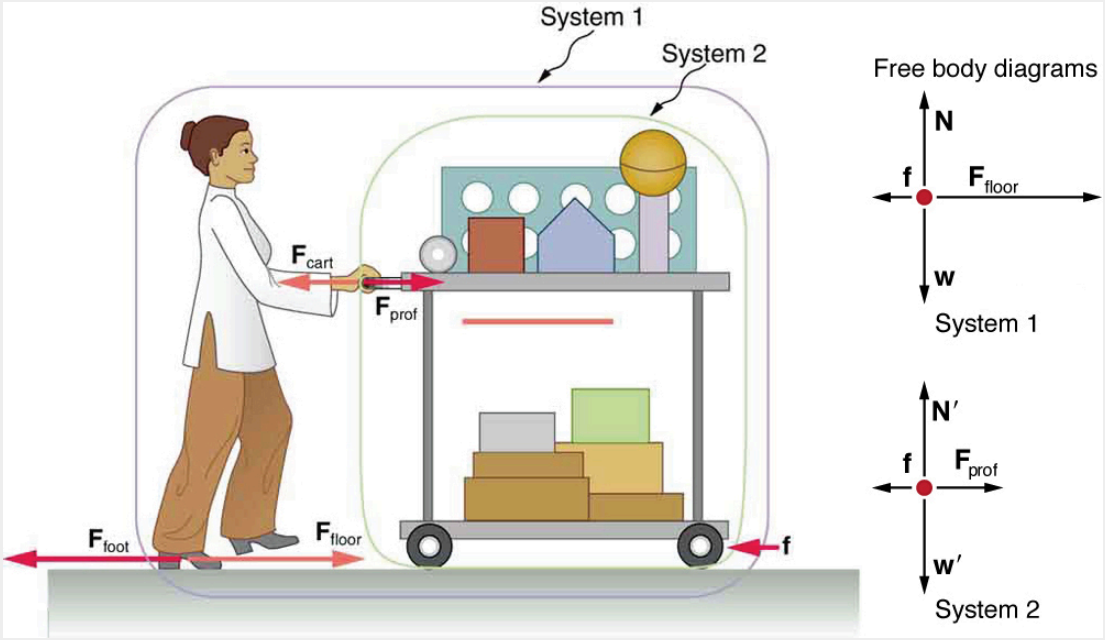


Figure 3. *A professor pushes a cart of demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for* **f***, since it is too small to draw to scale). Different questions are asked in each example; thus, the system of interest must be defined differently for each. System 1 is appropriate for Example 2, since it asks for the acceleration of the entire group of objects. Only* F_{floor} *and f are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for this example so that* F_{prof} *will be an external force and enter into Newton's second law. Note that the free-body diagrams, which allow us to apply Newton's second law, vary with the system chosen.*

Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in Figure 3. The professor pushes backward with a force F_{foot} of 150 N. According to Newton's third law, the floor exerts a forward reaction force F_{floor} of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. The problem is therefore one-dimensional along the horizontal direction. As noted, f opposes the motion and is thus in the opposite direction of F_{floor}. Note that we do not include the forces F_{prof} or F_{cart} because these are internal forces, and we do not include F_{foot} because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

Solution

Newton's second law is given by

$$a = \frac{F_{\text{net}}}{m}.$$

The net external force on System 1 is deduced from Figure 3 and the discussion above to be

$$F_{\text{net}} = F_{\text{floor}} - \text{f} = 150\ \text{N} - 24.0\ \text{N} = 126\ \text{N}.$$

The mass of System 1 is

$$m = (65.0 + 12.0 + 7.0)\ \text{kg} = 84\ \text{kg}.$$

These values of F_{net} and m produce an acceleration of

$$\begin{aligned} a &= \frac{F_{\text{net}}}{m} \\ &= \frac{126\ \text{N}}{84\ \text{kg}} = 1.5\ \text{m/s}^2 \end{aligned}$$

Discussion

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is to note that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart results in an equal and opposite force back on her. In this case both forces act on the same system and, therefore, cancel. Thus internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

Example 2: Force of the Cart—Choosing a New System

Calculate the force the professor exerts on the cart in Figure 3 using data from the previous example if needed.

Strategy

If we now define the system of interest to be the cart plus equipment (System 2 in Figure 3), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, F_{prof}, is an external force acting on System 2. F_{prof} was internal to System 1, but it is external to System 2 and will enter Newton's second law for System 2.

Solution

Newton's second law can be used to find F_{prof}. Starting with

$$a = \frac{F_{\text{net}}}{m}$$

and noting that the magnitude of the net external force on System 2 is

$$F_{\text{net}} = F_{\text{prof}} - f,$$

we solve for F_{prof}, the desired quantity:

$$F_{\text{prof}} = F_{\text{net}} + f.$$

The value of f is given, so we must calculate net F_{net}. That can be done since both the acceleration and mass of System 2 are known. Using Newton's second law we see that

$$F_{\text{net}} = ma,$$

where the mass of System 2 is 19.0 kg (m = 12.0 kg + 7.0 kg) and its acceleration was found to be a = 1.5 m/s^2 in the previous example. Thus,

$$F_{\text{net}} = ma,$$
$$F_{\text{net}} = (19.0\ \text{kg})(1.5\ \text{m/s}^2) = 29\ \text{N}.$$

Now we can find the desired force:

$$F_{\text{prof}} = F_{\text{net}} + f,$$
$$F_{\text{prof}} = 29\ \text{N} + 24.0\ \text{N} = 53\ \text{N}.$$

Discussion

It is interesting that this force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor.

The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which is not necessarily the same thing).

PHET EXPLORATIONS: GRAVITY FORCE LAB

Visualize the gravitational force that two objects exert on each other. Change properties of the objects in order to see how it changes the gravity force.

PhET Interactive Simulation

Gravity Force Lab

Section Summary

- **Newton's third law of motion** represents a basic symmetry in nature. It states: Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts.
- A **thrust** is a reaction force that pushes a body forward in response to a backward force. Rockets, airplanes, and cars are pushed forward by a thrust reaction force.

Glossary

Newton's third law of motion
whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts

thrust
a reaction force that pushes a body forward in response to a backward force; rockets, airplanes, and cars are pushed forward by a thrust reaction force

Conceptual Questions

1: When you take off in a jet aircraft, there is a sensation of being pushed back into the seat. Explain why you move backward in the seat—is there really a force backward on you? (The same reasoning explains whiplash injuries, in which the head is apparently thrown backward.)

2: A device used since the 1940s to measure the kick or recoil of the body due to heart beats is the "ballistocardiograph." What physics principle(s) are involved here to measure the force of cardiac contraction? How might we construct such a device?

3: Describe a situation in which one system exerts a force on another and, as a consequence, experiences a force that is equal in magnitude and opposite in direction. Which of Newton's laws of motion apply?

4: Why does an ordinary rifle recoil (kick backward) when fired? The barrel of a recoilless rifle is open at both ends. Describe how Newton's third law applies when one is fired. Can you safely stand close behind one when it is fired?

5: An American football lineman reasons that it is senseless to try to out-push the opposing player, since no matter how hard he pushes he will experience an equal and opposite force from the other player. Use Newton's laws and draw a free-body diagram of an appropriate system to explain how he can still out-push the opposition if he is strong enough.

6: Newton's third law of motion tells us that forces always occur in pairs of equal and opposite magnitude. Explain how the choice of the "system of interest" affects whether one such pair of forces cancels.

Problems & Exercises

1: What net external force is exerted on a 1100-kg artillery shell fired from a battleship if the shell is accelerated at 2.40×10^4 m/s^2? What is the magnitude of the force exerted on the ship by the artillery shell?

2: A brave but inadequate rugby player is being pushed backward by an opposing player who is exerting a force of 800 N on him. The mass of the losing player plus equipment is 90.0 kg, and he is accelerating at 1.20 m/s^2 backward. (a) What is the force of friction between the losing player's feet and the grass? (b) What force does the winning player exert on the ground to move forward if his mass plus equipment is 110 kg? (c) Draw a sketch of the situation showing the system of interest used to solve each part. For this situation, draw a free-body diagram and write the net force equation.

Solutions

Problems & Exercises

1: Force on shell: 2.64×10^7 N. Force exerted on ship = -2.64×10^7 N, by Newton's third law

4.5 Normal, Tension, and Spring Forces

Summary

- Define normal, tension and spring forces.
- Apply Newton's laws of motion to solve problems involving a variety of forces.
- Use trigonometric identities to resolve weight into components.

Forces are given many names, such as push, pull, thrust, lift, weight, friction, and tension. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. Several of these categories (normal forces, tensions forces, and spring forces) are discussed in this section, together with some interesting applications. Further examples of forces are discussed later in this chapter.

Normal Forces

Weight (also called force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You definitely notice that you must support the weight of a heavy object by pushing up on it when you hold it stationary, as illustrated in Figure 1(a). But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in Figure 1(b)? When the bag of dog food is placed on the table, the table actually sags slightly under the load. This would be noticeable if the load were placed on a card table, but even rigid objects deform when a force is applied to them. Unless the object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or trampoline or diving board). The greater the deformation, the greater the restoring force. So when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly, and the sag is slight so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.

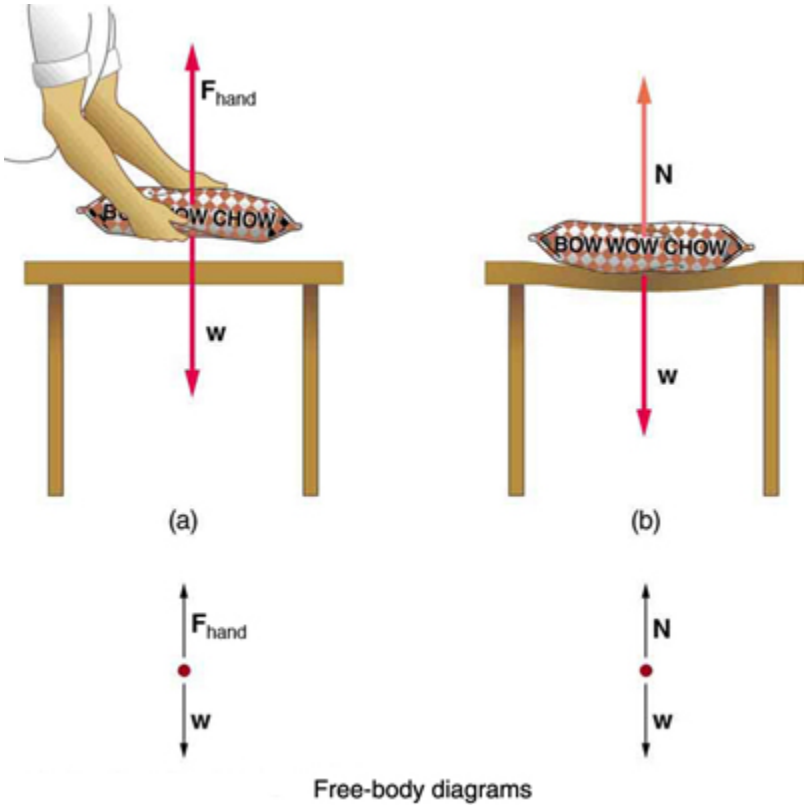


Figure 1. *(a) The person holding the bag of dog food must supply an upward force F_{hand} equal in magnitude and opposite in direction to the weight of the food w. (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring forces in the table grow as it sags until they supply a force N equal in magnitude and opposite in direction to the weight of the load.*

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a **normal force** and here is given the symbol *N*. (This is not the unit for force N.) The word normal means perpendicular to a surface. The normal force can be less than the object's weight (for example, if the object is on an incline as you will see in the next example) or it can be larger than the object's weight (for example, the normal force on a roller-coaster car at the bottom of a loop).

COMMON MISCONCEPTIONS: NORMAL FORCE (N) VS. NEWTON (N)

In this section we have introduced the quantity normal force, which is represented by the variable *N*. This should not be confused with the symbol for the newton, which is also represented by the letter N. These symbols are particularly important to distinguish because the units of a normal force (*N*) happen to be newtons (N). For example, the normal force *N* that the floor exerts on a chair might be *N* = 100 N. One important difference is that normal force is a vector, while the newton is simply a unit.

Be careful not to confuse these letters in your calculations! You will encounter more similarities among variables and units as you proceed in physics. Another example of this is the quantity work (*W*) and the unit watts (W).

Example 1: Weight on an Incline, a Two-Dimensional Problem

Consider the skier on a slope shown in Figure 2. Her mass including equipment is 60.0 kg. (a) What is her acceleration if friction is negligible? (b) What is her acceleration if friction is known to be 45.0 N? The details of the friction force are discussed in more detail in Chapter 4.6. For this question, we are only dealing the magnitude of the friction force (and our not concerned with why this force exists, or why the magnitude is the given value).

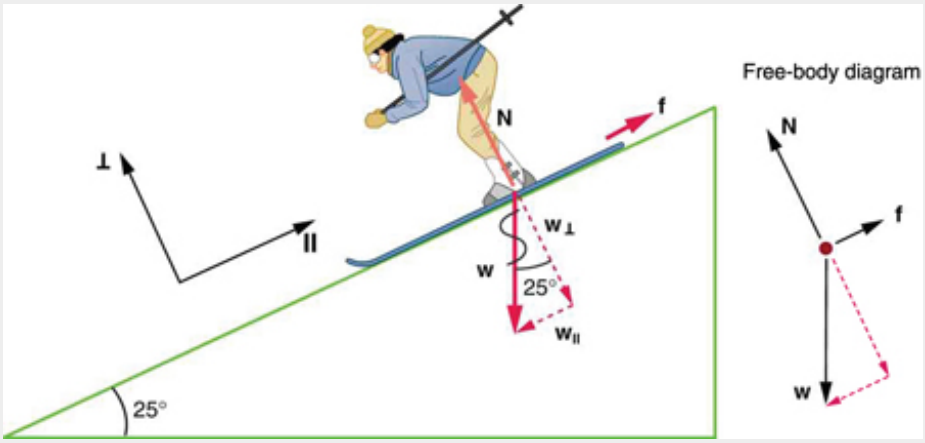


Figure 2. *Since motion and friction are parallel to the slope, it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). N is perpendicular to the slope and f is parallel to the slope, but w has components along both axes, namely* $w_\perp$ *and* $w_\parallel$*. N is equal in magnitude to the perpendicular component or* $w_\perp$*, so that there is no motion perpendicular to the slope, but f is less than* $w_\parallel$*, so that there is a downslope acceleration (along the parallel axis).*

Strategy

This is a two-dimensional problem, since the forces on the skier (the system of interest) are not parallel. The approach we have used in two-dimensional kinematics also works very well here. Choose a convenient coordinate system and project the vectors onto its axes, creating *two* connected *one*-dimensional problems to solve. The most convenient coordinate system for motion on an incline is one that has one coordinate parallel to the slope and one perpendicular to the slope. (Remember that motions along mutually perpendicular axes are independent.) We use the symbols $\perp$ and $\parallel$ to

represent perpendicular and parallel, respectively. This choice of axes simplifies this type of problem, because there is no motion perpendicular to the slope and because friction is always parallel to the surface between two objects. The only external forces acting on the system are the skier's weight, friction, and the support of the slope, respectively labeled *w*, and *N* in Figure 2. *N* is always perpendicular to the slope, and f is parallel to it. But *w* is not in the direction of either axis, and so the first step we take is to project it into components along the chosen axes, defining $w_{\parallel}$ to be the component of weight parallel to the slope and $w_{\perp}$ the component of weight perpendicular to the slope. Once this is done, we can consider the two separate problems of forces parallel to the slope and forces perpendicular to the slope.

Solution

The magnitude of the component of the weight parallel to the slope is $w_{\parallel} = w \sin\ 25^o = mg \sin\ 25^o$, and the magnitude of the component of the weight perpendicular to the slope is $w_{\perp} = w \cos\ 25^o = mg \cos\ 25^o$.

(a) Neglecting friction. Since the acceleration is parallel to the slope, we need only consider forces parallel to the slope. (Forces perpendicular to the slope add to zero, since there is no acceleration in that direction.) The forces parallel to the slope are the amount of the skier's weight parallel to the slope $w_{\parallel}$ and friction f. Using Newton's second law, with subscripts to denote quantities parallel to the slope,

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m}$$

where $F_{\text{net}\parallel} = w_{\parallel} = mg \sin\ 25^o$, assuming no friction for this part, so that

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{mg \sin\ 25^o}{m} = g \sin\ 25^o = (9.80\ \text{m/s}^2)(0.4226) = 4.14\ \text{m/s}^2$$

is the acceleration.

(b) Including friction. We now have a given value for friction, and we know its direction is parallel to the slope and it opposes motion between surfaces in contact. So the net external force is now

$$F_{\text{net}\parallel} = w_{\parallel} - f,$$

and substituting this into Newton's second law, $a_{\parallel} = \frac{F_{\text{net}}}{m}$, gives

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{w_{\parallel} - f}{m} = \frac{mg \sin\ 25^o - f}{m}.$$

We substitute known values to obtain

$$a_{\parallel} = \frac{(60.0\ \text{kg})(9.80\ \text{m/s}^2)(0.4226) - 45.0\ \text{N}}{60.0\ \text{kg}},$$

which yields

$$a_{\parallel} = 3.39\ \text{m/s}^2,$$

which is the acceleration parallel to the incline when there is 45.0 N of opposing friction.

Discussion

Since friction always opposes motion between surfaces, the acceleration is smaller when there is friction than when there is none. In fact, it is a general result that if friction on an incline is negligible, then the acceleration down the incline is $a = g \sin \theta$, *regardless of mass*. This is related to the previously discussed fact that all objects fall with the same acceleration in the absence of air resistance. Similarly, all objects, regardless of mass, slide down a frictionless incline with the same acceleration (if the angle is the same).

RESOLVING WEIGHT INTO COMPONENTS

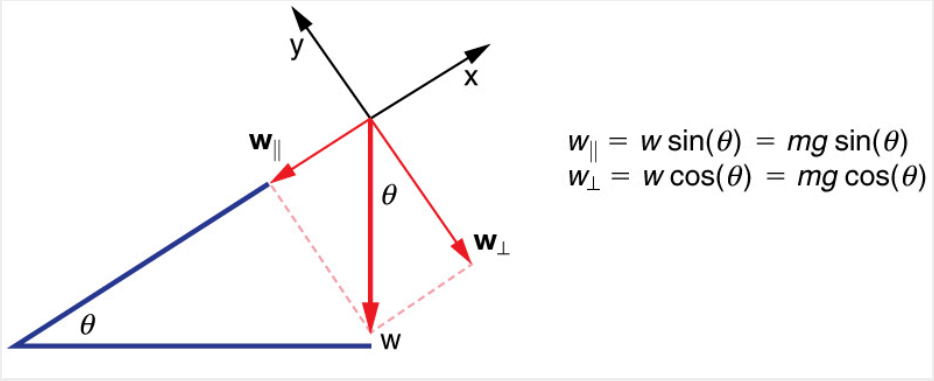


Figure 3. *An object rests on an incline that makes an angle θ with the horizontal.*

When an object rests on an incline that makes an angle θ with the horizontal, the force of gravity acting on the object is divided into two components: a force acting perpendicular to the plane, $w_{\perp}$, and a force acting parallel to the plane, $w_{\parallel}$. The perpendicular force of weight, $w_{\perp}$, is typically equal in magnitude and opposite in direction to the normal force, *N*. The force acting parallel to the plane, $w_{\parallel}$, causes the object to accelerate down the incline. The force of friction, f, opposes the motion of the object, so it acts upward along the plane.

It is important to be careful when resolving the weight of the object into components. If the angle of the incline is at an angle $\boldsymbol{\theta}$ to the horizontal, then the magnitudes of the weight components are

$$w_{\parallel} = w \sin \theta = mg \sin \theta$$

and

$$w_{\perp} = w \cos \theta = mg \cos \theta.$$

Instead of memorizing these equations, it is helpful to be able to determine them from reason. To do this, draw the right triangle formed by the three weight vectors. Notice that the θ of the incline is the same as the angle formed between w and $w_{\perp}$. Knowing this property, you can use trigonometry to determine the magnitude of the weight components:

$$\begin{aligned}\cos\ \theta &= \frac{w_{\perp}}{w} \\ w_{\perp} &= w\cos\ \theta \\ &= mg\cos\ \theta\end{aligned}$$

$$\begin{aligned}\sin\ \theta &= \frac{w_{\parallel}}{w} \\ w_{\parallel} &= w\sin\ \theta \\ &= mg\sin\ \theta\end{aligned}$$

TAKE-HOME EXPERIMENT: FORCE PARALLEL

To investigate how a force parallel to an inclined plane changes, find a rubber band, some objects to hang from the end of the rubber band, and a board you can position at different angles. How much does the rubber band stretch when you hang the object from the end of the board? Now place the board at an angle so that the object slides off when placed on the board. How much does the rubber band extend if it is lined up parallel to the board and used to hold the object stationary on the board? Try two more angles. What does this show?

Tension Forces

A **tension** is a force along the length of a medium, especially a force carried by a flexible medium, such as a rope or cable. The word “tension” comes from a Latin word meaning “to stretch.” Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called *tendons*. Any flexible connector, such as a string, rope, chain, wire, or cable, can exert pulls only parallel to its length; thus, a force carried by a flexible connector is a tension with direction parallel to the connector. It is important to understand that tension is a pull in a connector. In contrast, consider the phrase: “You can’t push a rope.” The tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope as shown in Figure 4.

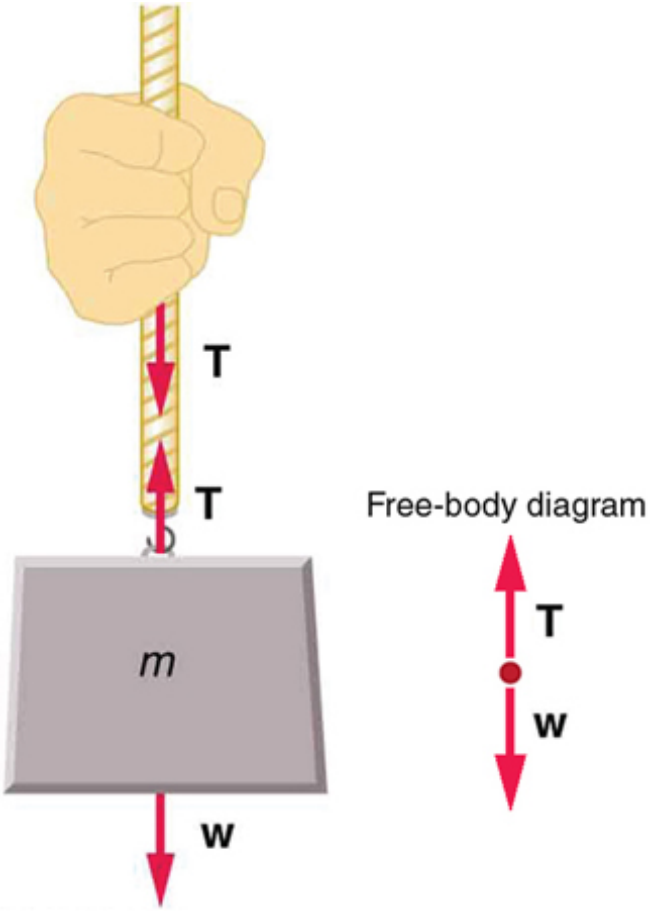


Figure 4. *When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force T, that force must be parallel to the length of the rope, as shown. The pull such a flexible connector exerts is a tension. Note that the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). This is an example of Newton's third law. The rope is the medium that carries the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.*

Tension in the rope must equal the weight of the supported mass, as we can prove using Newton's second law. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero, and thus $F_{\text{net}} = 0$. The only external forces acting on the mass are its weight w and the tension T supplied by the rope. Thus,

$$F_{\text{net}} = T - w = 0,$$

where T and w are the magnitudes of the tension and weight and their signs indicate direction, with up being positive here. Thus, just as you would expect, the tension equals the weight of the supported mass:

$$T = w = mg$$

For a 5.00-kg mass, then (neglecting the mass of the rope) we see that

$$T = mg = (5.00\ \text{kg})(9.80\ \text{m/s}^2) = 49.0\ \text{N}.$$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a finger joint, or a bicycle brake cable. If there is no friction, the tension is transmitted undiminished. Only its direction changes, and it is always parallel to the flexible connector. This is illustrated in Figure 5 (a) and (b).

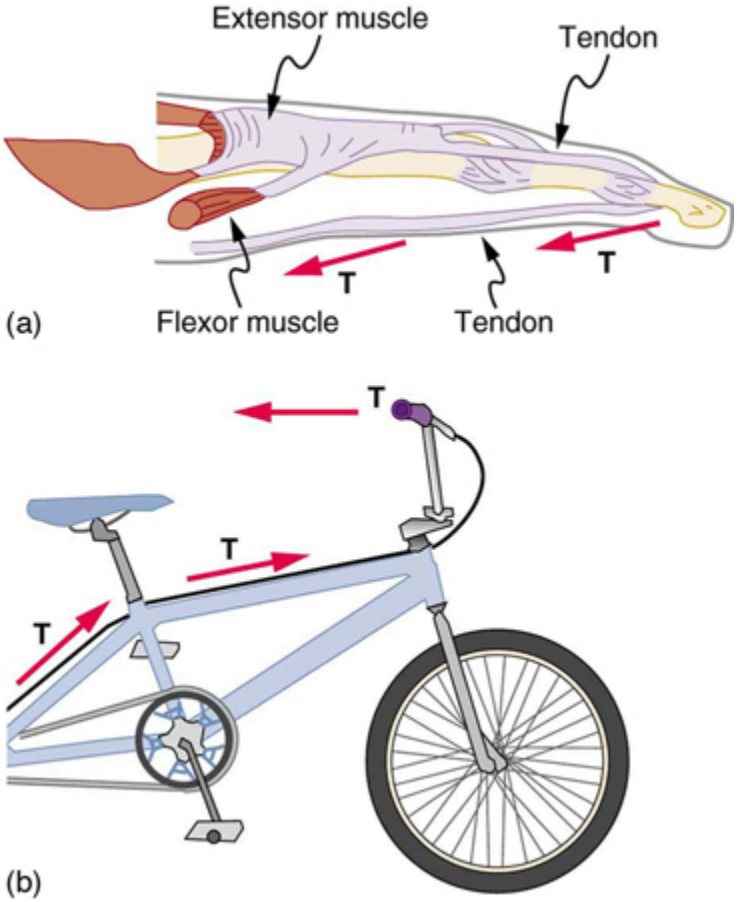


Figure 5. *(a) Tendons in the finger carry force T from the muscles to other parts of the finger, usually changing the force's direction, but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension T from the handlebars to the brake mechanism. Again, the direction but not the magnitude of T is changed.*

Example 2: What Is the Tension in a Tightrope?

Calculate the tension in the wire supporting the 70.0-kg tightrope walker shown in Figure 6.

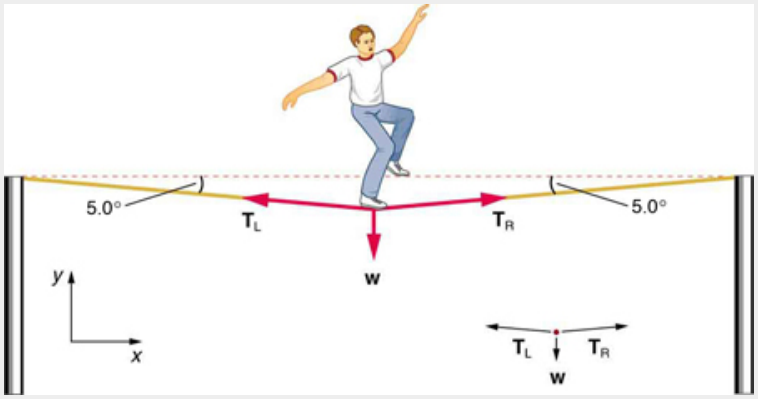


Figure 6. *The weight of a tightrope walker causes a wire to sag by 5.0 degrees. The system of interest here is the point in the wire at which the tightrope walker is standing.*

Strategy

As you can see in the figure, the wire is not perfectly horizontal (it cannot be!), but is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As usual, forces are vectors represented pictorially by arrows having the same directions as the forces and lengths proportional to their magnitudes. The system is the tightrope walker, and the only external forces acting on him are his weight ***w*** and the two tensions T_L (left tension) and T_R (right tension), as illustrated. It is reasonable to neglect the weight of the wire itself. The net external force is zero since the system is stationary. A little trigonometry can now be used to find the tensions. One conclusion is possible at the outset—we can see from part (b) of the figure that the magnitudes of the tensions T_L and T_R must be equal. This is because there is no horizontal acceleration in the rope, and the only forces acting to the left and right are T_L and T_R. Thus, the magnitude of those forces must be equal so that they cancel each other out.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case the best coordinate system has one axis horizontal and the other vertical. We call the horizontal the *x*-axis and the vertical the *y*-axis.

Solution

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to draw a new free-body diagram showing all of the horizontal and vertical components of each force acting on the system.

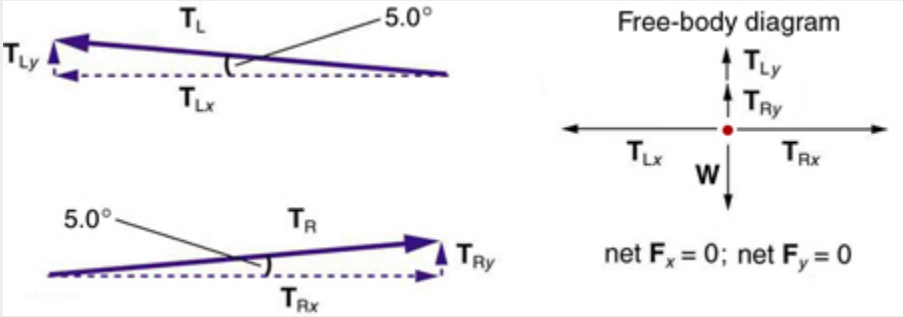


Figure 7. *When the vectors are projected onto vertical and horizontal axes, their components along those axes must add to zero, since the tightrope walker is stationary. The small angle results in T being much greater than w.*

Consider the horizontal components of the forces (denoted with a subscript x):

$$F_{\text{net}x} = T_{\text{L}x} - T_{\text{R}x}.$$

The net external horizontal force $F_{\text{net}x} = 0$, since the person is stationary. Thus,

$$F_{\text{net}x} = 0 = T_{\text{L}x} - T_{\text{R}x}$$

Now, observe Figure 7. You can use trigonometry to determine the magnitude of T_{L} and T_{R}. Notice that:

$$\cos\ 5.0^o = \frac{T_{\text{L}}x}{T_{\text{L}}}$$

$$T_{\text{L}}x = T_{\text{L}}\cos\ 5.0^o$$

$$\cos\ 5.0^o = \frac{T_{\text{R}}x}{T_{\text{R}}}$$

$$T_{\text{R}}x = T_{\text{R}}\cos\ 5.0^o$$

Equating $T_{\text{L}x}$ and $T_{\text{R}x}$:

$$T_{\text{L}}\cos\ 5.0^0 = T_{\text{R}}\cos\ 5.0^0.$$

Thus,

$$T_{\text{L}} = T_{\text{R}} = T,$$

as predicted. Now, considering the vertical components (denoted by a subscript y), we can solve for T. Again, since the person is stationary, Newton's second law implies that net $F_{\text{y}} = 0$. Thus, as illustrated in the free-body diagram in Figure 7,

$$F_{\text{net}y} = T_{\text{L}y} + T_{\text{R}y} - w = 0.$$

Observing Figure 7, we can use trigonometry to determine the relationship between $T_{\text{L}y}$, $T_{\text{R}y}$, and T. As we determined from the analysis in the horizontal direction, $T_{\text{L}} = T_{\text{R}} = T$:

\begin{align*} \sin~5.0^o &= \dfrac{T_\text{L}y}{T_\text{L}} \\[2mm] T_\text{L}y &= T_\text{L} \sin~5.0^o \\[5mm] \sin~5.0^o &= \dfrac{T_\text{R}y}{T_\text{R}} \\[2mm] T_\text{R}y &= T_\text{R} \sin~5.0^o \end{align*}

Now, we can substitute the values for $T_{\text{L}y}$ and $T_{\text{R}y}$, into the net force equation in the vertical direction:

$$\begin{aligned} F_{\text{net}y} &= T_{\text{R}y} + T_{\text{L}y} - w = 0 \\ &= T \sin\ 5.0^o + T \sin\ 5.0^o - w = 0 \\ &= 2T \sin\ 5.0^o - w = 0 \\ &= 2T \sin\ 5.0^o = w \end{aligned}$$

and

$$T = \frac{w}{2 \sin\ 5.0^o} = \frac{mg}{2 \sin\ 5.0^0},$$

so that

$$T = \frac{(70.0\ \text{kg})(9.80\ \text{m/s}^2)}{2(0.0872)},$$

and the tension is

$$T = 3900\ \text{N}.$$

Discussion

Note that the vertical tension in the wire acts as a normal force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a small fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, and so most of the tension in the wire is not used to support the weight of the tightrope walker.

If we wish to *create* a very large tension, all we have to do is exert a force perpendicular to a flexible connector, as illustrated in Figure 8. As we saw in the last example, the weight of the tightrope walker acted as a force perpendicular to the rope. We saw that the tension in the roped related to the weight of the tightrope walker in the following way:

$$T = \frac{w}{2 \sin\ \theta}.$$

We can extend this expression to describe the tension T created when a perpendicular force ($F_\perp$) is exerted at the middle of a flexible connector:

$$T = \frac{F_\perp}{2 \sin \theta}.$$

Note that θ is the angle between the horizontal and the bent connector. In this case, T becomes very large as θ approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e., θ = 0 and sin θ = 0). (See Figure 8.)

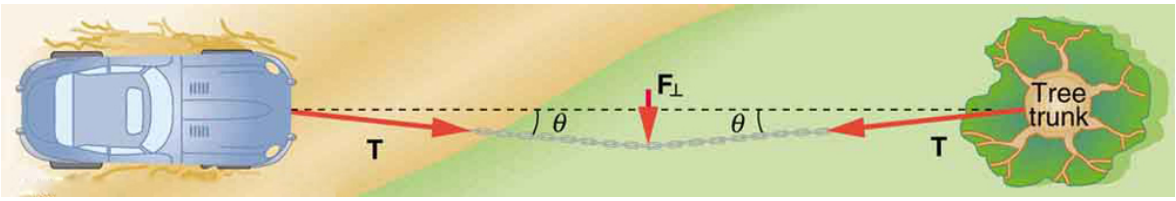


Figure 8. *We can create a very large tension in the chain by pushing on it perpendicular to its length, as shown. Suppose we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as nearly straight as possible. The tension in the chain is given by T = F[latex]\perp[/latex] / 2 sin(θ); since θ is small, T is very large. This situation is analogous to the tightrope walker shown in Figure 6, except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where F[latex]\perp[/latex] is applied.*

Figure 9. *Unless an infinite tension is exerted, any flexible connector—such as the chain at the bottom of the picture—will sag under its own weight, giving a characteristic curve when the weight is evenly distributed along the length. Suspension bridges—such as the Golden Gate Bridge shown in this image—are essentially very heavy flexible connectors. The weight of the bridge is evenly distributed along the length of flexible connectors, usually cables, which take on the characteristic shape. (credit: Leaflet, Wikimedia Commons)*

Spring Forces and Hooke's Law

Forces can affect an object's shape. If a bulldozer pushes a car into a wall, the car will not move but it will noticeably change shape. A change in shape due to the application of a force is a **deformation**. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation

is proportional to the force—that is, for small deformations, **Hooke's law** is obeyed. In equation form, Hooke's law is given by

$$|F| = k\Delta L,$$

where |F| is the magnitude of the force that produced the amount of deformation ΔL (the change in length, for example), and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force. Note that this force is a function of the deformation ΔL — it is not constant as a kinetic friction force is. Sometimes we use Δx instead of ΔL. The deformation can be along any axis. Rearranging this to

$$\Delta L = \frac{F}{k}$$

makes it clear that the deformation is proportional to the applied force. Figure 9 shows the Hooke's law relationship between the extension ΔL of a spring or of a human bone. For metals or springs, the straight line region in which Hooke's law pertains is much larger. Bones are brittle and the elastic region is small and the fracture abrupt. Eventually a large enough stress to the material will cause it to break or fracture. **Tensile strength** is the breaking stress that will cause permanent deformation or fracture of a material.

HOOKE'S LAW

$$F = k\Delta L,$$

where ΔL is the amount of deformation (the change in length, for example) produced by the force F, and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force.

$$\Delta L = \frac{F}{k}$$

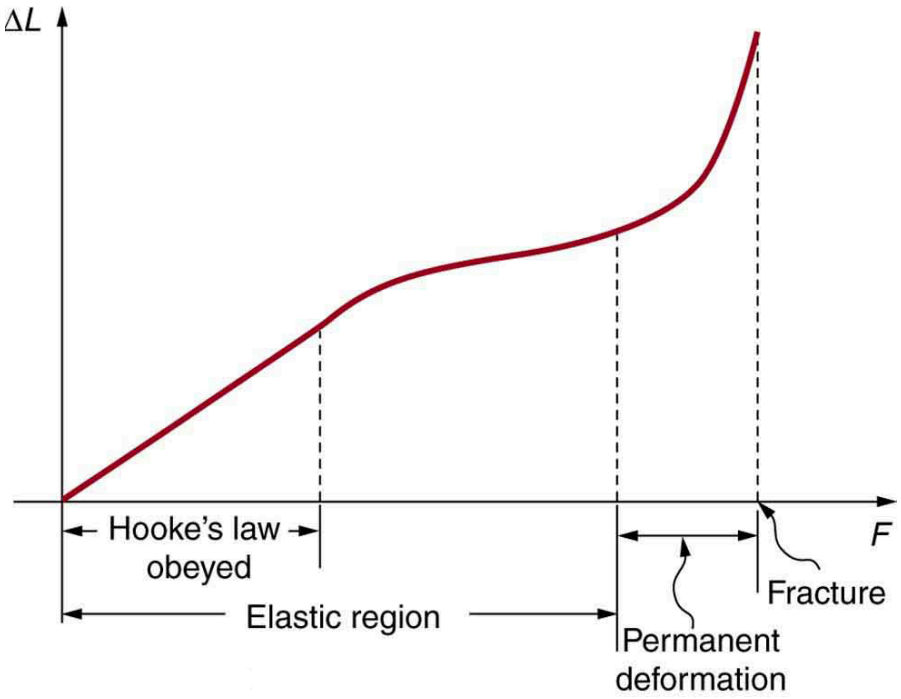


Figure 9. *A graph of deformation ΔL versus applied force F. The straight segment is the linear region where Hooke's law is obeyed. The slope of the straight region is 1 / k. For larger forces, the graph is curved but the deformation is still elastic—ΔL will return to zero if the force is removed. Still greater forces permanently deform the object until it finally fractures. The shape of the curve near fracture depends on several factors, including how the force F is applied. Note that in this graph the slope increases just before fracture, indicating that a small increase in F is producing a large increase in L near the fracture.*

The proportionality constant k depends upon a number of factors for the material. For example, a guitar string made of nylon stretches when it is tightened, and the elongation ΔL is proportional to the force applied (at least for small deformations). Thicker nylon strings and ones made of steel stretch less for the same applied force, implying they have a larger k (see Figure 10). Finally, all three strings return to their normal lengths when the force is removed, provided the deformation is small. Most materials will behave in this manner if the deformation is less than about 0.1% or about 1 part in 10^3.

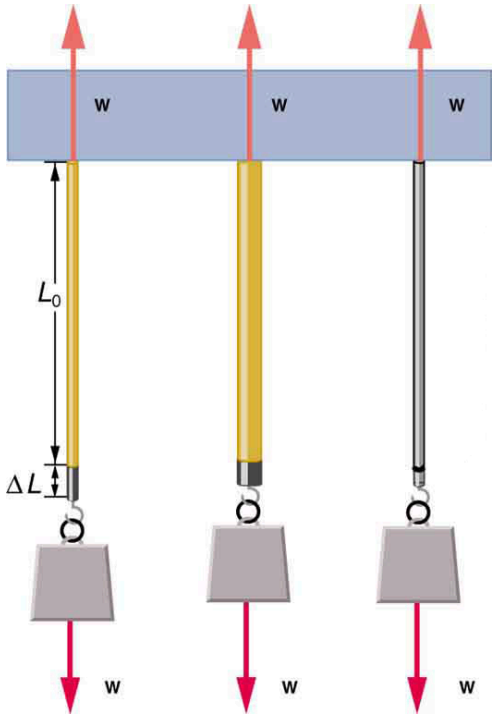


Figure 10. *The same force, in this case a weight (w), applied to three different guitar strings of identical length produces the three different deformations shown as shaded segments. The string on the left is thin nylon, the one in the middle is thicker nylon, and the one on the right is steel.*

STRETCH YOURSELF A LITTLE

How would you go about measuring the proportionality constant *k* of a rubber band? If a rubber band stretched 3 cm when a 100-g mass was attached to it, then how much would it stretch if two similar rubber bands were attached to the same mass—even if put together in parallel or alternatively if tied together in series?

As we have seen, the application of a force can cause the deformation of an object. However, Newton's Third Law says that the object will also exert a force (called a *restoring force*). For example, consider the mass and spring system shown in Figure 11. The mass experiences a restoring force (or spring force), which is given by Hooke's Law. The force is

$$\vec{F}_s = -k\Delta\vec{x},$$

where $\vec{F}_s$ is the spring force, *k* is the spring constant and $\Delta\vec{x}$ is the stretch or compression of the spring measured from its unstretched length. Note that the spring force is a *vector*, where the sign of the force

indicates the direction. Likewise, the stretch or compression of the spring is also a vector, and indicates the direction of the stretch or compression. The negative sign in the equation means that the spring forces *acts in the opposite direction to the stretch or compression.* This relationship between the spring force and the direction of the stretch or compression is shown in Figure 3. In many cases, the stretch or compression will simply be written as *x*, so the equation of the spring force is written as $\vec{F}_s = -kx$. However, *x* still stands for a *change* in length.

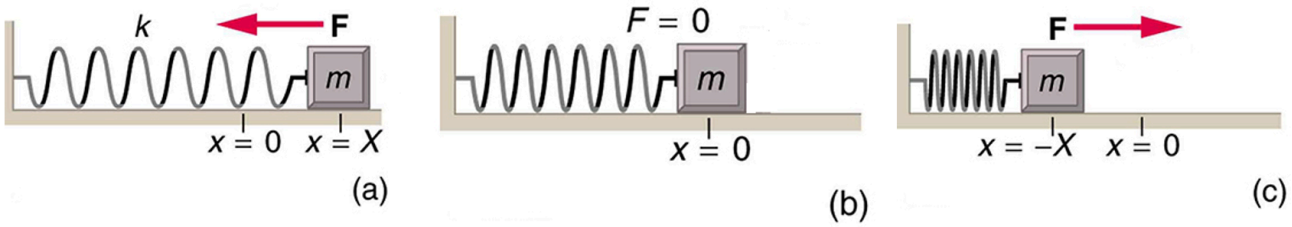


Figure 11.*The spring exerts a force on the mass due to Hooke's Law. In (a), the spring is stretched in the positive direction (to the right), so the restoring forces acts in the negative direction (to the left). In (c), the spring is stretched in the negative direction (to the left), so the restoring force acts in the positive direction (to the right). In (b), the spring is not stretched so there is no restoring force.*

In some applications, it is better to work with the *magnitude* of the spring force

$$|F| = k|x|$$

where **|*x*|** is the magnitude of the stretch or compression. To determine the direction of this force, remember that the spring forces *acts in the opposite direction to the stretch or compression.* This can be useful when determining the direction of the force on a free body diagram that involves a spring force.

Optional: Real Forces and Inertial Frames

There is another distinction among forces in addition to the types already mentioned. Some forces are real, whereas others are not. *Real forces* are those that have some physical origin, such as the gravitational pull. Contrastingly, *fictitious forces* are those that arise simply because an observer is in an accelerating frame of reference, such as one that rotates (like a merry-go-round) or undergoes linear acceleration (like a car slowing down). For example, if a satellite is heading due north above Earth's northern hemisphere, then to an observer on Earth it will appear to experience a force to the west that has no physical origin. Of course, what is happening here is that Earth is rotating toward the east and moves east under the satellite. In Earth's frame this looks like a westward force on the satellite, or it can be interpreted as a violation of Newton's first law (the law of inertia). An **inertial frame of reference** is one in which all forces are real and, equivalently, one in which Newton's laws have the simple forms given in this chapter.

Earth's rotation is slow enough that Earth is nearly an inertial frame. You ordinarily must perform precise experiments to observe fictitious forces and the slight departures from Newton's laws, such as the effect just described. On the large scale, such as for the rotation of weather systems and ocean currents, the effects can be easily observed.

The crucial factor in determining whether a frame of reference is inertial is whether it accelerates or rotates relative to a known inertial frame. Unless stated otherwise, all phenomena discussed in this text are considered in inertial frames.

All the forces discussed in this section are real forces, but there are a number of other real forces, such as lift and thrust, that are not discussed in this section. They are more specialized, and it is not necessary to discuss every type of force. It is natural, however, to ask where the basic simplicity we seek to find in physics is in the long list of forces. Are some more basic than others? Are some different manifestations of the same underlying force? The answer to both questions is yes, as will be seen in the next (extended) section and in the treatment of modern physics later in the text.

PHET EXPLORATIONS: FORCES IN 1 DIMENSION

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. View a free-body diagram of all the forces (including gravitational and normal forces).

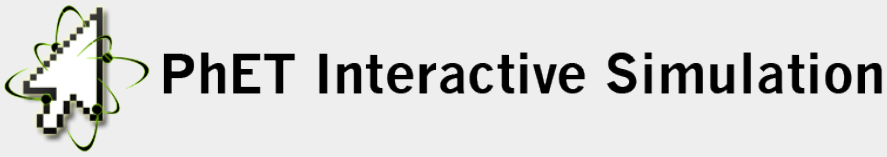


Figure 10. Forces in 1 Dimension

Section Summary

- When objects rest on a surface, the surface applies a force to the object that supports the weight of the object. This supporting force acts perpendicular to and away from the surface. It is called a normal force, *N*.
- When objects rest on a non-accelerating horizontal surface, the magnitude of the normal force is equal to the weight of the object:
$$N = mg.$$
- When objects rest on an inclined plane that makes an angle θ with the horizontal surface, the weight of the object can be resolved into components that act perpendicular ($w_{\perp}$) and parallel ($w_{\parallel}$) to the surface of the plane. These components can be calculated using:
$$w_{\parallel} = w \sin\ \theta = mg \sin\ \theta$$
$$w_{\perp} = w \cos\ \theta = mg \cos\ \theta.$$
- The pulling force that acts along a stretched flexible connector, such as a rope or cable, is called tension, *T*. When a rope supports the weight of an object that is at rest, the tension in the rope is equal to the weight of the object:

$$T = mg.$$

- Hooke's law is given by

$$F = k\Delta L \text{ or } F = k\Delta x$$

  where ΔL is the amount of deformation (the change in length), F is the applied force, and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force.

- The spring force is the restoring force provided by a spring. It has a magnitude of $|F| = k|x|$ and acts in the opposite direction to the stretch or compression.
- In any inertial frame of reference (one that is not accelerated or rotated), Newton's laws have the simple forms given in this chapter and all forces are real forces having a physical origin.

Glossary

inertial frame of reference
: a coordinate system that is not accelerating; all forces acting in an inertial frame of reference are real forces, as opposed to fictitious forces that are observed due to an accelerating frame of reference

normal force
: the force that a surface applies to an object to support the weight of the object; acts perpendicular to the surface on which the object rests

tension
: the pulling force that acts along a medium, especially a stretched flexible connector, such as a rope or cable; when a rope supports the weight of an object, the force on the object due to the rope is called a tension force

Conceptual Questions

1: If a leg is suspended by a traction setup as shown in Figure 11, what is the tension in the rope?

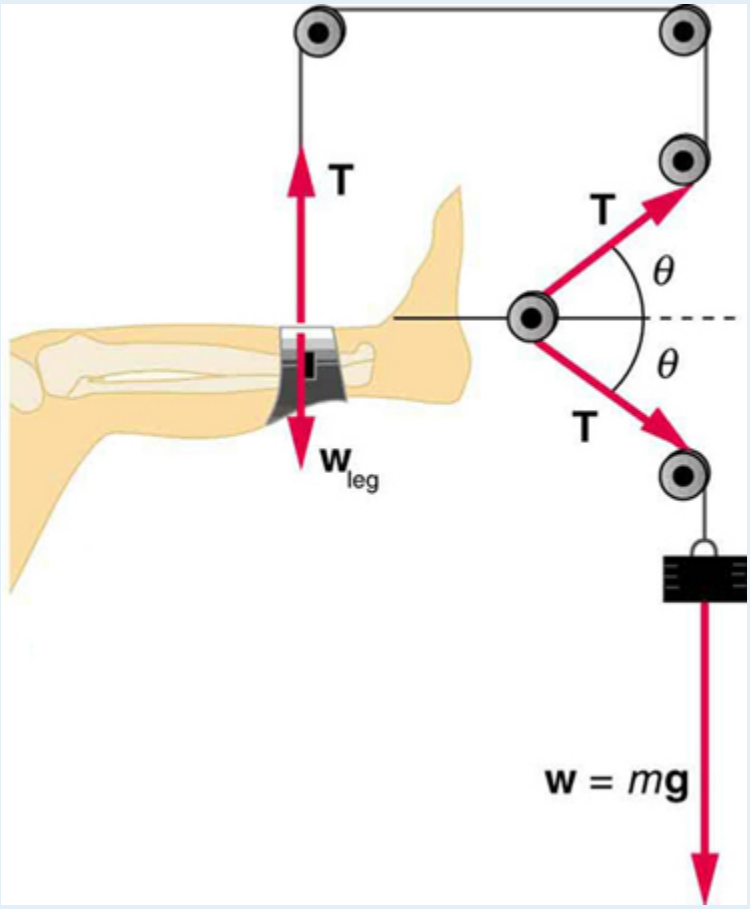


Figure 12. *A leg is suspended by a traction system in which wires are used to transmit forces. Frictionless pulleys change the direction of the force T without changing its magnitude.*

2: In a traction setup for a broken bone, with pulleys and rope available, how might we be able to increase the force along the tibia using the same weight? (See Figure 12.) (Note that the tibia is the shin bone shown in this image.)

3: How is a bow and arrow like a spring?

Problems & Exercises

1: Two teams of nine members each engage in a tug of war. Each of the first team's members has an average mass of 68 kg and exerts an average force of 1350 N horizontally. Each of the second team's members has an average mass of 73 kg and exerts an average force of 1365 N horizontally. (a) What is magnitude of the acceleration of the two teams? (b) What is the tension in the section of rope between the teams?

2: What force does a trampoline have to apply to a 45.0-kg gymnast to accelerate her straight up at 7.50 m/s^2? Note that the answer is independent of the velocity of the gymnast—she can be moving either up or down, or be stationary.

3: (a) Calculate the tension in a vertical strand of spider web if a spider of mass 8.00×10^{-5} kg hangs motionless on it. (b) Calculate the tension in a horizontal strand of spider web if the same spider sits motionless in the middle of it much like the tightrope walker in Figure 6. The strand sags at an angle of 12° below the horizontal. Compare this with the tension in the vertical strand (find their ratio).

4: Suppose a 60.0-kg gymnast climbs a rope. (a) What is the tension in the rope if he climbs at a constant speed? (b) What is the tension in the rope if he accelerates upward at a rate of 1.50 m/s^2?

5: Show that, as stated in the text, a force $F_\perp$ exerted on a flexible medium at its center and perpendicular to its length (such as on the tightrope wire in Figure 6) gives rise to a tension of magnitude $T = \dfrac{F_\perp}{2 \sin \theta}$.

6: A mass is suspended from a vertical spring such that it exerts a downwards force of magnitude 5.02 newtons on the spring. The spring stretches 0.0456 metres when the mass is attached to it. What is the spring constant in newton/metres? Hint: Suspended means hanging in the air not moving. What is the net force on a mass that is not moving?

7: The shock absorbers on my car have a spring constant of 97,722 newton/metres. When a certain person sits in the car they exert a downwards force of 987 newtons on the car. How far "down" does the car move after they sit in it? (Watch the movie "The Italian Job" and see how they used this type of physics to figure out which truck was actually carrying the gold.)

8: A spring balance has a spring constant of 34.5 newton/metres and it stretches 3.21 centimetres when an unknown mass is attached to it. What is the force in newtons exerted on the spring by the unknown mass?

9: If an applied force of 12 newtons stretches a certain spring 4.0 centimetres, how much stretch will occur for an applied force of 18 newtons?

10: Consider the baby being weighed in Figure 13. (a) What is the mass of the child and basket if a scale reading of 55 N is observed? (b) What is the tension T_1 in the cord attaching the baby to the scale? (c) What is the tension T_2 in the cord attaching the scale to the ceiling, if the scale has a mass of 0.500 kg? (d) Draw a sketch of the situation indicating the system of interest used to solve each part. The masses of the cords are negligible.

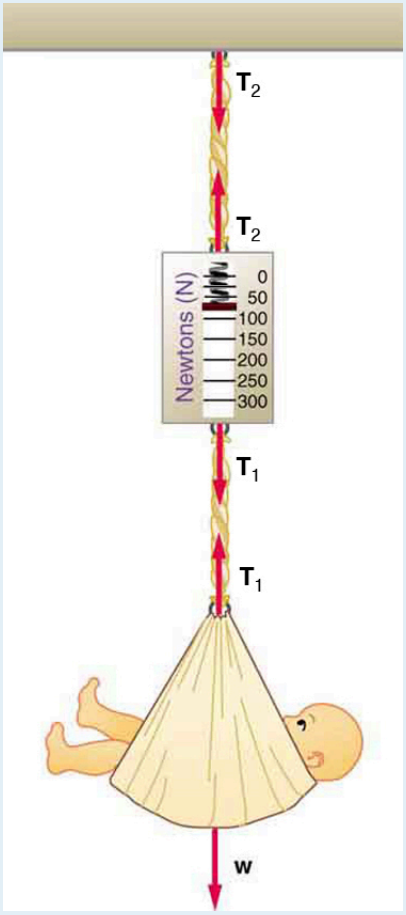


Figure 13. *A baby is weighed using a spring scale.*

Solutions

Problems & Exercises

1: (a) 0.11 m/s^2 (b) 1.2 x 10^4 N

3: (a) 7.84 x 10^{-4} N (b) 1.89 x 10^{-3} N. This is 2.41 times the tension in the vertical strand.

5: Newton's second law applied in vertical direction gives

$$F_y = F - 2T\sin\ \theta = 0$$

$$F = 2T\sin\ \theta$$

$$T = \frac{F}{2\sin\ \theta}$$

7: 0.0101 m = 1.01 cm

9: 6.0 cm

4.6 Friction

Summary

- Discuss the general characteristics of friction.
- Describe the various types of friction.
- Calculate the magnitude of static and kinetic friction.

Friction is a force that is around us all the time that opposes relative motion between systems in contact but also allows us to move (which you have discovered if you have ever tried to walk on ice). While a common force, the behavior of friction is actually very complicated and is still not completely understood. We have to rely heavily on observations for whatever understandings we can gain. However, we can still deal with its more elementary general characteristics and understand the circumstances in which it behaves.

FRICTION

Friction is a force that opposes relative motion between systems in contact.

One of the simpler characteristics of friction is that it is parallel to the contact surface between systems and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, **static friction** can act between them; the static friction is usually greater than the kinetic friction between the objects.

KINETIC FRICTION

If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you may push harder and

harder on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. But if you finally push hard enough, the crate seems to slip suddenly and starts to move. Once in motion it is easier to keep it in motion than it was to get it started, indicating that the kinetic friction force is less than the static friction force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it to be easier to get the crate started and keep it going (as you might expect).

Figure 1 is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows them to be rough. So when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, break off the points, or do both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explain the dependence of friction on the nature of the substances. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.

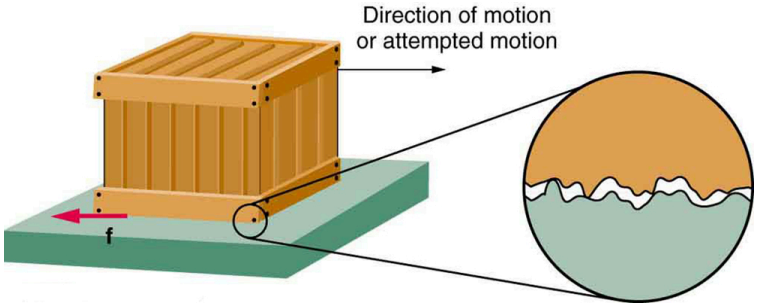


Figure 1. *Frictional forces, such as f, always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. In order for the object to move, it must rise to where the peaks can skip along the bottom surface. Thus a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. Such adhesive forces also depend on the substances the surfaces are made of, explaining, for example, why rubber-soled shoes slip less than those with leather soles.*

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for when there is motion (kinetic friction).

When there is no motion between the objects, the **magnitude of static friction** f_s is

$$f_s \leq \mu_s N,$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force (the force perpendicular to the surface).

MAGNITUDE OF STATIC FRICTION

Magnitude of static friction f_s is

$$f_s \leq \mu_s N,$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force.

The symbol ≤ means *less than or equal to*, implying that static friction can have a minimum value of 0 (in other words, there can be no static friction) and a maximum value of $\mu_s N$. Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds $f_{s(max)}$, the object will move. Thus

$$f_{s(max)} = \mu_s N.$$

Once an object is moving, the **magnitude of kinetic friction** f_k is given by

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction. A system in which $f_k = \mu_k N$ is described as a system in which *friction behaves simply*.

MAGNITUDE OF KINETIC FRICTION

The magnitude of kinetic friction f_k is given by

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction.

As seen in Table 1, the coefficients of kinetic friction are less than their static counterparts. That values of μ in Table 1 are stated to only one or, at most, two digits is an indication of the approximate description of friction given by the above two equations.

System	Static friction, μ_s	Kinetic friction, μ_k
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.7	0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.4	0.02

Table 1. Coefficients of Static and Kinetic Friction.

The equations given earlier include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force would be equal to its weight, $W = mg$ = (100 kg)(9.80 m/s^2) = 980 N, perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than $f_{s(max)} = \mu_s N$ = (0.45)(980 N) = 440 N to move the crate. Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only 290 N ($f_k = \mu_k N$ = (0.30)(980 N) = 290 N) would keep it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unit less quantity with a magnitude usually between 0 and 1.0. The coefficient of the friction depends on the two surfaces that are in contact.

TAKE-HOME EXPERIMENT

Find a small plastic object (such as a food container) and slide it on a kitchen table by giving it a gentle tap. Now spray water on the table, simulating a light shower of rain. What happens now when you give the object the same-sized tap? Now add a few drops of (vegetable or olive) oil on the surface of the water and give the same tap. What happens now? This latter situation is particularly important for drivers to note, especially after a light rain shower. Why?

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost glassy surface. The joints also produce a fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint (Figure 2). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.

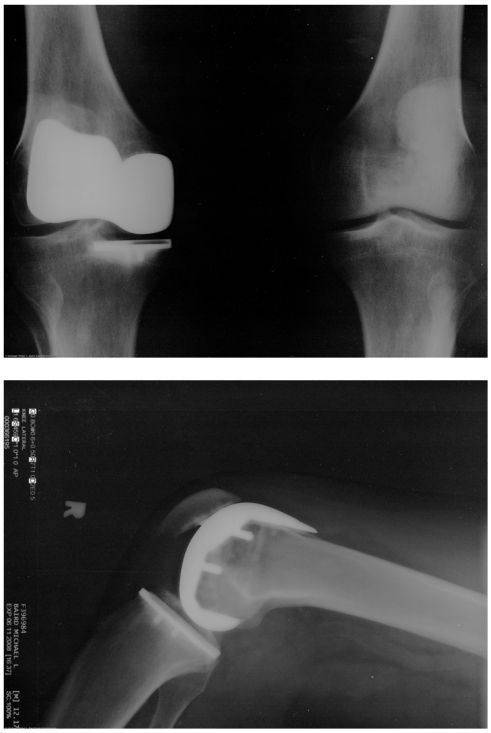


Figure 2. *Artificial knee replacement is a procedure that has been performed for more than 20 years. In this figure, we see the post-op x rays of the right knee joint replacement. (credit: Mike Baird, Flickr)*

Other natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Artificial lubricants are also common in hospitals and doctor’s clinics. For example, when ultrasonic imaging is carried out, the gel that couples the transducer to the skin also serves to to lubricate the surface between the transducer and the skin—thereby reducing the coefficient of friction between the two surfaces. This allows the transducer to mover freely over the skin.

Example 1: Skiing Exercise

A skier with a mass of 62 kg is sliding down a snowy slope. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

Strategy

The magnitude of kinetic friction was given in to be 45.0 N. Kinetic friction is related to the normal force N as $f_k = \mu_k N$; thus, the coefficient of kinetic friction can be found if we can find the normal force of the skier on a slope. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should equal the component of the skier's weight perpendicular to the slope. (See the skier and free-body diagram in Figure 3.)

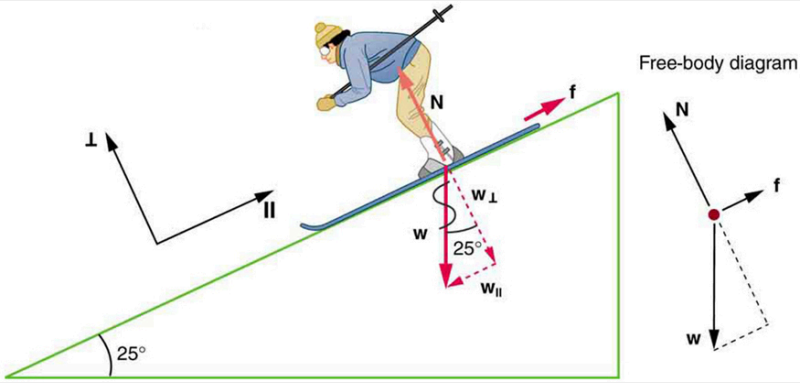


Figure 3. *The motion of the skier and friction are parallel to the slope and so it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). N (the normal force) is perpendicular to the slope, and f (the friction) is parallel to the slope, but w (the skier's weight) has components along both axes, namely $w_\perp$ and $w_\parallel$. N is equal in magnitude to $w_\perp$, so there is no motion perpendicular to the slope. However, f is less than $w_\parallel$ in magnitude, so there is acceleration down the slope (along the x-axis).*

That is,

$$N = w_\perp = w \cos\ 25^o = mg \cos\ 25^o.$$

Substituting this into our expression for kinetic friction, we get

$$f_k = \mu_k mg \cos\ 25^0,$$

which can now be solved for the coefficient of kinetic friction μ_k.

Solution

Solving for μ_k gives

$$\mu_k = \frac{f_k}{N} = \frac{f_k}{w \cos\ 25^o} = \frac{f_k}{mg \cos\ 25^o}$$

Substituting known values on the right-hand side of the equation,

$$\mu_{\text{k}} = \frac{45.0\ \text{N}}{(62\ \text{kg})(9.80\ \text{m/s}^2)(0.906)} = 0.082.$$

Discussion

This result is a little smaller than the coefficient listed in Table 1 for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass m slides down a slope that makes an angle θ with the horizontal, friction is given by $f_{\text{k}} = \mu_{\text{k}} mg \cos \theta$. All objects will slide down a slope with constant acceleration under these circumstances. Proof of this is left for this chapter's Problems and Exercises.

TAKE-HOME EXPERIMENT

An object will slide down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in Example 1, the kinetic friction on a slope $f_{\text{k}} = \mu_{\text{k}} mg \cos \theta$. The component of the weight down the slope is equal to $mg \sin \theta$ (see the free-body diagram in Figure 3). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out:

$$f_{\text{k}} = Fg_{\text{x}}$$
$$\mu_{\text{k}} mg \cos\ \theta = mg \sin\ \theta.$$

Solving for μ_{k}, we find that

$$\mu_{\text{k}} = \frac{mg \sin\ \theta}{mg \cos\ \theta} = \tan\ \theta.$$

Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find μ_{k}. Note that the coin will not start to slide at all until an angle greater than θ is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Discuss how this may affect the value for μ_{k} and its uncertainty.

We have discussed that when an object rests on a horizontal surface, there is a normal force supporting it equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force.

MAKING CONNECTIONS: SUBMICROSCOPIC EXPLANATIONS OF FRICTION

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) to heat.

Figure 4 illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area since only high spots touch. When a greater normal force is exerted, the actual contact area increases, and it is found that the friction is proportional to this area.

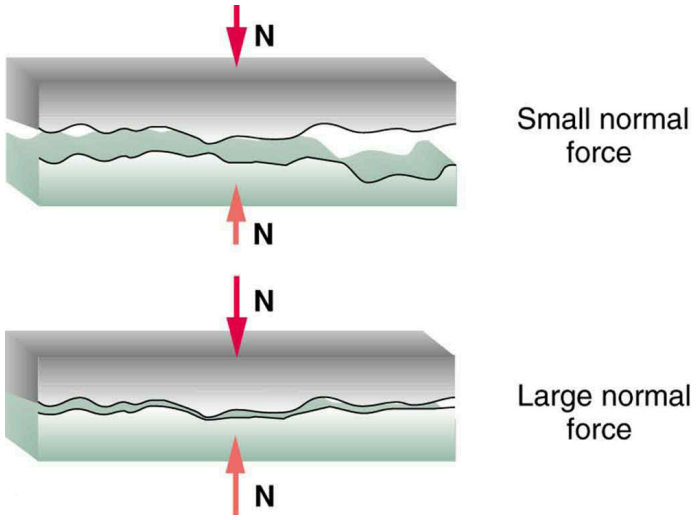


Figure 4. *Two rough surfaces in contact have a much smaller area of actual contact than their total area. When there is a greater normal force as a result of a greater applied force, the area of actual contact increases as does friction.*

But the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur between atoms and molecules on the surfaces. Figure 5 shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which will be discussed later in this chapter. The variation in shear stress is remarkable (more

than a factor of 10^{12}) and difficult to predict theoretically, but shear stress is yielding a fundamental understanding of a large-scale phenomenon known since ancient times—friction.

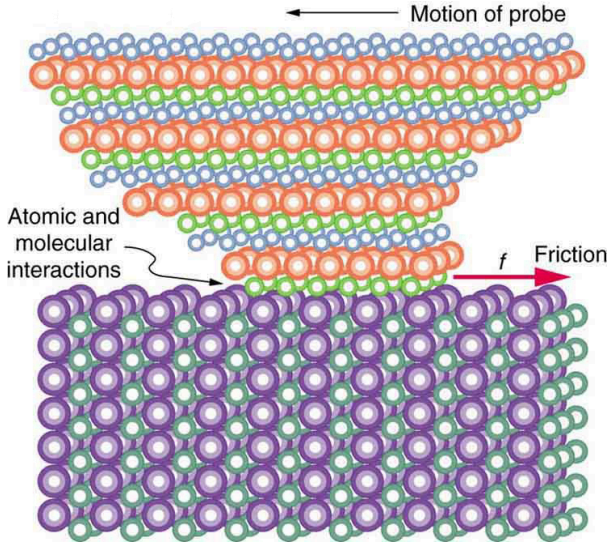


Figure 5. *The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.*

PHET EXPLORATIONS: FORCES AND MOTION

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. Draw a free-body diagram of all the forces (including gravitational and normal forces).

PhET Interactive Simulation

Figure 6. *Forces and Motion*

Section Summary

- Friction is a contact force between systems that opposes the motion or attempted motion between them. Simple friction is proportional to the normal force N pushing the systems together. (A normal force is always perpendicular to the contact surface between systems.) Friction depends on both of the materials involved. The magnitude of static friction f_s

between systems stationary relative to one another is given by

$$f_{\text{s}} \leq \mu \, \text{N},$$

where μ_{s} is the coefficient of static friction, which depends on both of the materials.

- The kinetic friction force f_{k} between systems moving relative to one another is given by

$$f_{\text{k}} = \mu_{\text{k}} \, \text{N},$$

where μ_{k} is the coefficient of kinetic friction, which also depends on both materials.

Glossary

friction
a force that opposes relative motion or attempts at motion between systems in contact

kinetic friction
a force that opposes the motion of two systems that are in contact and moving relative to one another

static friction
a force that opposes the motion of two systems that are in contact and are not moving relative to one another

magnitude of static friction
$f_{\text{s}} \leq \mu_{\text{s}} N$, where μ_{s} is the coefficient of static friction and N is the magnitude of the normal force

magnitude of kinetic friction
$f_{\text{k}} = \mu_{\text{k}} N$, where μ_{k} is the coefficient of kinetic friction

Conceptual Questions

1: Define normal force. What is its relationship to friction when friction behaves simply?

2: The glue on a piece of tape can exert forces. Can these forces be a type of simple friction? Explain, considering especially that tape can stick to vertical walls and even to ceilings.

3: When you learn to drive, you discover that you need to let up slightly on the brake pedal as you come to a stop or the car will stop with a jerk. Explain this in terms of the relationship between static and kinetic friction.

4: When you push a piece of chalk across a chalkboard, it sometimes screeches because it rapidly alternates between slipping and sticking to the board. Describe this process in more detail, in particular explaining how it is related to the fact that kinetic friction is less than static friction. (The same slip-grab process occurs when tires screech on pavement.)

Problems & Exercises

1: A physics major is cooking breakfast when he notices that the frictional force between his steel spatula and his Teflon frying pan is only 0.200 N. Knowing the coefficient of kinetic friction between the two materials, he quickly calculates the normal force. What is it?

2: (a) When rebuilding her car's engine, a physics major must exert 300 N of force to insert a dry steel piston into a steel cylinder. What is the magnitude of the normal force between the piston and cylinder? (b) What is the magnitude of the force would she have to exert if the steel parts were oiled?

3: (a) What is the maximum frictional force in the knee joint of a person who supports 66.0 kg of her mass on that knee? (b) During strenuous exercise it is possible to exert forces to the joints that are easily ten times greater than the weight being supported. What is the maximum force of friction under such conditions? The frictional forces in joints are relatively small in all circumstances except when the joints deteriorate, such as from injury or arthritis. Increased frictional forces can cause further damage and pain.

4: Suppose you have a 120-kg wooden crate resting on a wood floor. (a) What maximum force can you exert horizontally on the crate without moving it? (b) If you continue to exert this force once the crate starts to slip, what will the magnitude of its acceleration then be?

5: (a) If half of the weight of a small 1.00×10^3 kg utility truck is supported by its two drive wheels, what is the magnitude of the maximum acceleration it can achieve on dry concrete? (b) Will a metal cabinet lying on the wooden bed of the truck slip if it accelerates at this rate? (c) Solve both problems assuming the truck has four-wheel drive.

6: A team of eight dogs pulls a sled with waxed wood runners on wet snow (mush!). The dogs have average masses of 19.0 kg, and the loaded sled with its rider has a mass of 210 kg. (a) Calculate the magnitude of the acceleration starting from rest if each dog exerts an average force of 185 N backward on the snow. (b) What is the magnitude of the acceleration once the sled starts to move? (c) For both situations, calculate the magnitude of the force in the coupling between the dogs and the sled.

7: Consider the 65.0-kg ice skater being pushed by two others shown in Figure 7. (a) Find the direction and magnitude of F_{tot}, the total force exerted on her by the others, given that the magnitudes F_1 and F_2 are 26.4 N and 18.6 N, respectively. (b) What is her initial acceleration if she is initially stationary and wearing steel-bladed skates that point in the direction of F_{tot}? (c) What is her acceleration assuming she is already moving in the direction of F_{tot}? (Remember that friction always acts in the direction opposite that of motion or attempted motion between surfaces in contact.)

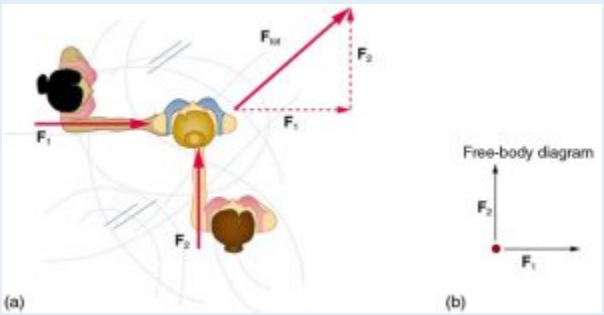


Figure 7.

8: Show that the acceleration of any object down a frictionless incline that makes an angle θ with the horizontal is $a = g \sin \theta$. (Note that this acceleration is independent of mass.)

9: Show that the acceleration of any object down an incline where friction behaves simply (that is, where $f_k = \mu_k N$) is $a = g\ (\sin \theta - \mu_k \cos \theta)$. Note that the acceleration is independent of mass and reduces to the expression found in the previous problem when friction becomes negligibly small ($\mu_k = 0$).

10: Calculate the deceleration of a snow boarder going up a 5.0°, slope assuming the coefficient of friction for waxed wood on wet snow. The result of Exercise 9 may be useful, but be careful to consider the fact that the snow boarder is going uphill. Explicitly show how you follow the steps in Chapter 4.7 Problem-Solving Strategies.

11: (a) Calculate the acceleration of a skier heading down a 10.0° slope, assuming the coefficient of friction for waxed wood on wet snow. (b) Find the angle of the slope down which this skier could coast at a constant velocity. You can neglect air resistance in both parts, and you will find the result of Exercise 9 to be useful. Explicitly show how you follow the steps in the Chapter 4.7 Problem-Solving Strategies.

12: If an object is to rest on an incline without slipping, then friction must equal the component of the weight of the object parallel to the incline. This requires greater and greater friction for steeper slopes. Show that the maximum angle of an incline above the horizontal for which an object will not slide down is $\theta = \tan^{-1} \mu_s$. You may use the result of the previous problem. Assume that a=0 and that static friction has reached its maximum value.

13: Calculate the maximum deceleration of a car that is heading down a 6° slope (one that makes an angle of 6° with the horizontal) under the following road conditions. You may assume that the weight of the car is evenly distributed on all four tires and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the deceleration. (Ignore rolling.) Calculate for a car: (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that $\mu_s = 0.100$, the same as for shoes on ice.

14: Calculate the maximum acceleration of a car that is heading up a 4° slope (one that makes an angle of 4° with the horizontal) under the following road conditions. Assume that only half the weight of the car is supported by the two drive wheels and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the acceleration. (Ignore rolling.) (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that $\mu_s = 0.100$, the same as for shoes on ice.

15: Repeat Exercise 14 for a car with four-wheel drive.

16: A freight train consists of two 8.00×10^5-kg engines and 45 cars with average masses of 5.50×10^5 kg. (a) What force must each engine exert backward on the track to accelerate the train at a rate of 5.00×10^{-2} m/s^2 if the force of friction is 7.50×10^5 N, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the magnitude of the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

17: Consider the 52.0-kg mountain climber in Figure 8. (a) Find the tension in the rope and the force that the mountain climber must exert with her feet on the vertical rock face to remain stationary. Assume that the force is exerted parallel to her legs. Also, assume negligible force exerted by her arms. (b) What is the minimum coefficient of friction between her shoes and the cliff?

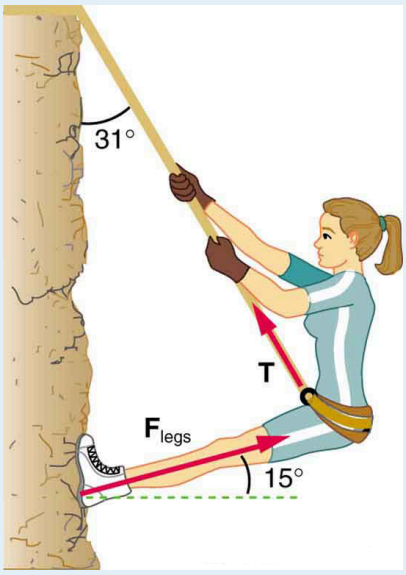


Figure 8. *Part of the climber's weight is supported by her rope and part by friction between her feet and the rock face.*

18: A contestant in a winter sporting event pushes a 45.0-kg block of ice across a frozen lake as shown in Figure 9(a). (a) Calculate the minimum force *F* he must exert to get the block moving. (b) What is the magnitude of its acceleration once it starts to move, if that force is maintained?

19: Repeat Exercise 18 with the contestant pulling the block of ice with a rope over his shoulder at the same angle above the horizontal as shown in Figure 9(b).

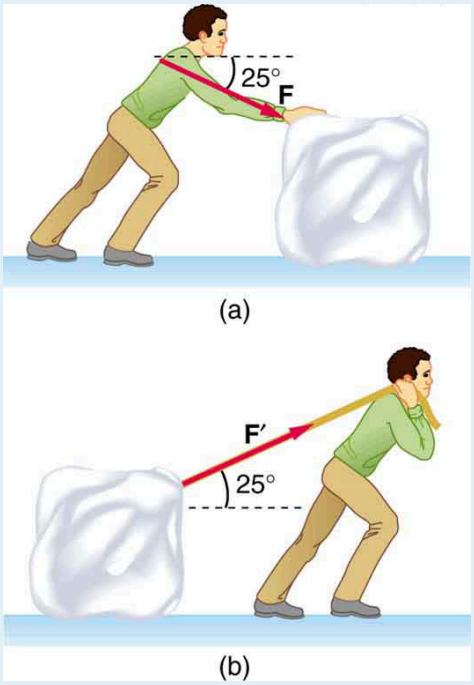


Figure 9. *Which method of sliding a block of ice requires less force—(a) pushing or (b) pulling at the same angle above the horizontal?*

Solutions

Problems & Exercises

1: 5.00 N

4: (a) 588 N (b) 1.96 m/s^2

6: (a) 3.29 m/s^2 (b) 3.52 m/s^2 (c) 980 N and 945 N

10: 1.83 m/s^2

14: (a) 4.20 m/s^2 (b) 2.74 m/s^2 (c) -0.195 m/s^2

16: (a) 1.03×10^6 N (b) 3.48×10^5 N

18: (a) 51.0 N (b) 0.720 m/s^2

4.7 Problem-Solving Strategies

Summary

- Understand and apply a problem-solving procedure to solve problems using Newton's laws of motion.

Success in problem solving is obviously necessary to understand and apply physical principles, not to mention the more immediate need of passing exams. The basics of problem solving, presented earlier in this text, are followed here, but specific strategies useful in applying Newton's laws of motion are emphasized. These techniques also reinforce concepts that are useful in many other areas of physics. Many problem-solving strategies are stated outright in the worked examples, and so the following techniques should reinforce skills you have already begun to develop.

Problem-Solving Strategy for Newton's Laws of Motion

Step 1. As usual, it is first necessary to identify the physical principles involved. *Once it is determined that Newton's laws of motion are involved (if the problem involves forces), it is particularly important to draw a careful sketch of the situation*. Such a sketch is shown in Figure 1(a). Then, as in Figure 1(b), use arrows to represent all forces, label them carefully, and make their lengths and directions correspond to the forces they represent (whenever sufficient information exists).

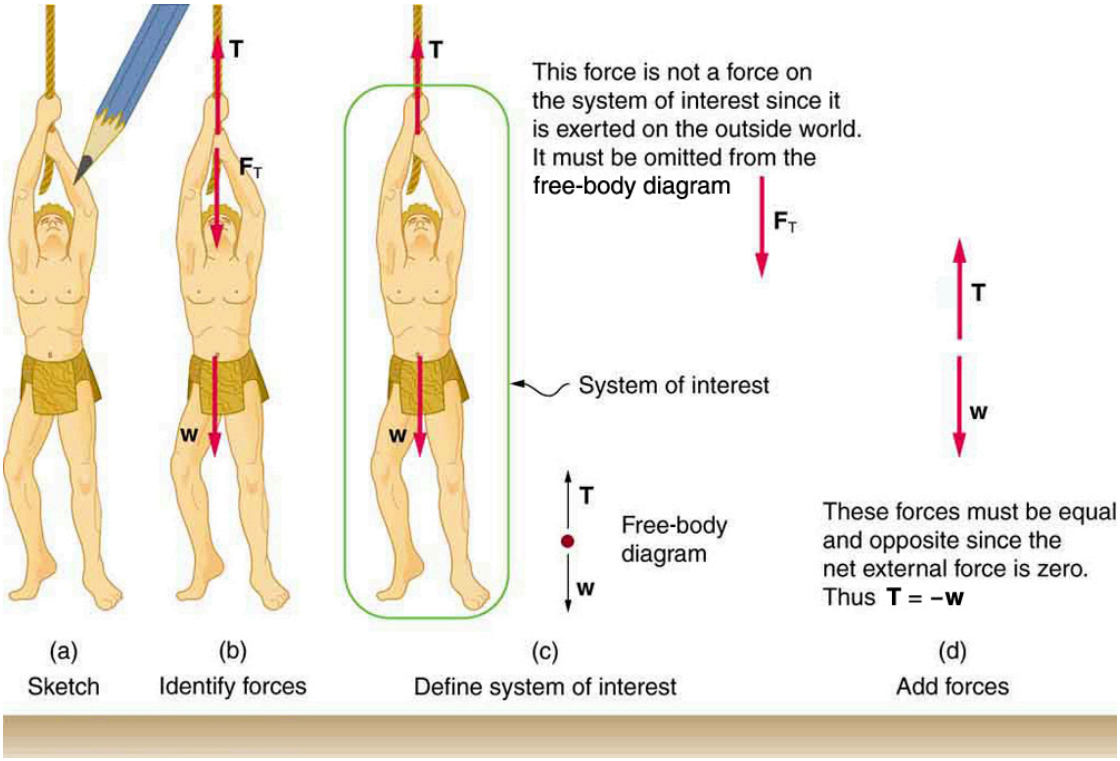


Figure 1. *(a) A sketch of Tarzan hanging from a vine. (b) Arrows are used to represent all forces. T is the tension in the vine above Tarzan, F_T is the force he exerts on the vine, and w is his weight. All other forces, such as the nudge of a breeze, are assumed negligible. (c) Suppose we are given the ape man's mass and asked to find the tension in the vine. We then define the system of interest as shown and draw a free-body diagram. F_T is no longer shown, because it is not a force acting on the system of interest; rather, F_T acts on the outside world. (d) Showing only the arrows, the head-to-tail method of addition is used. It is apparent that T = -w, if Tarzan is stationary.*

Step 2. Identify what needs to be determined and what is known or can be inferred from the problem as stated. That is, make a list of knowns and unknowns. *Then carefully determine the system of interest*. This decision is a crucial step, since Newton's second law involves only external forces. Once the system of interest has been identified, it becomes possible to determine which forces are external and which are internal, a necessary step to employ Newton's second law. (See Figure 1(c).) Newton's third law may be used to identify whether forces are exerted between components of a system (internal) or between the system and something outside (external). As illustrated earlier in this chapter, the system of interest depends on what question we need to answer. This choice becomes easier with practice, eventually developing into an almost unconscious process. Skill in clearly defining systems will be beneficial in later chapters as well.

A diagram showing the system of interest and all of the external forces is called a **free-body diagram**. Only forces are shown on free-body diagrams, not acceleration or velocity. We have drawn several of these in worked examples. Figure 1(c) shows a free-body diagram for the system of interest. Note that no internal forces are shown in a free-body diagram.

Step 3. Once a free-body diagram is drawn, *Newton's second law can be applied to solve the problem*. This is done in Figure 1(d) for a particular situation. In general, once external forces are clearly identified in free-body diagrams, it should be a straightforward task to put them into equation form and solve for the unknown, as done in all previous examples. If the problem is one-dimensional—that is, if all forces

are parallel—then they add like scalars. If the problem is two-dimensional, then it must be broken down into a pair of one-dimensional problems. This is done by projecting the force vectors onto a set of axes chosen for convenience. As seen in previous examples, the choice of axes can simplify the problem. For example, when an incline is involved, a set of axes with one axis parallel to the incline and one perpendicular to it is most convenient. It is almost always convenient to make one axis parallel to the direction of motion, if this is known.

Applying Newton's Second Law

Before you write net force equations, it is critical to determine whether the system is accelerating in a particular direction. If the acceleration is zero in a particular direction, then the net force is zero in that direction. Similarly, if the acceleration is nonzero in a particular direction, then the net force is described by the equation: $F_{\text{net}} = ma$. For example, if the system is accelerating in the horizontal direction, but it is not accelerating in the vertical direction, then you will have the following conclusions:

$$F_{\text{net}x} = ma,$$
$$F_{\text{net}y} = 0.$$

You will need this information in order to determine unknown forces acting in a system.

Step 4. As always, *check the solution to see whether it is reasonable*. In some cases, this is obvious. For example, it is reasonable to find that friction causes an object to slide down an incline more slowly than when no friction exists. In practice, intuition develops gradually through problem solving, and with experience it becomes progressively easier to judge whether an answer is reasonable. Another way to check your solution is to check the units. If you are solving for force and end up with units of m/s, then you have made a mistake.

Section Summary

- To solve problems involving Newton's laws of motion, follow the procedure described:
    1. Draw a sketch of the problem.
    2. Identify known and unknown quantities, and identify the system of interest. Draw a free-body diagram, which is a sketch showing all of the forces acting on an object. The object is represented by a dot, and the forces are represented by vectors extending in different directions from the dot. If vectors act in directions that are not horizontal or vertical, resolve the vectors into horizontal and vertical components and draw them on the free-body diagram.
    3. Write Newton's second law in the horizontal and vertical directions and add the forces acting on the object. If the object does not accelerate in a particular direction (for example, the x-direction) then $F_{\text{net}x} = 0$. If the object does accelerate in that

direction, $F_{netx} = ma$.

4. Check your answer. Is the answer reasonable? Are the units correct?

Problems & Exercises

1: A 5.00×10^5-kg rocket is accelerating straight up. Its engines produce 1.250×10^7 N of thrust, and air resistance is 4.50×10^6 N. What is the rocket's acceleration? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

2: The wheels of a midsize car exert a force of 2100 N backward on the road to accelerate the car in the forward direction. If the force of friction including air resistance is 250 N and the acceleration of the car is 1.80 m/s^2, what is the mass of the car plus its occupants? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion. For this situation, draw a free-body diagram and write the net force equation.

3: Calculate the force a 70.0-kg high jumper must exert on the ground to produce an upward acceleration 4.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

4: When landing after a spectacular somersault, a 40.0-kg gymnast decelerates by pushing straight down on the mat. Calculate the force she must exert if her deceleration is 7.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

5: A freight train consists of two 8.00×10^4-kg engines and 45 cars with average masses of 5.50×10^4 kg. (a) What force must each engine exert backward on the track to accelerate the train at a rate of 5.00×10^{-2} m/s^2 if the force of friction is 7.50×10^5 N, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

6: Commercial airplanes are sometimes pushed out of the passenger loading area by a tractor. (a) An 1800-kg tractor exerts a force of 1.75×10^4 N backward on the pavement, and the system experiences forces resisting motion that total 2400 N. If the acceleration is 0.150 m/s^2, what is the mass of the airplane? (b) Calculate the force exerted by the tractor on the airplane, assuming 2200 N of the friction is experienced by the airplane. (c) Draw two sketches showing the systems of interest used to solve each part, including the free-body diagrams for each.

7: A 1100-kg car pulls a boat on a trailer. (a) What total force resists the motion of the car, boat, and trailer, if the car exerts a 1900-N force on the road and produces an acceleration of 0.550 m/s^2? The mass of the boat plus trailer is 700 kg. (b) What is the force in the hitch between the car and the trailer if 80% of the resisting forces are experienced by the boat and trailer?

8: (a) Find the magnitudes of the forces F_1 and F_2 that add to give the total force F_{tot} shown in Figure 2. This may be done either graphically or by using trigonometry. (b) Show graphically that the same total force is obtained independent of the order of addition of F_1 and F_2. (c) Find the direction and magnitude of some other pair of vectors that add to give F_{tot}. Draw these to scale on the same drawing used in part (b) or a similar picture.

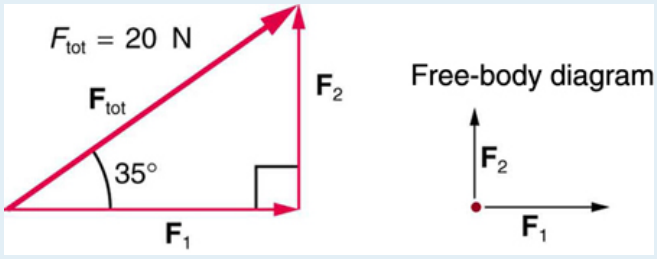


Figure 2.

9: Two children pull a third child on a snow saucer sled exerting forces F_1 and F_2 as shown from above in Figure 3. Find the acceleration of the 49.00-kg sled and child system. Note that the direction of the frictional force is unspecified; it will be in the opposite direction of the sum of F_1 and F_2.

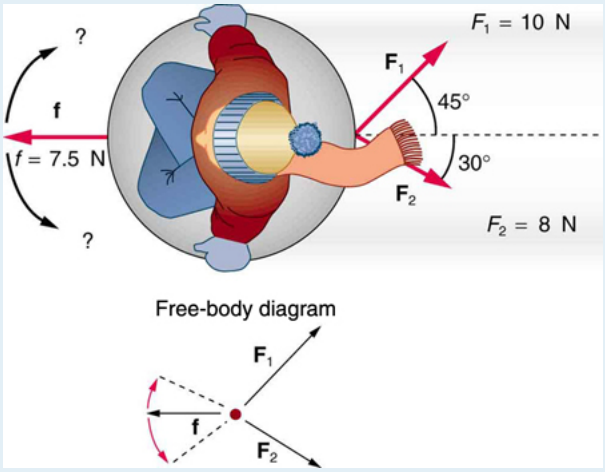


Figure 3. *An overhead view of the horizontal forces acting on a child's snow saucer sled.*

10: Suppose your car was mired deeply in the mud and you wanted to use the method illustrated in Figure 4 to pull it out. (a) What force would you have to exert perpendicular to the center of the rope to produce a force of 12,000 N on the car if the angle is 2.00°? In this part, explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion. (b) Real ropes stretch under such forces. What force would be exerted on the car if the angle increases to 7.00° and you still apply the force found in part (a) to its center?

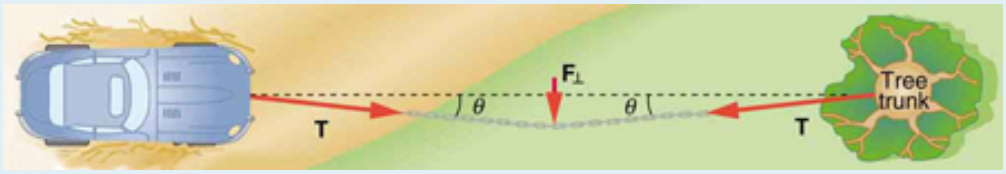


Figure 4.

11: What force is exerted on the tooth in Figure 5 if the tension in the wire is 25.0 N? Note that the force applied to the tooth is smaller than the tension in the wire, but this is necessitated by practical considerations of how force can be applied in the mouth. Explicitly show how you follow steps in the Problem-Solving Strategy for Newton's laws of motion.

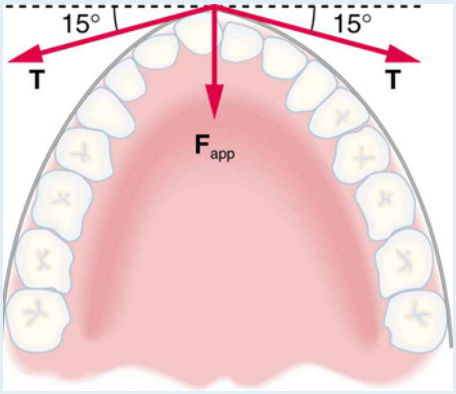


Figure 5. *Braces are used to apply forces to teeth to realign them. Shown in this figure are the tensions applied by the wire to the protruding tooth. The total force applied to the tooth by the wire, F_{app}, points straight toward the back of the mouth.*

12: Figure 6 shows Superhero and Trusty Sidekick hanging motionless from a rope. Superhero's mass is 90.0 kg, while Trusty Sidekick's is 55.0 kg, and the mass of the rope is negligible. (a) Draw a free-body diagram of the situation showing all forces acting on Superhero, Trusty Sidekick, and the rope. (b) Find the tension in the rope above Superhero. (c) Find the tension in the rope between Superhero and Trusty Sidekick. Indicate on your free-body diagram the system of interest used to solve each part.

Figure 6. *Superhero and Trusty Sidekick hang motionless on a rope as they try to figure out what to do next. Will the tension be the same everywhere in the rope?*

13: A nurse pushes a cart by exerting a force on the handle at a downward angle 35.0° below the horizontal. The loaded cart has a mass of 28.0 kg, and the force of friction is 60.0 N. (a) Draw a free-body diagram for the system of interest. (b) What force must the nurse exert to move at a constant velocity?

14: Construct Your Own Problem Consider the tension in an elevator cable during the time the elevator starts from rest and accelerates its load upward to some cruising velocity. Taking the elevator and its load to be the system of interest, draw a free-body diagram. Then calculate the tension in the cable. Among the things to consider are the mass of the elevator and its load, the final velocity, and the time taken to reach that velocity.

15: Construct Your Own Problem Consider two people pushing a toboggan with four children on it up a snow-covered slope. Construct a problem in which you calculate the acceleration of the toboggan and its load. Include a free-body diagram of the appropriate system of interest as the basis for your analysis. Show vector forces and their components and explain the choice of coordinates. Among the things to be considered are the forces exerted by those pushing, the angle of the slope, and the masses of the toboggan and children.

16: Unreasonable Results (a) Repeat Exercise 7, but assume an acceleration of 1.20 m/s^2 is produced. (b) What is unreasonable about the result? (c) Which premise is unreasonable, and why is it unreasonable?

17: Unreasonable Results (a) What is the initial acceleration of a rocket that has a mass of 1.50×10^6 kg at takeoff, the engines of which produce a thrust of 2.00×10^6 N? Do not neglect gravity. (b) What is unreasonable about the result? (This result has been unintentionally achieved by several real rockets.) (c) Which premise is unreasonable, or which premises are inconsistent? (You may find it useful to compare this problem to the rocket problem earlier in this section.)

Solutions

Problems & Exercises

1:

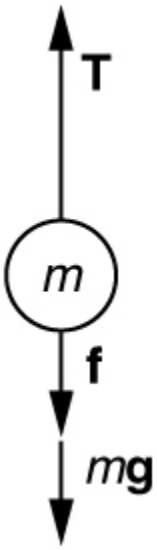


Figure 7.

Using the free-body diagram: $F_{\text{net}} = T - f - mg = ma$, so that

$$a = \frac{T - f - mg}{m} = \frac{1.250 \times 10^7 \text{ N} - 4.50 \times 10^6 \text{ N} - (5.00 \times 10^5 \text{ kg})(9.80 \text{ m/s}^2)}{5.00 \times 10^5 \text{ kg}} = 6.20 \text{ m/s}^2$$

3:

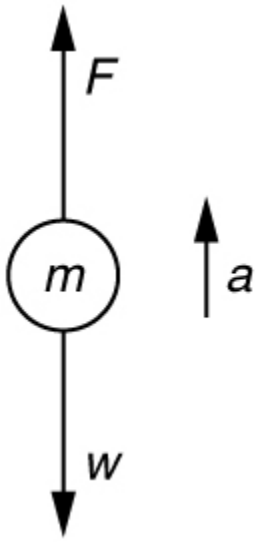


Figure 8.

1. Use Newton's laws of motion
2. Given : $a = 4.00g = (4.00)(9.80\ \text{m/s}^2) = 39.2\ \text{m/s}^2; m = 70.0\ \text{kg},$ Find: $F.$
3. $\sum F = +F - w = ma,$ so that
   $F = ma + w = ma + mg = m(a + g).$
   $F = (70.0\ \text{kg})[(39.2\ \text{m/s}^2) + (9.80\ \text{m/s}^2)] = 3.43 \times 10^3\ \text{N}.$ The force exerted by the high-jumper is actually down on the ground, but F is up from the ground and makes him jump.
4. This result is reasonable, since it is quite possible for a person to exert a force of the magnitude of $10^3\ \text{N}.$

5: (a) 4.41 x 10^5 N (b) 1.50 x 10^5 N

7: (a) 910 N (b) 1.11 x 10^3 N

9: a = 0.139 m/s, θ = 12.4^o

11:

1. Use Newton's laws since we are looking for forces.
2. Draw a free-body diagram:

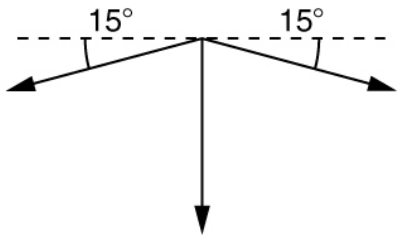


Figure 9.

3. The tension is given as T = 25.0 N. Find F_{app}. Using Newton's laws gives: $\sum F_y = 0,$ so that applied force is due to the *y*-components of the two tensions:
   $F_{\text{app}} = 2T \sin\ \theta = 2(25.0\ \text{N}) \sin\ 15^o = 12.9\ \text{N}$
   The *x*-components of the tension cancel. $\sum F_x = 0.$
4. This seems reasonable, since the applied tensions should be greater than the force applied to the tooth.

4.8 Further Applications of Newton's Laws of Motion

Summary

- Apply problem-solving techniques to solve for quantities in more complex systems of forces.
- Integrate concepts from kinematics to solve problems using Newton's laws of motion.

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills.

Example 1: Drag Force on a Barge

Suppose two tugboats push on a barge at different angles, as shown in Figure 1. The first tugboat exerts a force of 2.7×10^5 N in the *x*-direction, and the second tugboat exerts a force of 3.6×10^5 N in the *y*-direction.

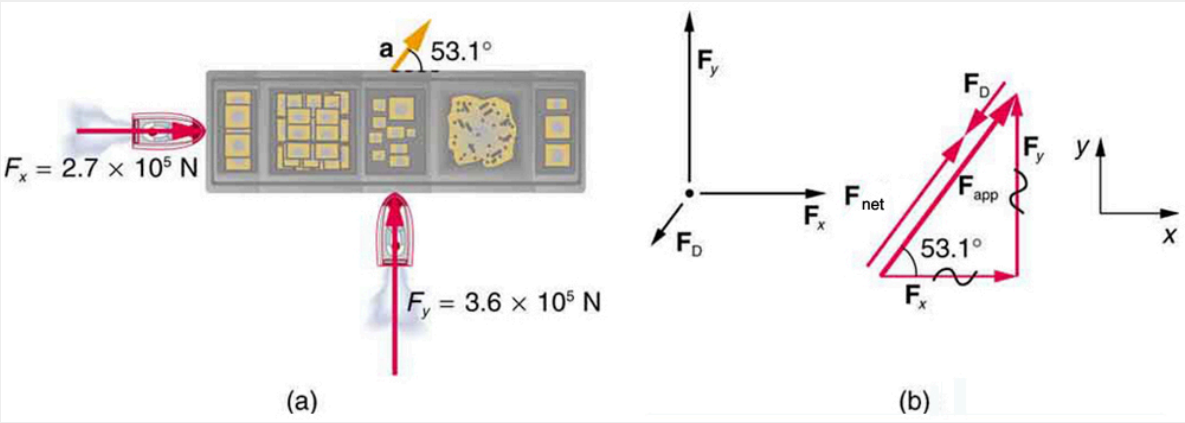


Figure 1. *(a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the x- and y-axes are in the same direction as F_x and F_y. The problem quickly becomes a one-dimensional problem along the direction of F_{app}, since friction is in the direction opposite to F_{app}.*

If the mass of the barge is 5.0×10^6 kg and its acceleration is observed to be 7.5×10^{-2} m/s^2 in the direction shown, what is the drag force of the water on the barge resisting the motion? (Note: drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object.)

Strategy

The directions and magnitudes of acceleration and the applied forces are given in Figure 1(a). We will define the total force of the tugboats on the barge as F_{app} so that:

$$F_{\text{app}} = F_x + F_y$$

Since the barge is flat bottomed, the drag of the water F_{D} will be in the direction opposite to F_{app}, as shown in the free-body diagram in Figure 1(b). The system of interest here is the barge, since the forces on *it* are given as well as its acceleration. Our strategy is to find the magnitude and direction of the net applied force F_{app}, and then apply Newton's second law to solve for the drag force F_{D}.

Solution

Since F_{x} and F_{y} are perpendicular, the magnitude and direction of F_{app} are easily found. First, the resultant magnitude is given by the Pythagorean theorem:

$$F_{\text{app}} = \sqrt{F_x^2 + F_y^2}$$

$$F_{\text{app}} = \sqrt{(2.7 \times 10^5 \text{ N})^2 + (3.6 \times 10^5 \text{ N})^2} = 4.5 \times 10^5 \text{ N}.$$

The angle is given by

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

$$\theta = \tan^{-1}\left(\frac{3.6 \times 10^5 \text{ N}}{2.7 \times 10^5 \text{ N}}\right) = 53^o,$$

which we know, because of Newton's first law, is the same direction as the acceleration. F_{D} is in the opposite direction of F_{app}, since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as F_{app}, but its magnitude is slightly less than F_{app}. The problem is now one-dimensional. From Figure 1(b), we can see that

$$F_{\text{net}} = F_{\text{app}} - F_{\text{D}}.$$

But Newton's second law states that

$$F_{\text{net}} = ma.$$

Thus,

$$F_{\text{app}} - F_{\text{D}} = ma.$$

This can be solved for the magnitude of the drag force of the water F_{D} in terms of known quantities:

$$F_{\text{D}} = F_{\text{app}} - ma.$$

Substituting known values gives

$$F_{\text{D}} = (4.5 \times 10^5 \text{ N}) - (5.0 \times 10^6 \text{ kg})(7.5 \times 10^{-2} \text{ m/s}^2) = 7.5 \times 10^4 \text{ N}.$$

The direction of F_{D} has already been determined to be in the direction opposite to F_{app}, or at an angle of 53° south of west.

Discussion

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge

into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where F_D is less than 1/600th of the weight of the ship.

In the earlier example of a tightrope walker we noted that the tensions in wires supporting a mass were equal only because the angles on either side were equal. Consider the following example, where the angles are not equal; slightly more trigonometry is involved.

Example 2: Different Tensions at Different Angles

Consider the traffic light (mass 15.0 kg) suspended from two wires as shown in Figure 2. Find the tension in each wire, neglecting the masses of the wires.

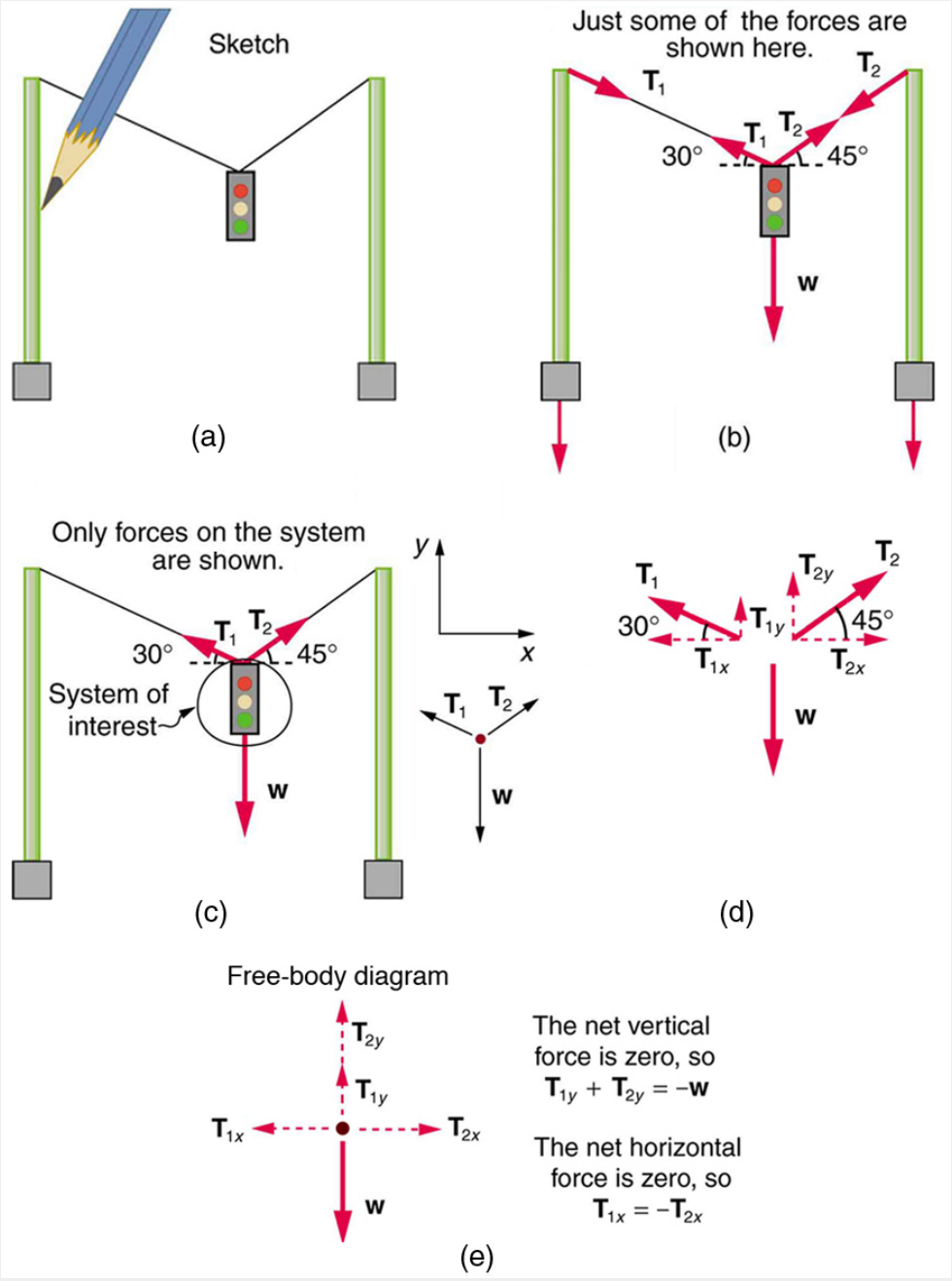


Figure 2. *A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical (y) and horizontal (x) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.*

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in Figure 2(c). The three

forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in part (d) of the figure. There are two unknowns in this problem (T_1 and T_2), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or *x*-axis:

$$F_{\text{net}x} = T_{2x} - T_{1x} = 0.$$

Thus, as you might expect,

$$T_{1x} = T_{2x}.$$

This gives us the following relationship between T_1 and T_2:

$$T_1 \cos(30^0) = T_2 \cos(45^0).$$

Thus,

$$T_2 = (1.225)T_1.$$

Note that T_1 and T_2 are not equal in this case, because the angles on either side are not equal. It is reasonable that T_2 ends up being greater than T_1 , because it is exerted more vertically than T_1

Now consider the force components along the vertical or *y*-axis:

$$F_{\text{net}y} = T_{1y} + T_{2y} - w = 0.$$

This implies

$$T_{1y} + T_{2y} = w.$$

Substituting the expressions for the vertical components gives

$$T_1 \sin(30^0) + T_2 \sin(45^0) = w.$$

There are two unknowns in this equation, but substituting the expression for T_2 in terms of T_1 reduces this to one equation with one unknown:

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg,$$

which yields

$$(1.366)T_1 = (15.0 \text{ kg})(9.80 \text{ m/s}^2).$$

Solving this last equation gives the magnitude of T_1 to be

$$T_1 = 108 \text{ N}.$$

Finally, the magnitude of T_2 is determined using the relationship between them, T_2 = 1.225 T_1 found above. Thus we obtain

$$T_2 = 132 \text{ N}.$$

Discussion

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker).

The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride? Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed: will the scale still read more than your weight at rest? Consider the following example.

Example 3: What Does the Bathroom Scale Read in an Elevator?

Figure 3 shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of 1.20 m/s^2, and (b) if the elevator moves upward at a constant speed of 1 m/s.

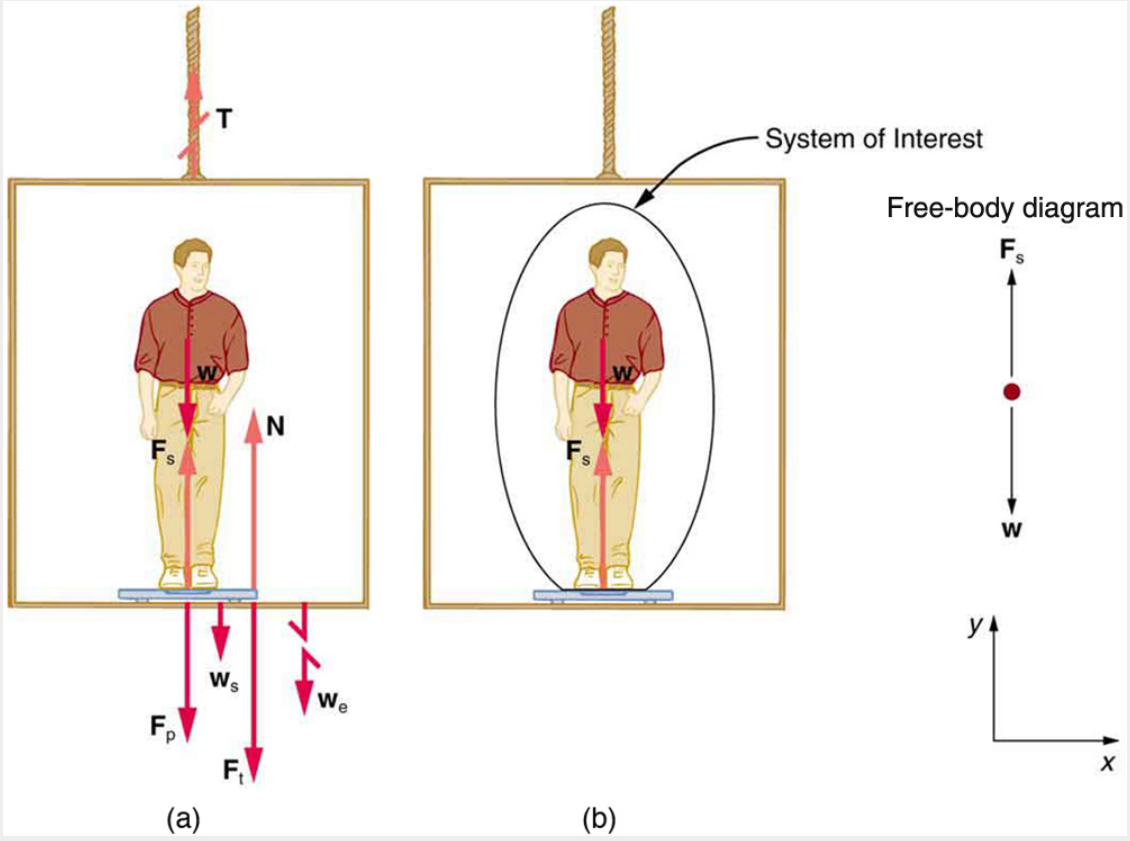


Figure 3. *(a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. T is the tension in the supporting cable, w is the weight of the person, w_s is the weight of the scale, $w_e$is the weight of the elevator, F_s is the force of the scale on the person, F_p is the force of the person on the scale, F_t is the force of the scale on the floor of the elevator, and N is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.*

Strategy

If the scale is accurate, its reading will equal F_p, the magnitude of the force the person exerts

downward on it. Figure 3(a) shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in Figure 3(b). Analysis of the free-body diagram using Newton's laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight *w* and the upward force of the scale F_s. According to Newton's third law F_p and F_s are equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,

$$F_{\text{net}} = ma.$$

From the free-body diagram we see that $F_{\text{net}} = F_s - w$, so that

$$F_s - w = ma.$$

Solving for F_s gives an equation with only one unknown:

$$F_s = ma + w,$$

or, because *weight = gravity force downwards = mg*, simply

$$F_s = ma + mg.$$

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

Solution for (a)

In this part of the problem, $a = 1.20\ \text{m/s}^2$, so that

$$F_s = (75.0\ \text{kg})(1.20\ \text{m/s}^2) + (75.0\ \text{kg})(9.80\ \text{m/s}^2) = 825N,$$

Discussion for (a)

This is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

$$F_{\text{net}} = ma = 0 = F_s - w$$
$$F_s = w = mg$$
$$F_s = (75.0\ \text{kg})(9.80\ \text{m/s}^2) = 735N$$

So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

Solution for (b)

Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight? For any constant velocity—up, down, or stationary—acceleration is zero because $a = \dfrac{\Delta v}{\Delta t}$, and $\Delta v = 0$.

Thus,

$$F_s = ma + mg = 0 + mg$$

Now

$$F_{\text{s}} = (75.0 \text{ kg})(9.80 \text{ m/s}^2) = 735N$$

Discussion for (b)

The scale reading is 735 N, which equals the person's weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward,a is negative, and the scale reading is *less* than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person's weight. If the elevator is in free-fall and accelerating downward at *g*, then the scale reading will be zero and the person will *appear* to be weightless.

Integrating Concepts: Newton's Laws of Motion and Kinematics

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton's laws of motion can also be integrated with other concepts that have been discussed previously in this text to solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters. When approaching problems that involve various types of forces, acceleration, velocity, and/or position, use the following steps to approach the problem:

Problem-Solving Strategy

Step 1. *Identify which physical principles are involved*. Listing the givens and the quantities to be calculated will allow you to identify the principles involved.

Step 2. *Solve the problem using strategies outlined in the text*. If these are available for the specific topic, you should refer to them. You should also refer to the sections of the text that deal with a particular topic. The following worked example illustrates how these strategies are applied to an integrated concept problem.

Example 4: What Force Must a Soccer Player Exert to Reach Top Speed?

A soccer player starts from rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What was his average acceleration? (b) What average force did he exert backward on the ground to achieve this acceleration? The player's mass is 70.0 kg, and air resistance is negligible.

Strategy

1. *To solve an integrated concept problem,* we must first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example considers *acceleration* along a straight line. This is a topic of *kinematics.* Part (b) deals with force, a topic of *dynamics* found in this chapter.
2. The following solutions to each part of the example illustrate how the specific problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so forth.

Solution for (a)

We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is Δv = 8.00 m/s. We are given the elapsed time, and so Δt = 2.50 s. The unknown is acceleration, which can be found from its definition:

$$a = \frac{\Delta v}{\Delta t}.$$

Substituting the known values yields

$$\begin{aligned} a &= \frac{8.00 \text{ m/s}}{2.50 \text{ s}} \\ &= 3.20 \text{ m/s}^2 \end{aligned}$$

Discussion for (a)

This is an attainable acceleration for an athlete in good condition.

Solution for (b)

Here we are asked to find the average force the player exerts backward to achieve this forward acceleration. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes his acceleration. Since we now know the player's acceleration and are given his mass, we can use Newton's second law to find the force exerted. That is,

$$F_{\text{net}} = ma.$$

Substituting the known values of *m* and *a* gives

$$F_{\text{net}} = (70.0 \text{ kg})(3.20 \text{ m/s}^2) = 224N$$

Discussion for (b)

This is about 50 pounds, a reasonable average force.

This worked example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These strategies are found throughout the text, and many worked examples show how to use them for single topics. You will find these techniques for integrated concept problems useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

Summary

- Newton's laws of motion can be applied in numerous situations to solve problems of motion.
- Some problems will contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the direction in which an object accelerates so that you can determine whether $F_{\text{net}} = ma$ or $F_{\text{net}} = 0$.
- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating, the normal force will be less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force will always be less than the full weight of the object.
- Some problems will contain various physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and dynamics in order to solve these problems of motion.

Conceptual Questions

1: To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at g. Why will they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

2: A cartoon shows the toupee coming off the head of an elevator passenger when the elevator rapidly stops during an upward ride. Can this really happen without the person being tied to the floor of the elevator? Explain your answer.

Problems & Exercises

1: A flea jumps by exerting a force of 1.20×10^{-5} N straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of 0.500×10^{-6} N on the flea. Find the direction and magnitude of the acceleration of the flea if its mass is 6.00×10^{-7} kg. Do not neglect the gravitational force.

2: Two muscles in the back of the leg pull upward on the Achilles tendon, as shown in Figure 4. (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?

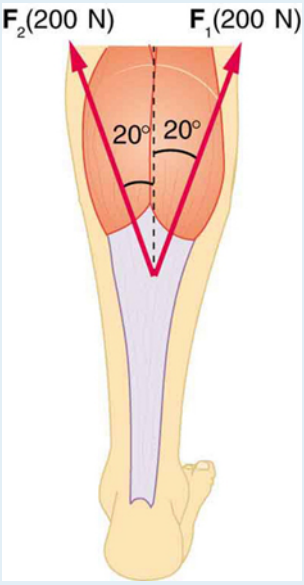


Figure 4. *Achilles tendon*

3: A 76.0-kg person is being pulled away from a burning building as shown in Figure 5. Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.

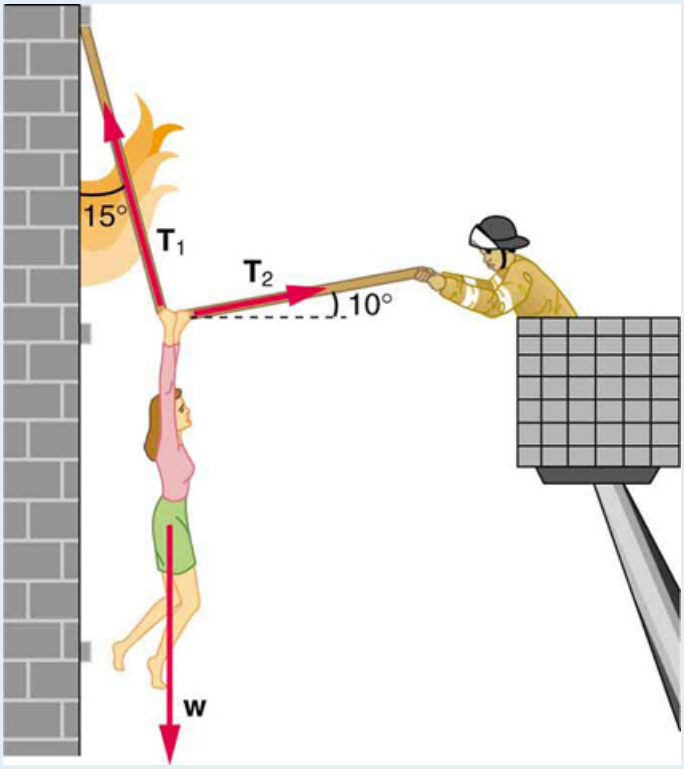


Figure 5. *The force T_2 needed to hold steady the person being rescued from the fire is less than her weight and less than the force T_1 in the other rope, since the more vertical rope supports a greater part of her weight (a vertical force)*

4: Integrated Concepts A 35.0-kg dolphin decelerates from 12.0 to 7.50 m/s in 2.30 s to join another dolphin in play. What average force was exerted to slow him if he was moving horizontally? (The gravitational force is balanced by the buoyant force of the water.)

5: Integrated Concepts When starting a foot race, a 70.0-kg sprinter exerts an average force of 650 N backward on the ground for 0.800 s. (a) What is his final speed? (b) How far does he travel? Hint: Find his acceleration first then use kinematics.

6: Integrated Concepts A large rocket has a mass of 2.00×10^6 kg at takeoff, and its engines produce a thrust of 3.50×10^7 N. (a) Find its initial acceleration if it takes off vertically. (b) How long does it take to reach a velocity of 120 km/h straight up, assuming constant mass and thrust? (c) In reality, the mass of a rocket decreases significantly as its fuel is consumed. Describe qualitatively how this affects the acceleration and time for this motion.

7: Integrated Concepts A basketball player jumps straight up for a ball. To do this, he lowers his body 0.300 m and then accelerates through this distance by forcefully straightening his legs. This player leaves the floor with a vertical velocity sufficient to carry him 0.900 m above the floor. (a) Calculate his velocity when he leaves the floor. (b) Calculate his acceleration while he is straightening his legs. He goes from zero to the velocity found in part (a) in a distance of 0.300 m. (c) Calculate the force he exerts on the floor to do this, given that his mass is 110 kg.

8: Integrated Concepts A 2.50-kg fireworks shell is fired straight up from a mortar and reaches a height of 110 m. (a) Neglecting air resistance (a poor assumption, but we will make it for this example), calculate the shell's velocity when it leaves the mortar. (b) The mortar itself is a tube 0.450 m long. Calculate the average acceleration of the shell in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the shell in the mortar? Express your answer in newtons and as a ratio to the weight of the shell.

9: Integrated Concepts Repeat Exercise 8 for a shell fired at an angle 10.0° from the vertical.

10: Integrated Concepts An elevator filled with passengers has a mass of 1700 kg. (a) The elevator accelerates upward from rest at a rate of 1.20 m/s^2 for 1.50 s. Calculate the tension in the cable supporting the elevator. (b) The elevator continues upward at constant velocity for 8.50 s. What is the tension in the cable during this time? (c) The elevator decelerates at a rate of 0.600{ m/s^2 for 3.00 s. What is the tension in the cable during deceleration? (d) How high has the elevator moved above its original starting point, and what is its final velocity?

11: Unreasonable Results (a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of 0.400 m/s^2 for 50.0 s? (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

12: Unreasonable Results A 75.0-kg man stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with his weight. (The scale exerts an upward force on him equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

Solutions

Problems & Exercises

1: 10.2 m/s^2, 4.67° from vertical

3: T_1 = 736 N T_2 = 194 N as net force is 0 N so using magnitudes only

T1 cos 15^o + T2 sin 10^o = Weight = mg and T1 sin 15^o = T2 cos10^o

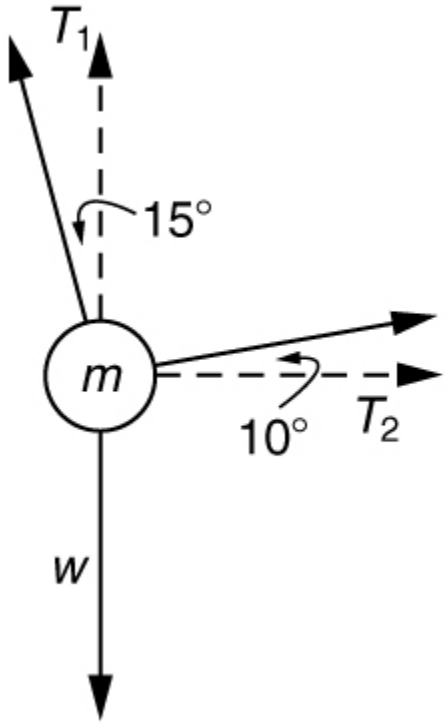


Figure 6.

5: (a) 7.43 m/s (b) 2.97 m as the acceleration is 650 N / 70.0 kg = 9.29 m/s^2

7: (a) 4.20 m/s (b) 29.4 m/s^2 (c) 4.31 x 10^3 N

9: (a) 47.1 m/s (b) 2.47 x 10^3 m/s^2 (c) 6.18 x 10^3 N. The average force is 252 times the shell's weight.

4.9 Optional: Drag Forces

Summary

- Express mathematically the drag force.
- Discuss the applications of drag force.
- Define terminal velocity.
- Determine the terminal velocity given mass.

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion. Like friction, the **drag force** always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. For most large objects such as bicyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force F_{D} is found to be proportional to the square of the speed of the object. We can write this relationship mathematically as F_{D} being proportional to v^2. When taking into account other factors, this relationship becomes

$$F_{\mathrm{D}} = \frac{1}{2}C\rho \mathrm{A}v^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid. (Recall that density is mass per unit volume.) This equation can also be written in a more generalized fashion as $F_{\mathrm{D}} = bv^2$, where b is a constant equivalent to $0.5C\rho A$. We have set the exponent for these equations as 2 because, when an object is moving at high velocity through air, the magnitude of the drag force is proportional to the square of the speed. As we shall see in a few pages on fluid dynamics, for small particles moving at low speeds in a fluid, the exponent is equal to 1.

DRAG FORCE

Drag force F_{D} is found to be proportional to the square of the speed of the object. Mathematically

$$F_{\mathrm{D}} \propto v^2$$

$$F_{\mathrm{D}} = \frac{1}{2} C \rho \mathrm{A} v^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid.

Athletes as well as car designers seek to reduce the drag force to lower their race times. (See Figure 1). “Aerodynamic” shaping of an automobile can reduce the drag force and so increase a car’s gas mileage.

Figure 1. *From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed. They are shaped like a bullet with tapered fins. (credit: U.S. Army, via Wikimedia Commons)*

The value of the drag coefficient, C, is determined empirically, usually with the use of a wind tunnel. (See Figure 2).

Figure 2. *NASA researchers test a model plane in a wind tunnel. (credit: NASA/Ames)*

The drag coefficient can depend upon velocity, but we will assume that it is a constant here. Table 2 lists some typical drag coefficients for a variety of objects. Notice that the drag coefficient is a dimensionless quantity. At highway speeds, over 50% of the power of a car is used to overcome air drag. The most fuel-efficient cruising speed is about 70–80 km/h (about 45–50 mi/h). For this reason, during the 1970s oil crisis in the United States, maximum speeds on highways were set at about 90 km/h (55 mi/h).

Object	C
Airfoil	0.05
Toyota Camry	0.28
Ford Focus	0.32
Honda Civic	0.36
Ferrari Testarossa	0.37
Dodge Ram pickup	0.43
Sphere	0.45
Hummer H2 SUV	0.64
Skydiver (feet first)	0.70
Bicycle	0.90
Skydiver (horizontal)	1.0
Circular flat plate	1.12

Table 2. Drag Coefficient Values Typical values of drag coefficient C.

Substantial research is under way in the sporting world to minimize drag. The dimples on golf balls are being redesigned as are the clothes that athletes wear. Bicycle racers and some swimmers and runners wear full bodysuits. Australian Cathy Freeman wore a full body suit in the 2000 Sydney Olympics, and won the gold medal for the 400 m race. Many swimmers in the 2008 Beijing Olympics wore (Speedo) body suits; it might have made a difference in breaking many world records (See Figure 3). Most elite swimmers (and cyclists) shave their body hair. Such innovations can have the effect of slicing away milliseconds in a race, sometimes making the difference between a gold and a silver medal. One consequence is that careful and precise guidelines must be continuously developed to maintain the integrity of the sport.

Figure 3. *Body suits, such as this LZR Racer Suit, have been credited with many world records after their release in 2008. Smoother "skin" and more compression forces on a swimmer's body provide at least 10% less drag. (credit: NASA/Kathy Barnstorff)*

Some interesting situations connected to Newton's second law occur when considering the effects of drag forces upon a moving object. For instance, consider a skydiver falling through air under the influence of gravity. The two forces acting on him are the force of gravity and the drag force (ignoring the buoyant force). The downward force of gravity remains constant regardless of the velocity at which the person is moving. However, as the person's velocity increases, the magnitude of the drag force increases until the magnitude of the drag force is equal to the gravitational force, thus producing a net force of zero. A zero net force means that there is no acceleration, as given by Newton's second law. At this point, the person's velocity remains constant and we say that the person has reached his *terminal velocity* (*vt*). Since F_{D} is proportional to the speed, a heavier skydiver must go faster for F_{D} to equal his weight. Let's see how this works out more quantitatively.

At the terminal velocity,

$$F_{\mathrm{net}} = mg - F_{\mathrm{D}} = ma = 0.$$

Thus,

$$mg = F_{\mathrm{D}}.$$

Using the equation for drag force, we have

$$mg = \frac{1}{2}\rho CAv^2.$$

Solving for the velocity, we obtain

$$v = \sqrt{\frac{2mg}{\rho CA}}.$$

Assume the density of air is ρ = 1.21 kg/m^3. A 75-kg skydiver descending head first will have an area approximately A = 0.18 m^2 and a drag coefficient of approximately C = 0.70. We find that

$$v = \sqrt{\frac{2(75\text{ kg})(9.80\text{ m/s}^2)}{(1.21\text{ kg/m}^3)(0.70)(0.18\text{ m}^2)}} = 98\text{ m/s} = 350\text{ km/h}.$$

This means a skydiver with a mass of 75 kg achieves a maximum terminal velocity of about 350 km/h while traveling in a pike (head first) position, minimizing the area and his drag. In a spread-eagle position, that terminal velocity may decrease to about 200 km/h as the area increases. This terminal velocity becomes much smaller after the parachute opens.

TAKE-HOME EXPERIMENT

This interesting activity examines the effect of weight upon terminal velocity. Gather together some nested coffee filters. Leaving them in their original shape, measure the time it takes for one, two, three, four, and five nested filters to fall to the floor from the same height (roughly 2 m). (Note that, due to the way the filters are nested, drag is constant and only mass varies.) They obtain terminal velocity quite quickly, so find this velocity as a function of mass. Plot the terminal velocity v versus mass. Also plot v^2 versus mass. Which of these relationships is more linear? What can you conclude from these graphs?

Example 1: A Terminal Velocity

Find the terminal velocity of an 85-kg skydiver falling in a spread-eagle position.

Strategy

At terminal velocity, F_{net} = 0. Thus the drag force on the skydiver must equal the force of gravity (the person's weight). Using the equation of drag force, we find $mg = \frac{1}{2}\rho C\text{A}v^2$.

Thus the terminal velocity v_{t} can be written as

$$v_{\text{t}} = \sqrt{\frac{2mg}{\rho CA}}.$$

Solution

All quantities are known except the person's projected area. This is an adult (82 kg) falling spread eagle. We can estimate the frontal area as

$$A = (2\ \text{m})(0.35\ \text{m}) = 0.70\ \text{m}^2.$$

Using our equation for v_t, we find that

$$v_t = \sqrt{\frac{2(85\ \text{kg})(9.80\ \text{m/s}^2)}{(1.21\ \text{kg/m}^3)(1.0)(0.70\ \text{m}^2)}} = 44\ \text{m/s}.$$

Discussion

This result is consistent with the value for v_t mentioned earlier. The 75-kg skydiver going feet first had a v = 98 m/s. He weighed less but had a smaller frontal area and so a smaller drag due to the air.

The size of the object that is falling through air presents another interesting application of air drag. If you fall from a 5-m high branch of a tree, you will likely get hurt—possibly fracturing a bone. However, a small squirrel does this all the time, without getting hurt. You don't reach a terminal velocity in such a short distance, but the squirrel does.

The following interesting quote on animal size and terminal velocity is from a 1928 essay by a British biologist, J.B.S. Haldane, titled "On Being the Right Size."

To the mouse and any smaller animal, [gravity] presents practically no dangers. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, and a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force.

The above quadratic dependence of air drag upon velocity does not hold if the object is very small, is going very slow, or is in a denser medium than air. Then we find that the drag force is proportional just to the velocity. This relationship is given by **Stokes' law**, which states that

$$F_s = 6\pi r\eta v,$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

STOKE'S LAW

$$F_s = 6\pi r\eta v,$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

Good examples of this law are provided by microorganisms, pollen, and dust particles. Because each of these objects is so small, we find that many of these objects travel unaided only at a constant (terminal) velocity. Terminal velocities for bacteria (size about 1 μm) can be about 2 μm/s. To move at a greater speed, many bacteria swim using flagella (organelles shaped like little tails) that are powered by little motors embedded in the cell. Sediment in a lake can move at a greater terminal velocity (about 5 μm/s), so it can take days to reach the bottom of the lake after being deposited on the surface.

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fishes, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spear head as the flock forms a streamlined pattern (see Figure 4). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.

Figure 4. *Geese fly in a V formation during their long migratory travels. This shape reduces drag and energy consumption for individual birds, and also allows them a better way to communicate. (credit: Julo, Wikimedia Commons)*

GALILEO'S EXPERIMENT

Galileo is said to have dropped two objects of different masses from the Tower of Pisa. He measured how long it took each to reach the ground. Since stopwatches weren't readily available, how do you think he measured their fall time? If the objects were the same size, but with different masses, what do you think he should have observed? Would this result be different if done on the Moon?

PHET EXPLORATIONS: MASSES & SPRINGS

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energy for each spring.

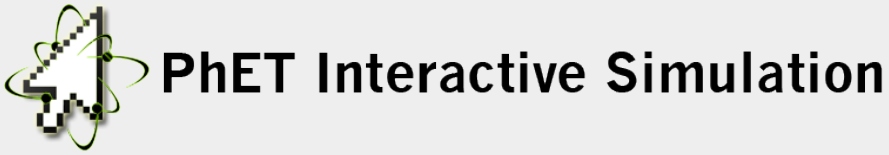


Figure 5. *Masses & Springs*

Section Summary

- Drag forces acting on an object moving in a fluid oppose the motion. For larger objects (such as a baseball) moving at a velocity v in air, the drag force is given by

$$F_{\text{D}} = \frac{1}{2}C\rho Av^2,$$

  where C is the drag coefficient (typical values are given in Table 2), A is the area of the object facing the fluid, and ρ is the fluid density.

- For small objects (such as a bacterium) moving in a denser medium (such as water), the drag force is given by Stokes' law,

$$F_{\text{s}} = 6\pi\eta rv,$$

  where r is the radius of the object, η is the fluid viscosity, and v is the object's velocity.

Glossary

drag force
F_{D}, found to be proportional to the square of the speed of the object; mathematically

$$F_{\text{D}} \propto v^2$$

$$F_{\text{D}} = \frac{1}{2}C\rho Av^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid

Stokes' law
$F_{\text{s}} = 6\pi r\eta v$, where r is the radius of the object, η is the viscosity of the fluid, and v is the object's

velocity

Conceptual Questions

1: Athletes such as swimmers and bicyclists wear body suits in competition. Formulate a list of pros and cons of such suits.

2: Two expressions were used for the drag force experienced by a moving object in a liquid. One depended upon the speed, while the other was proportional to the square of the speed. In which types of motion would each of these expressions be more applicable than the other one?

3: As cars travel, oil and gasoline leaks onto the road surface. If a light rain falls, what does this do to the control of the car? Does a heavy rain make any difference?

4: Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

Problems & Exercises

1: The terminal velocity of a person falling in air depends upon the weight and the area of the person facing the fluid. Find the terminal velocity (in meters per second and kilometers per hour) of an 80.0-kg skydiver falling in a pike (headfirst) position with a surface area of 0.140 m^2.

2: A 60-kg and a 90-kg skydiver jump from an airplane at an altitude of 6000 m, both falling in the pike position. Make some assumption on their frontal areas and calculate their terminal velocities. How long will it take for each skydiver to reach the ground (assuming the time to reach terminal velocity is small)? Assume all values are accurate to three significant digits.

3: A 560-g squirrel with a surface area of 930 cm^2 falls from a 5.0-m tree to the ground. Estimate its terminal velocity. (Use a drag coefficient for a horizontal skydiver.) What will be the velocity of a 56-kg person hitting the ground, assuming no drag contribution in such a short distance?

4: To maintain a constant speed, the force provided by a car's engine must equal the drag force plus the force of friction of the road (the rolling resistance). (a) What are the magnitudes of drag forces at 70 km/h and 100 km/h for a Toyota Camry? (Drag area is 0.70 m^2) (b) What is the magnitude of drag force at 70 km/h and 100 km/h for a Hummer H2? (Drag area is 2.44 m^2) Assume all values are accurate to three significant digits.

5: By what factor does the drag force on a car increase as it goes from 65 to 110 km/h?

6: Calculate the speed a spherical rain drop would achieve falling from 5.00 km (a) in the absence of air drag (b) with air drag. Take the size across of the drop to be 4 mm, the density to be 1.00×10^3 kg/m^3, and the surface area to be πr^2.

7: Using Stokes' law, verify that the units for viscosity are kilograms per meter per second.

8: Find the terminal velocity of a spherical bacterium (diameter 2.00 μm) falling in water. You will first need

to note that the drag force is equal to the weight at terminal velocity. Take the density of the bacterium to be 1.10×10^3 kg/m^3.

9: Stokes' law describes seimentation of particles in liquids and can be used to measure viscosity. Particles in liquids achieve terminal velocity quickly. One can measure the time it takes for a particle to fall a certain distance and then use Stokes' law to calculate the viscosity of the liquid. Suppose a steel ball bearing (density 7.8×10^3 kg/m^3, diameter 3.0 mm) is dropped in a container of motor oil. It takes 12 s to fall a distance of 0.60 m. Calculate the viscosity of the oil.

Solutions

Problems & Exercises

1: 115 m/s; 414 km/hr

3: 25 m/s; 9.9 m/s

5: 2.9

7: $[\eta] = \dfrac{[F_\text{s}]}{[r][v]} = \dfrac{\text{kg}\cdot\text{m/s}^2}{\text{m}\cdot\text{m/s}} = \dfrac{\text{kg}}{\text{m}\cdot\text{s}}$

9: 0.76 kg/m s

Chapter 5 Uniform Circular Motion and Gravitation

Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. In some ways, this chapter is a continuation of Chapter 4 Dynamics: Newton's Laws of Motion as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, **uniform circular motion**, motion in a circular path at constant speed. Studying this topic illustrates most concepts associated with rotational motion and leads to the study of many new topics we group under the name *rotation*. Pure *rotational motion* occurs when points in an object move in circular paths centered on one point. Pure *translational motion* is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving along ice.

Glossary

uniform circular motion
: the motion of an object in a circular path at constant speed

5.1 Centripetal Acceleration

Summary

- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

Figure 1 shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration** (a_C); centripetal means “toward the center” or “center seeking.”

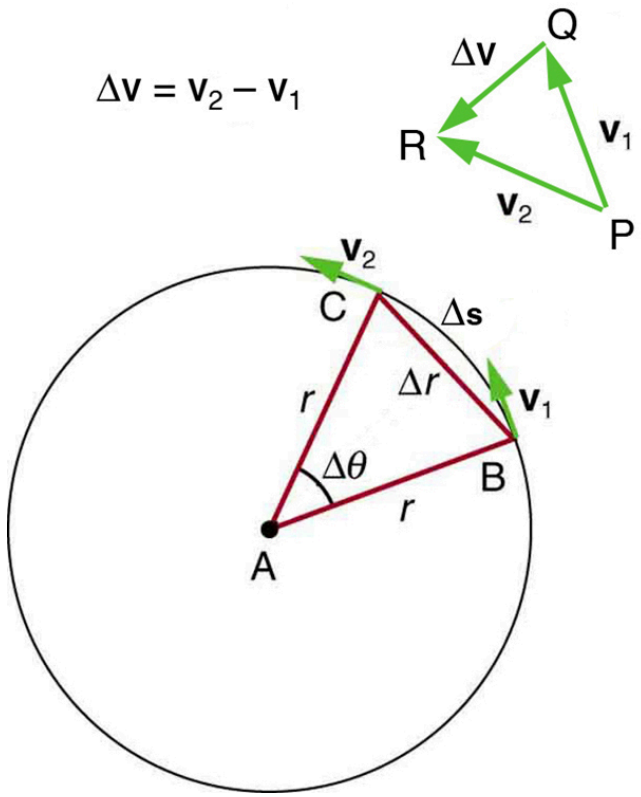


Figure 1. *The directions of the velocity of an object at two different points are shown, and the change in velocity Δv is seen to point directly toward the center of curvature. (See small inset.) Because ac = Δv/Δt, the acceleration is also toward the center; ac is called centripetal acceleration. (Because Δθ is very small, the arc length Δs is equal to the chord length Δr for small time differences.)*

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii r and Δs are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}.$$

Acceleration is $\Delta v/\Delta t$, and so we first solve this expression for Δv:

$$\Delta v = \frac{v}{r}\Delta s.$$

Then we divide this by Δt, yielding

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}.$$

Finally, noting that $\Delta v/\Delta t = a_c$ and that $\Delta s/\Delta t = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

$$a_{\mathrm{c}} = \frac{v^2}{r},$$

which is the acceleration of an object in a circle of radius *r* at a speed *v*. So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that a_{c} is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that a_{c} is greater for tighter turns, as you have probably noticed.

A centrifuge (see Figure 2b) is a rotating device used to separate specimens of different densities. High centripetal acceleration significantly decreases the time it takes for separation to occur, and makes separation possible with small samples. Centrifuges are used in a variety of applications in science and medicine, including the separation of single cell suspensions such as bacteria, viruses, and blood cells from a liquid medium and the separation of macromolecules, such as DNA and protein, from a solution. Centrifuges are often rated in terms of their centripetal acceleration relative to acceleration due to gravity (*g*); maximum centripetal acceleration of several hundred thousand *g* is possible in a vacuum. Human centrifuges, extremely large centrifuges, have been used to test the tolerance of astronauts to the effects of accelerations larger than that of Earth's gravity.

Example 1: How Does the Centripetal Acceleration of a Car Around a Curve Compare with That Due to Gravity?

What is the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h)? Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed. See Figure 2(a).

Strategy

Because *v* and *r* are given, we use the expression $a_{\mathrm{c}} = \frac{v^2}{r}$.

Solution

Entering the given values of *v* = 25.0 m/s and *r* = 500 m into the first expression for a_{c} gives

$$a_{\mathrm{c}} = \frac{v^2}{r} = \frac{25.0^2}{500} = 1.25\ \mathrm{m/s^2}$$

Discussion

To compare this with the acceleration due to gravity ($g = 9.80\ m/s^2$), we take the ratio of a_{c}/g = (1.25 m/s^2)/(9.80m/s^2) = 0.128. Thus, a_{c} = 0.128 *g* and is noticeable especially if you were not wearing a seat belt.

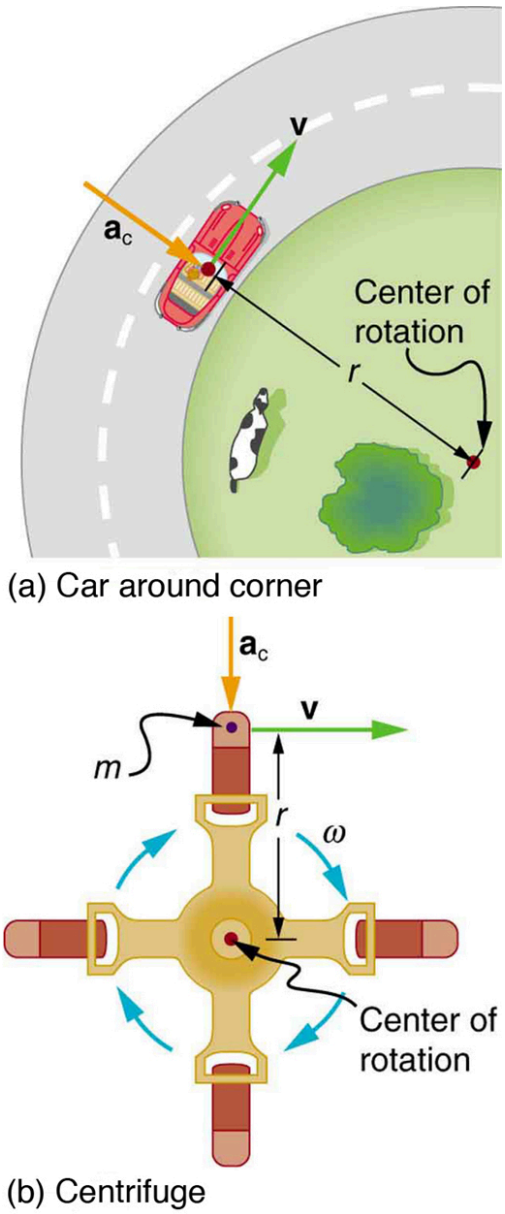


Figure 2. *(a) The car following a circular path at constant speed is accelerated perpendicular to its velocity, as shown. The magnitude of this centripetal acceleration is found in Example 1. (b) A particle of mass in a centrifuge is rotating at constant angular velocity . It must be accelerated perpendicular to its velocity or it would continue in a straight line. The magnitude of the necessary acceleration is found in Example 2.*

Example 2: What is the speed of an Ultracentrifuge?

Calculate the speed of a point 7.50 cm from the axis of a spinning **ultracentrifuge** if the centripetal acceleration at that point is $4.63 \times 10^6 \text{ m/s}^2$

Strategy

Both *r* and a_c are given, so we can use the equation $a_c = \frac{v^2}{r}$ to determine the speed of the centrifuge.

Solution

First we solve for the speed of the centrifuge

$$a_c = \frac{v^2}{r} \rightarrow v = \sqrt{a_c r}$$

Converting 7.50 cm to meters and substituting known values gives

$$v = \sqrt{a_c r} = \sqrt{(4.63 \times 10^6)(0.0750)} = 590 \text{ m/s}.$$

Discussion

The given centripetal acceleration is 472,000 times as strong as *g*. It is no wonder that such high velocity centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

Of course, a net external force is needed to cause any acceleration, just as Newton proposed in his second law of motion. So a net external force is needed to cause a centripetal acceleration. In the next section on centripetal force, we will consider the forces involved in circular motion.

PHET EXPLORATIONS: LADYBUG MOTION 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

PhET Interactive Simulation

Figure 3. *Ladybug Motion in 2D*

Section Summary

- Centripetal acceleration a_c is the acceleration experienced while in uniform circular motion. It always points toward the center of rotation. It is perpendicular to the linear velocity v and has the magnitude

$$a_c = \frac{v^2}{r}$$

- The unit of centripetal acceleration is m/s^2.

Glossary

centripetal acceleration
the acceleration of an object moving in a circle, directed toward the center

ultracentrifuge
a centrifuge optimized for spinning a rotor at very high speeds

Conceptual Questions

1: Can centripetal acceleration change the speed of circular motion? Explain.

Problems & Exercises

1: A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00 m radius, at how many revolutions per minute will the riders be subjected to a centripetal acceleration whose magnitude is 1.50 times that due to gravity?

2: A runner taking part in the 200 m dash must run around the end of a track that has a circular arc with a

radius of curvature of 30 m. If he completes the 200 m dash in 23.2 s and runs at constant speed throughout the race, what is the magnitude of his centripetal acceleration as he runs the curved portion of the track?

3: Taking the age of Earth to be about 4×10^9 years and assuming its orbital radius of 1.5×10^{11} has not changed and is circular, calculate the approximate total distance Earth has traveled since its birth (in a frame of reference stationary with respect to the Sun). *Recall that the circumference of a circle is* $2\pi r$.

4: The propeller of a World War II fighter plane is 2.30 m in diameter and the tip of the blade spins at 145 m/s. What is the centripetal acceleration of the propeller tip under these conditions? Calculate it in meters per second squared and convert to multiples of *g*.

5: Integrated Concepts

Riders in an amusement park ride shaped like a Viking ship hung from a large pivot are rotated back and forth like a rigid pendulum. Sometime near the middle of the ride, the ship is momentarily motionless at the top of its circular arc. The ship then swings down under the influence of gravity.

(a) Assuming negligible friction, find the speed of the riders at the bottom of its arc, given the system's center of mass travels in an arc having a radius of 14.0 m and the riders are near the center of mass.

(b) What is the centripetal acceleration at the bottom of the arc?

(c) Draw a free body diagram of the forces acting on a rider at the bottom of the arc.

(d) Find the force exerted by the ride on a 60.0 kg rider and compare it to her weight.

(e) Discuss whether the answer seems reasonable.

6: Unreasonable Results

A mother pushes her child on a swing so that his speed is 9.00 m/s at the lowest point of his path. The swing is suspended 2.00 m above the child's center of mass.

(a) What is the magnitude of the centripetal acceleration of the child at the low point?

(b) What is the magnitude of the force the child exerts on the seat if his mass is 18.0 kg?

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent?

Solutions

Problems & Exercises

1: 10.8 m/s

3: 4×20^{21} m

5: (a) 23.4 m/s (b) 39.1 m/s^2 (c)

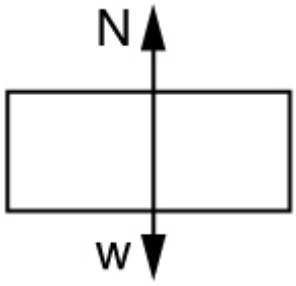


Figure 4.

(d) 1.76×10^3 N or 3.00 ω, that is, the normal force (upward) is three times her weight. (e) This answer seems reasonable, since she feels like she's being forced into the chair MUCH stronger than just by gravity.

6: (a) 40.5 m/s^2 (b) 905 N (c) The force in part (b) is very large. The acceleration in part (a) is too much, about 4 g. (d) The speed of the swing is too large. At the given velocity at the bottom of the swing, there is enough kinetic energy to send the child all the way over the top, ignoring friction.

5.2 Centripetal Force

Summary

- Calculate coefficient of friction on a car tire.
- Calculate ideal speed and angle of a car on a turn.

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing uniform circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's second law of motion, net force is mass times acceleration: net $F = ma$. For uniform circular motion, the acceleration is the centripetal acceleration— $a = a_{\text{c}}$. Thus, the magnitude of centripetal force F_{c} is

$$F_{\text{c}} = ma_{\text{c}}.$$

By using the expressions for centripetal acceleration a_{c} from $a_{\text{c}} = \dfrac{v^2}{r}$ we get an expression for the centripetal force F_{c} in terms of mass, velocity, and radius of curvature:

$$F_{\text{c}} = \frac{mv^2}{r}$$

The centripetal force F_{c} is always perpendicular to the path and pointing to the center of curvature, because a_{c} is perpendicular to the velocity and pointing to the center of curvature.

Note that if you solve the first expression for r, you get

$$r = \frac{mv^2}{F_{\text{c}}}.$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve.

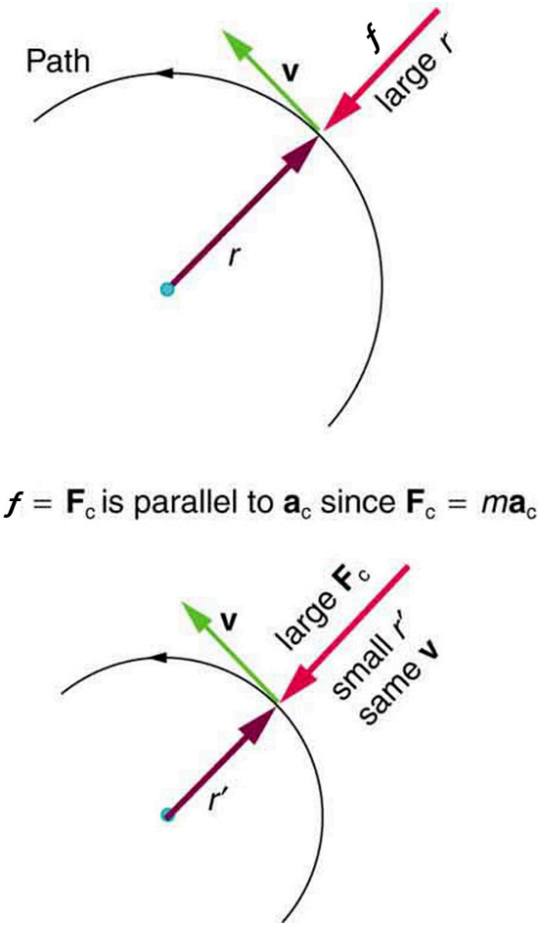


Figure 1. *The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c, the smaller the radius of curvature r and the sharper the curve. The second curve has the same v, but a larger F_c produces a smaller r'.*

Example 1: What Coefficient of Friction Do Care Tires Need on a Flat Curve?

(a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve at 25.0 m/s.

(b) Assuming an unbanked curve, find the minimum static coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (see Figure 2).

Strategy and Solution for (a)

We know that $F_c = \frac{mv^2}{r}$. Thus,

$$F_{\text{c}} = \frac{mv^2}{r} = \frac{(900\ \text{kg})(25.0\ \text{m/s})^2}{(500\ \text{m})} = 1125\ \text{N}.$$

Strategy for (b)

Figure 2 shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_{\text{s}}N$, where μ_{s} is the static coefficient of friction and N is the normal force. The normal force equals the car's weight on level ground, so that $N = mg$. Thus the centripetal force in this situation is

$$F_{\text{c}} = f = \mu_{\text{s}}\text{N} = \mu_{\text{s}}mg.$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for F_{c} from the equation

$$\begin{aligned} F_{\text{c}} &= m\frac{v^2}{r}\ , \\ &= mr\omega^2 \end{aligned}$$

$$m\frac{v^2}{r} = \mu_{\text{s}}mg.$$

We solve this for μ_{s}, noting that mass cancels, and obtain

$$\mu_{\text{s}} = \frac{v^2}{rg}.$$

Solution for (b)

Substituting the knowns,

$$\mu_{\text{s}} = \frac{(25.0\ \text{m/s})^2}{(500\ \text{m})(9.80\ \text{m/s}^2)} = 0.13.$$

(Because coefficients of friction are approximate, the answer is given to only two digits.)

Discussion

The coefficient of friction found in part (b) is much smaller than is typically found between tires and roads. The car will still negotiate the curve if the coefficient is greater than 0.13, because static friction is a responsive force, being able to assume a value less than but no more than $\mu_{\text{s}}N$. A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less as will be discussed below.

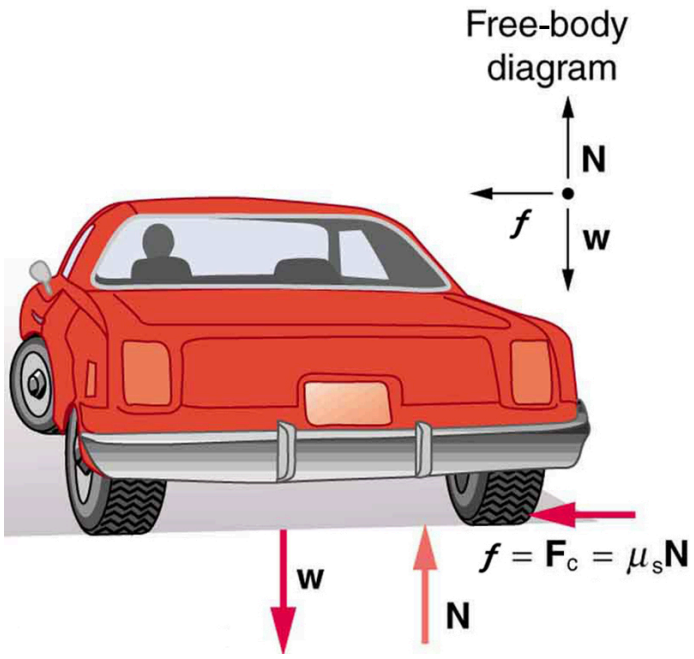


Figure 2. *This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction between the tires and the road. A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.*

Optional: Banked Curves

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See Figure 3. The greater the angle θ, the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

Figure 3 shows a free body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight w and the normal force of the road N. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude mv^2/r. Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force—that is,

$$N \sin \theta = \frac{mv^2}{r}.$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N \cos \theta$, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N \cos \theta = mg.$$

Now we can combine the last two equations to eliminate N and get an expression for θ, as desired. Solving the second equation for $N = mg/(\cos \theta)$, and substituting this into the first yields

$$mg \frac{\sin \theta}{\cos \theta} = \frac{mv^2}{r}$$

$$mg \tan \theta = \frac{mv^2}{r}$$

$$\tan \theta = \frac{v^2}{rg}.$$

Taking the inverse tangent gives

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \text{ (ideally banked curve, no friction).}$$

This expression can be understood by considering how θ depends on v and r. A large θ will be obtained for a large v and a small r. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that θ does not depend on the mass of the vehicle.

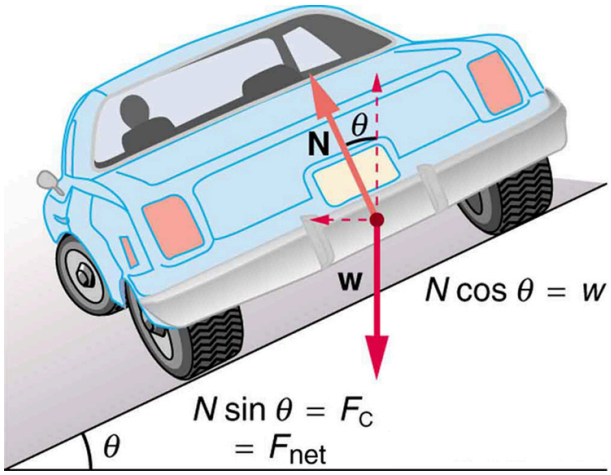


Figure 3. *The car on this banked curve is moving away and turning to the left.*

Example 2: What Is the Ideal Speed to Take a Steeply Banked Tight Curve?

Curves on some test tracks and race courses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at 65.0° should be driven if the road is frictionless.

Strategy

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

Solution

Starting with

$$\tan\theta = \frac{v^2}{rg}$$

we get

$$v = (rg\tan\theta)^{1/2}.$$

Noting that tan 65.0º = 2.14, we obtain

\begin{align*} v &= [(100~\text{m})(9.80~\text{m/s}^2)(2.14)]^{1/2} \\ &= 45.8~\text{m/s}. \end{align*}

Discussion

This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Calculations similar to those in the preceding examples can be performed for a host of interesting situations in which centripetal force is involved—a number of these are presented in this chapter's Problems and Exercises.

TAKE-HOME EXPERIMENT

Ask a friend or relative to swing a golf club or a tennis racquet. Take appropriate measurements to estimate the centripetal acceleration of the end of the club or racquet. You may choose to do this in slow motion.

PHET EXPLORATIONS: GRAVITY AND ORBITS

Move the sun, earth, moon and space station to see how it affects their gravitational forces and orbital paths. Visualize the sizes and distances between different heavenly bodies, and turn off gravity to see what would happen without it!

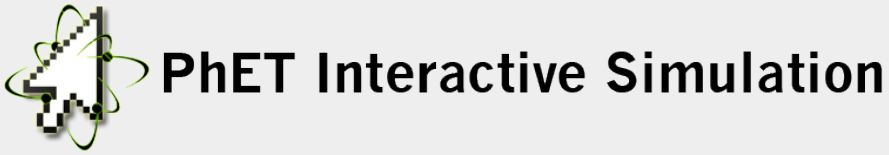


Figure 4. *Gravity and Orbits*

Section Summary

- Centripetal force F_c is any force causing uniform circular motion. It is a "center-seeking" force that always points toward the center of rotation. It is perpendicular to linear velocity v and has magnitude

$$F_c = ma_c,$$

which can also be expressed as

$$F_c = m\frac{v^2}{r}$$

Glossary

centripetal force
: any net force causing uniform circular motion

ideal banking
: the sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction

ideal speed
: the maximum safe speed at which a vehicle can turn on a curve without the aid of friction between the tire and the road

ideal angle
: the angle at which a car can turn safely on a steep curve, which is in proportion to the ideal speed

banked curve
: the curve in a road that is sloping in a manner that helps a vehicle negotiate the curve

Conceptual Questions

Conceptual Questions

1: If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

2: Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

3: If centripetal force is directed toward the center, why do you feel that you are 'thrown' away from the center as a car goes around a curve? Explain.

4: Race car drivers routinely cut corners as shown in Figure 5. Explain how this allows the curve to be taken at the greatest speed.

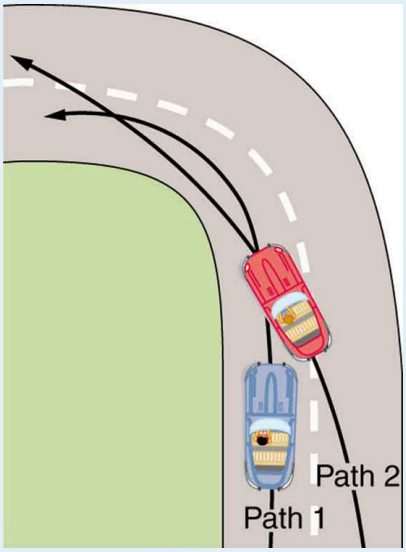


Figure 5. *Two paths around a race track curve are shown. Race car drivers will take the inside path (called cutting the corner) whenever possible because it allows them to take the curve at the highest speed.*

5: A number of amusement parks have rides that make vertical loops like the one shown in Figure 6. For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:

(a) The car goes over the top at faster than this speed?

(b)The car goes over the top at slower than this speed?

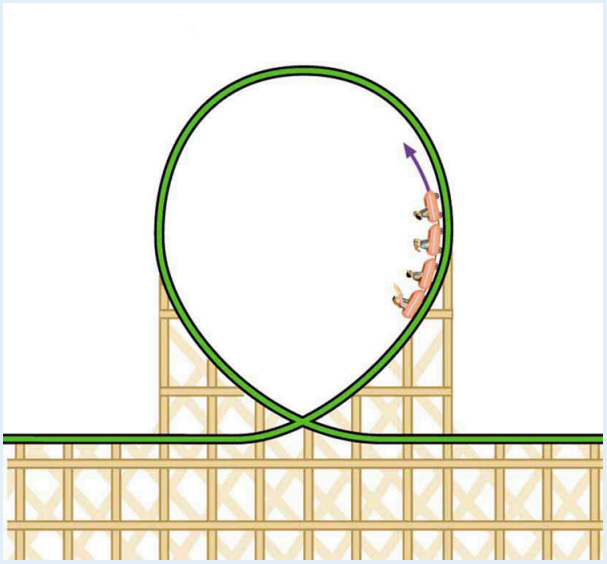


Figure 6. *Amusement rides with a vertical loop are an example of a form of curved motion.*

7: What is the direction of the force exerted by the car on the passenger as the car goes over the top of the amusement ride pictured in Figure 6 under the following circumstances:

(a) The car goes over the top at such a speed that the gravitational force is the only force acting?

(b) The car goes over the top faster than this speed?

(c) The car goes over the top slower than this speed?

8: As a skater forms a circle, what force is responsible for making her turn? Use a free body diagram in your answer.

9: Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown in Figure 7 will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.

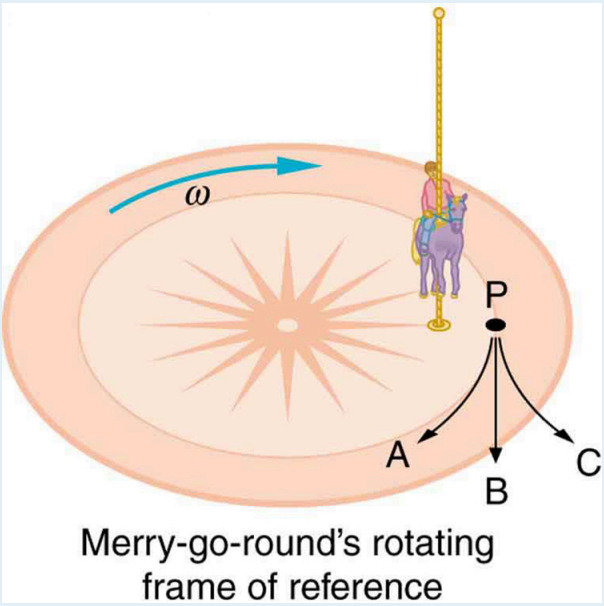


Figure 7. *A child riding on a merry-go-round releases her lunch box at point P. This is a view from above the clockwise rotation. Assuming it slides with negligible friction, will it follow path A, B, or C, as viewed from Earth's frame of reference? What will be the shape of the path it leaves in the dust on the merry-go-round?*

10: Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?

11: Suppose a mass is moving in a circular path on a frictionless table as shown in figure. In the Earth's frame of reference, there is no centrifugal force pulling the mass away from the centre of rotation, yet there is a very real force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its physical origin.

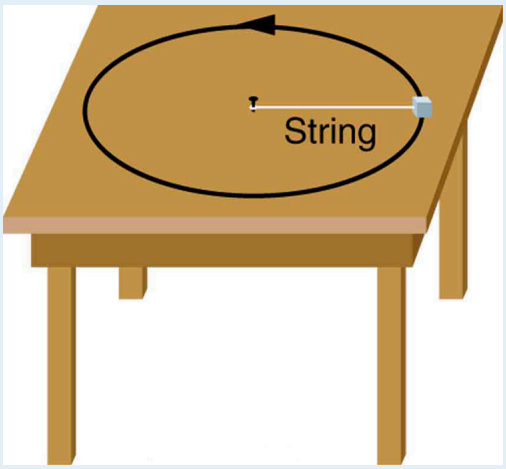


Figure 8. *A mass attached to a nail on a frictionless table moves in a circular path. The force stretching the string is real and not fictional. What is the physical origin of the force on the string?*

Problems & Exercises

1: A 22.0 kg child is riding a playground merry-go-round is moving at a speed of 0.0698 m/s when she is 1.25 m from the center. What is the centripetal force exerted on her? Compare this force to her weight.

2: What is the ideal banking angle for a gentle turn of 1.20 km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?

3: What is the ideal speed to take a 100 m radius curve banked at a 20.0° angle?

4: (a) What is the radius of a bobsled turn banked at 75.0° and taken at 30.0 m/s, assuming it is ideally banked?

(b) Calculate the centripetal acceleration.

(c) Does this acceleration seem large to you?

5: Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen in Figure 9. To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force), and the vertical normal force (which must equal the system's weight).

(a) Show that θ (as defined in the figure) is related to the speed v and radius of curvature r of the turn in the same way as for an ideally banked roadway—that is, $\theta = \tan^{-1} v^2/rg$.

(b) Calculate θ for a 12.0 m/s turn of radius 30.0 m (as in a race).

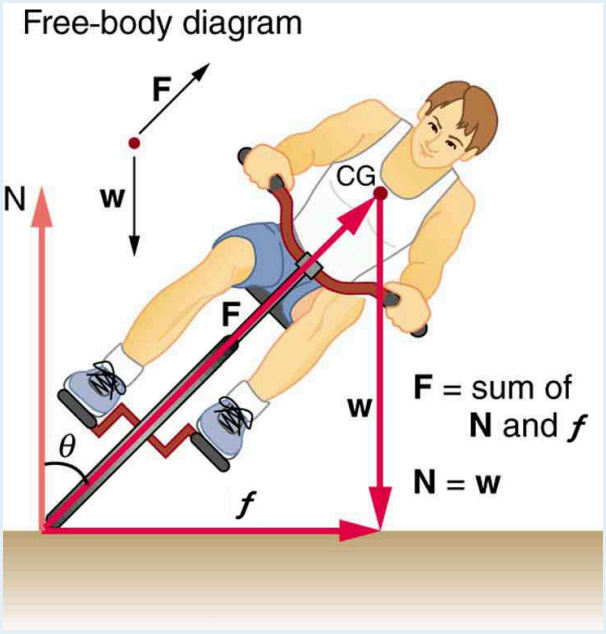


Figure 9. *A bicyclist negotiating a turn on level ground must lean at the correct angle—the ability to do this becomes instinctive. The force of the ground on the wheel needs to be on a line through the center of gravity. The net external force on the system is the centripetal force. The vertical component of the force on the wheel cancels the weight of the system while its horizontal component must supply the centripetal force. This process produces a relationship among the angle θ, the speed v, and the radius of curvature r of the turn similar to that for the ideal banking of roadways.*

6: Integrated Concepts

If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a real problem on icy mountain roads). (a) Calculate the ideal speed to take a 100 m radius curve banked at 15.0º. (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?

7: Modern roller coasters have vertical loops like the one shown in Figure 11. The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 g?

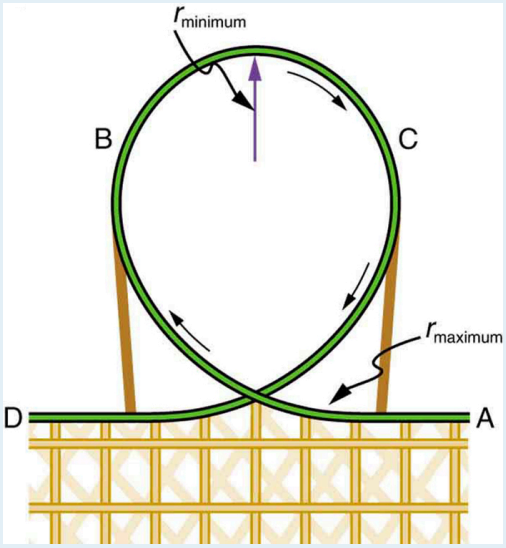


Figure 11. *Teardrop-shaped loops are used in the latest roller coasters so that the radius of curvature gradually decreases to a minimum at the top. This means that the centripetal acceleration builds from zero to a maximum at the top and gradually decreases again. A circular loop would cause a jolting change in acceleration at entry, a disadvantage discovered long ago in railroad curve design. With a small radius of curvature at the top, the centripetal acceleration can more easily be kept greater than g so that the passengers do not lose contact with their seats nor do they need seat belts to keep them in place.*

8: Unreasonable Results

(a) Calculate the minimum coefficient of friction needed for a car to negotiate an unbanked 50.0 m radius curve at 30.0 m/s.

(b) What is unreasonable about the result?

(c) Which premises are unreasonable or inconsistent?

Solutions

Problems & Exercises

1: (a) 483 N (b) 2.24 times her weight

2: 4.14^0

4: 24.6 m (b) $36.6\ \mathrm{m/s^2}$ (c) $a_\mathrm{c} = 3.73\ \mathrm{g}$. This does not seem too large, but it is clear that bobsledders feel a lot of force on them going through sharply banked turns.

6: (a) 16.2 m/s (b) 0.234

8: (a)1.84 (b) A coefficient of friction this much greater than 1 is unreasonable . (c) The assumed speed is too great for the tight curve.

5.3 Newton’s Universal Law of Gravitation

Summary

- Explain Earth’s gravitational force.
- Describe the gravitational effect of the Moon on Earth.
- Discuss weightlessness in space.
- Examine the Cavendish experiment

What do aching feet, a falling apple, and the orbit of the Moon have in common? Each is caused by the gravitational force. Our feet are strained by supporting our weight—the force of Earth’s gravity on us. An apple falls from a tree because of the same force acting a few meters above Earth’s surface. And the Moon orbits Earth because gravity is able to supply the necessary centripetal force at a distance of hundreds of millions of meters. In fact, the same force causes planets to orbit the Sun, stars to orbit the center of the galaxy, and galaxies to cluster together. Gravity is another example of underlying simplicity in nature. It is the weakest of the four basic forces found in nature, and in some ways the least understood. It is a force that acts at a distance, without physical contact, and is expressed by a formula that is valid everywhere in the universe, for masses and distances that vary from the tiny to the immense.

Sir Isaac Newton was the first scientist to precisely define the gravitational force, and to show that it could explain both falling bodies and astronomical motions. See Figure 1. But Newton was not the first to suspect that the same force caused both our weight and the motion of planets. His forerunner Galileo Galilei had contended that falling bodies and planetary motions had the same cause. Some of Newton’s contemporaries, such as Robert Hooke, Christopher Wren, and Edmund Halley, had also made some progress toward understanding gravitation. But Newton was the first to propose an exact mathematical form and to use that form to show that the motion of heavenly bodies should be conic sections—circles, ellipses, parabolas, and hyperbolas. This theoretical prediction was a major triumph—it had been known for some time that moons, planets, and comets follow such paths, but no one had been able to propose a mechanism that caused them to follow these paths and not others.

Figure 1. *According to early accounts, Newton was inspired to make the connection between falling bodies and astronomical motions when he saw an apple fall from a tree and realized that if the gravitational force could extend above the ground to a tree, it might also reach the Sun. The inspiration of Newton's apple is a part of worldwide folklore and may even be based in fact. Great importance is attached to it because Newton's universal law of gravitation and his laws of motion answered very old questions about nature and gave tremendous support to the notion of underlying simplicity and unity in nature. Scientists still expect underlying simplicity to emerge from their ongoing inquiries into nature.*

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

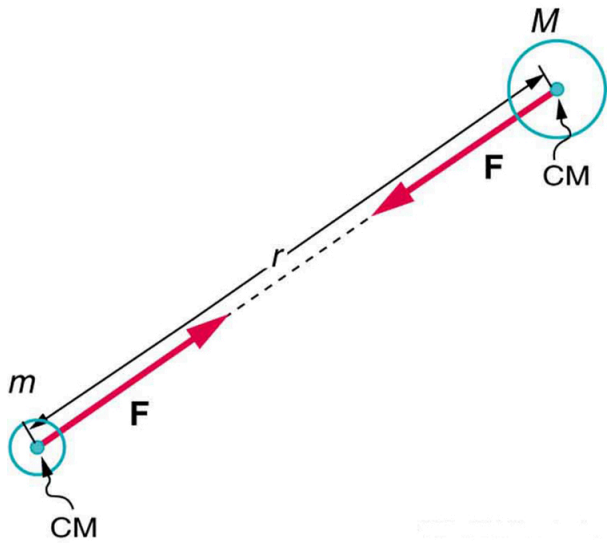


Figure 2. *Gravitational attraction is along a line joining the centers of mass of these two bodies. The magnitude of the force is the same on each, consistent with Newton's third law.*

MISCONCEPTION ALERT

The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's third law.

The bodies we are dealing with tend to be large. To simplify the situation we assume that the body acts as if its entire mass is concentrated at one specific point called the **center of mass** (CM). For two bodies having masses m_1 and m_2 with a distance $\boldsymbol{r}$ between their centers of mass (for this course, we will assume that objects have a uniform density, which means the center of mass is the geometric center of the object), the equation for Newton's universal law of gravitation is

$$F_{\text{g}} = \frac{Gm_1m_2}{r^2},$$

where F_{g} is the magnitude of the gravitational force and G is a proportionality factor called the **gravitational constant**. G is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be

$$G = 6.674 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$$

in SI units. Note that the units of G are such that a force in newtons is obtained from Newton's universal law of gravitation when considering masses in kilograms and distance in meters. For example, two 1.000 kg masses separated by 1.000 m will experience a gravitational attraction of 6.674×10^{-11} N. This is an

extraordinarily small force. The small magnitude of the gravitational force is consistent with everyday experience. We are unaware that even large objects like mountains exert gravitational forces on us. In fact, our body weight is the force of attraction of the *entire Earth* on us with a mass of 6×10^{24} kg.

Recall that the acceleration due to gravity g is about 9.80 m/s^2 on Earth. We can now determine why this is so. The weight of an object mg is the gravitational force between it and Earth. Substituting $m_1 g$ for F in Newton's universal law of gravitation gives

$$m_1 g = \frac{G m_1 m_2}{r^2},$$

where m_1 is the mass of the object, m_2 is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 3. The mass m_1 of the object cancels, leaving an equation for g:

$$g = \frac{G m_2}{r^2}.$$

Substituting known values for Earth's mass and radius (to three significant figures),

$$g = (6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}) \times \frac{5.98 \times 10^{24} \text{ kg}}{(6.38 \times 10^6 \text{ m})^2},$$

and we obtain a value for the acceleration of a falling body:

$$g = 9.80 \text{ m/s}^2.$$

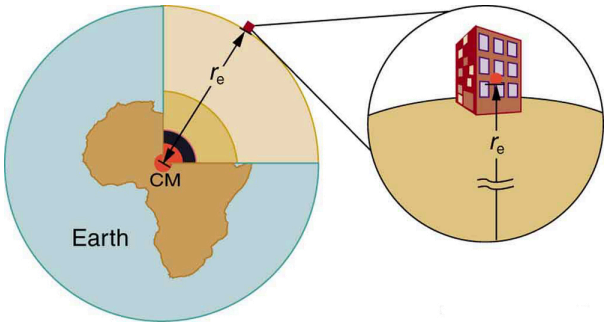


Figure 3. *The distance between the centers of mass of Earth and an object on its surface is very nearly the same as the radius of Earth, because Earth is so much larger than the object.*

This is the expected value *and is independent of the body's mass*. Newton's law of gravitation takes Galileo's observation that all masses fall with the same acceleration a step further, explaining the observation in terms of a force that causes objects to fall—in fact, in terms of a universally existing force of attraction between masses.

TAKE-HOME EXPERIMENT

Take a marble, a ball, and a spoon and drop them from the same height. Do they hit the floor at the same time? If you drop a piece of paper as well, does it behave like the other objects? Explain your observations.

MAKING CONNECTIONS

Attempts are still being made to understand the gravitational force. Modern physics is exploring the connections of gravity to other forces, space, and time. General relativity alters our view of gravitation, leading us to think of gravitation as bending space and time.

In the following example, we make a comparison similar to one made by Newton himself. He noted that if the gravitational force caused the Moon to orbit Earth, then the acceleration due to gravity should equal the centripetal acceleration of the Moon in its orbit. Newton found that the two accelerations agreed "pretty nearly."

Example 1: Earth's Gravitational Force Is the Centripetal Force Making the Moon Move in a Curved Path

(a) Find the acceleration due to Earth's gravity at the distance of the Moon.

(b) Calculate the centripetal acceleration needed to keep the Moon in its orbit (assuming a circular orbit about a fixed Earth), and compare it with the value of the acceleration due to Earth's gravity that you have just found. The orbital period of the Moon is 27.3 days.

Strategy for (a)

This calculation is the same as the one finding the acceleration due to gravity at Earth's surface, except that ***r*** is the distance from the center of Earth to the center of the Moon. The radius of the Moon's nearly circular orbit is 3.84×10^8 m.

Solution for (a)

Substituting known values into the expression for g found above, remembering that m_2 is the mass of Earth not the Moon, yields

$$g = G\frac{m_2}{r^2} = (6.67 \times 10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{kg^2}) \times \frac{5.98 \times 10^{24}\text{ kg}}{(3.84 \times 10^8\text{ m})^2}$$
$$= 2.70 \times 10^{-3}\text{ m/s}^2.$$

Strategy for (b)

Centripetal acceleration can be calculated using

$$a_\text{c} = \frac{v^2}{r}$$

where *r* the distance to the Moon and *v* is the orbital speed of the Moon.

Solution for (b)

Given that the period (the time it takes to make one complete rotation) of the Moon's orbit is 27.3 days, and using

$$27.3\text{ d} \times 24\frac{\text{hr}}{\text{d}} \times 60\frac{\text{min}}{\text{hr}} \times 60\frac{\text{s}}{\text{min}} = 2.34 \times 10^6\text{ s}$$

we see that

$$v = \frac{2\pi r}{T} = \frac{2\pi 3.84 \times 10^8}{2.34 \times 10^6} = 1030\text{ m/s}.$$

The centripetal acceleration is

$$a_\text{c} = \frac{v^2}{r} = \frac{(1030)^2}{3.84 \times 10^8} = 2.76 \times 10^{-3}\text{ m/s}^2$$

The direction of the acceleration is toward the center of the Earth.

Discussion

The centripetal acceleration of the Moon found in (b) differs by about 2% from the acceleration due to Earth's gravity found in (a). This agreement is approximate because the Moon's orbit is slightly elliptical, and Earth is not stationary (rather the Earth-Moon system rotates about its center of mass, which is located some 1700 km below Earth's surface). The clear implication is that Earth's gravitational force causes the Moon to orbit Earth.

Why does Earth not remain stationary as the Moon orbits it? This is because, as expected from Newton's third law, if Earth exerts a force on the Moon, then the Moon should exert an equal and opposite force on Earth (see Figure 4). We do not sense the Moon's effect on Earth's motion, because the Moon's gravity moves our bodies right along with Earth but there are other signs on Earth that clearly show the effect of the Moon's gravitational force as discussed in Chapter 5.4 Satellites and Kepler's Laws.

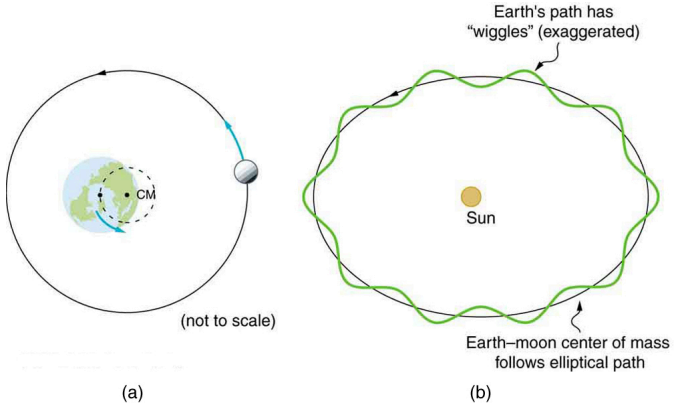


Figure 4. *(a) Earth and the Moon rotate approximately once a month around their common center of mass. (b) Their center of mass orbits the Sun in an elliptical orbit, but Earth's path around the Sun has "wiggles" in it. Similar wiggles in the paths of stars have been observed and are considered direct evidence of planets orbiting those stars. This is important because the planets' reflected light is often too dim to be observed.*

Optional: Tides

Ocean tides are one very observable result of the Moon's gravity acting on Earth. Figure 5 is a simplified drawing of the Moon's position relative to the tides. Because water easily flows on Earth's surface, a high tide is created on the side of Earth nearest to the Moon, where the Moon's gravitational pull is strongest. Why is there also a high tide on the opposite side of Earth? The answer is that Earth is pulled toward the Moon more than the water on the far side, because Earth is closer to the Moon. So the water on the side of Earth closest to the Moon is pulled away from Earth, and Earth is pulled away from water on the far side. As Earth rotates, the tidal bulge (an effect of the tidal forces between an orbiting natural satellite and the primary planet that it orbits) keeps its orientation with the Moon. Thus there are two tides per day (the actual tidal period is about 12 hours and 25.2 minutes), because the Moon moves in its orbit each day as well).

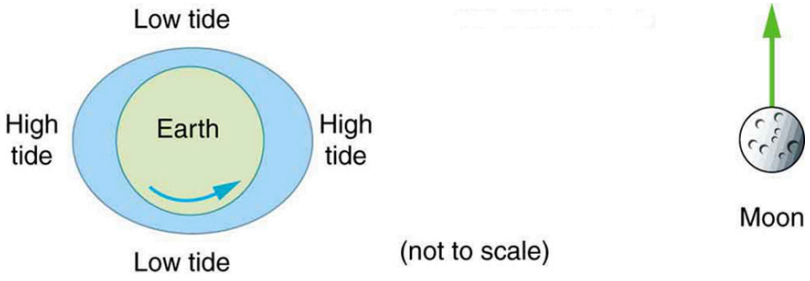


Figure 5. *The Moon causes ocean tides by attracting the water on the near side more than Earth, and by attracting Earth more than the water on the far side. The distances and sizes are not to scale. For this simplified representation of the Earth-Moon system, there are two high and two low tides per day at any location, because Earth rotates under the tidal bulge.*

The Sun also affects tides, although it has about half the effect of the Moon. However, the largest tides, called spring tides, occur when Earth, the Moon, and the Sun are aligned. The smallest tides, called neap tides, occur when the Sun is at a 90° angle to the Earth-Moon alignment.

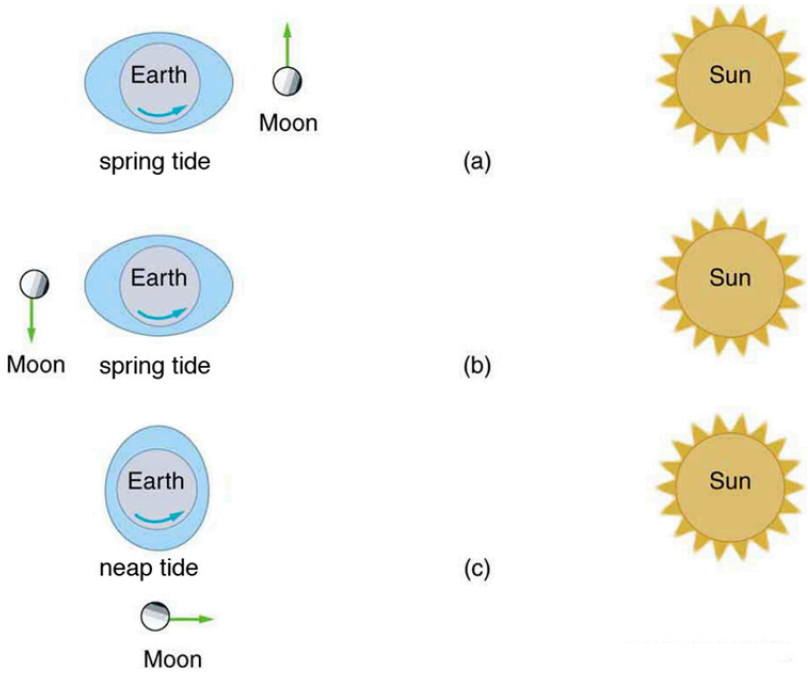


Figure 6. *(a, b) Spring tides: The highest tides occur when Earth, the Moon, and the Sun are aligned. (c) Neap tide: The lowest tides occur when the Sun lies at* 90^0 *to the Earth-Moon alignment. Note that this figure is not drawn to scale.*

Tides are not unique to Earth but occur in many astronomical systems. The most extreme tides occur where the gravitational force is the strongest and varies most rapidly, such as near black holes (see Figure 7). A few likely candidates for black holes have been observed in our galaxy. These have masses greater than the Sun but have diameters only a few kilometers across. The tidal forces near them are so great that they can actually tear matter from a companion star.

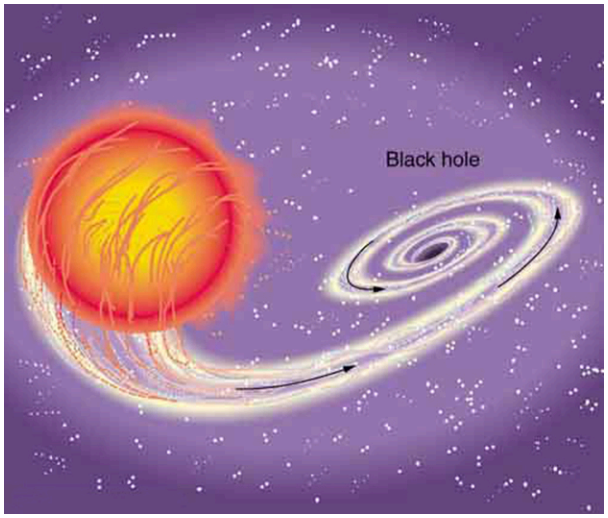


Figure 7. *A black hole is an object with such strong gravity that not even light can escape it. This black hole was created by the supernova of one star in a two-star system. The tidal forces created by the black hole are so great that it tears matter from the companion star. This matter is compressed and heated as it is sucked into the black hole, creating light and X-rays observable from Earth.*

"Weightlessness" and Microgravity

In contrast to the tremendous gravitational force near black holes is the apparent gravitational field experienced by astronauts orbiting Earth. What is the effect of "weightlessness" upon an astronaut who is in orbit for months? Or what about the effect of weightlessness upon plant growth? Weightlessness doesn't mean that an astronaut is not being acted upon by the gravitational force. There is no "zero gravity" in an astronaut's orbit. The term just means that the astronaut is in free-fall, accelerating with the acceleration due to gravity. If an elevator cable breaks, the passengers inside will be in free fall and will experience weightlessness. You can experience short periods of weightlessness in some rides in amusement parks.

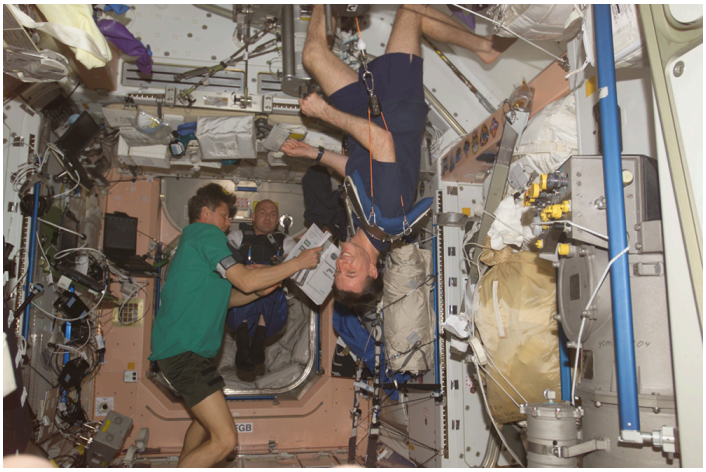


Figure 8. *Astronauts experiencing weightlessness on board the International Space Station. (credit: NASA)*

Microgravity refers to an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface. Many interesting biology and physics topics have been studied over the past three decades in the presence of microgravity. Of immediate concern is the effect on astronauts of extended times in outer space, such as at the International Space Station. Researchers have observed that muscles will atrophy (waste away) in this environment. There is also a corresponding loss of bone mass. Study continues on cardiovascular adaptation to space flight. On Earth, blood pressure is usually higher in the feet than in the head, because the higher column of blood exerts a downward force on it, due to gravity. When standing, 70% of your blood is below the level of the heart, while in a horizontal position, just the opposite occurs. What difference does the absence of this pressure differential have upon the heart?

Some findings in human physiology in space can be clinically important to the management of diseases back on Earth. On a somewhat negative note, spaceflight is known to affect the human immune system, possibly making the crew members more vulnerable to infectious diseases. Experiments flown in space also have shown that some bacteria grow faster in microgravity than they do on Earth. However, on a positive note, studies indicate that microbial antibiotic production can increase by a factor of two in space-grown cultures. One hopes to be able to understand these mechanisms so that similar successes can be achieved on the ground. In another area of physics space research, inorganic crystals and protein crystals have been grown in outer space that have much higher quality than any grown on Earth, so crystallography studies on their structure can yield much better results.

Plants have evolved with the stimulus of gravity and with gravity sensors. Roots grow downward and shoots grow upward. Plants might be able to provide a life support system for long duration space missions by regenerating the atmosphere, purifying water, and producing food. Some studies have indicated that plant growth and development are not affected by gravity, but there is still uncertainty about structural changes in plants grown in a microgravity environment.

The Cavendish Experiment: Then and Now

As previously noted, the universal gravitational constant G is determined experimentally. This definition

was first done accurately by Henry Cavendish (1731–1810), an English scientist, in 1798, more than 100 years after Newton published his universal law of gravitation. The measurement of *G* is very basic and important because it determines the strength of one of the four forces in nature. Cavendish's experiment was very difficult because he measured the tiny gravitational attraction between two ordinary-sized masses (tens of kilograms at most), using apparatus like that in Figure 9. Remarkably, his value for *G* differs by less than 1% from the best modern value.

One important consequence of knowing *G* was that an accurate value for Earth's mass could finally be obtained. This was done by measuring the acceleration due to gravity as accurately as possible and then calculating the mass of Earth m_2 from the relationship Newton's universal law of gravitation gives

$$m_1 g = G\frac{m_1 m_2}{r^2},$$

where m_1 is the mass of the object, m_2 is the mass of Earth, and *r* is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 2. The mass m_1 of the object cancels, leaving an equation for *g*:

$$g = G\frac{m_2}{r^2}.$$

Rearranging to solve for m_2 yields

$$m_2 = \frac{gr^2}{G}.$$

So m_2 can be calculated because all quantities on the right, including the radius of Earth *r*, are known from direct measurements. We shall see in Chapter 5.4 Satellites and Kepler's Laws that knowing *G* also allows for the determination of astronomical masses. Interestingly, of all the fundamental constants in physics, *G* is by far the least well determined.

The Cavendish experiment is also used to explore other aspects of gravity. One of the most interesting questions is whether the gravitational force depends on substance as well as mass—for example, whether one kilogram of lead exerts the same gravitational pull as one kilogram of water. A Hungarian scientist named Roland von Eötvös pioneered this inquiry early in the 20th century. He found, with an accuracy of five parts per billion, that the gravitational force does not depend on the substance. Such experiments continue today, and have improved upon Eötvös' measurements. Cavendish-type experiments such as those of Eric Adelberger and others at the University of Washington, have also put severe limits on the possibility of a fifth force and have verified a major prediction of general relativity—that gravitational energy contributes to rest mass. Ongoing measurements there use a torsion balance and a parallel plate (not spheres, as Cavendish used) to examine how Newton's law of gravitation works over sub-millimeter distances. On this small-scale, do gravitational effects depart from the inverse square law? So far, no deviation has been observed.

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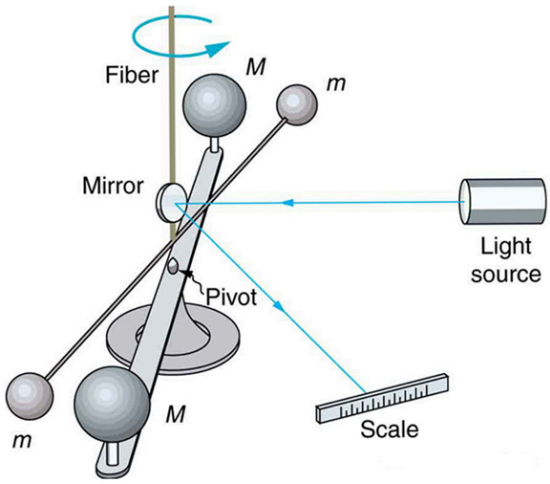


Figure 9. *Cavendish used an apparatus like this to measure the gravitational attraction between the two suspended spheres (m) and the two on the stand (M) by observing the amount of torsion (twisting) created in the fiber. Distance between the masses can be varied to check the dependence of the force on distance. Modern experiments of this type continue to explore gravity.*

Section Summary

- Newton's universal law of gravitation: Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is

$$F_{\text{g}} = \frac{Gm_1m_2}{r^2},$$

  where F is the magnitude of the gravitational force. G is the gravitational constant, given by

$$G = 6.674 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2$$

- Newton's law of gravitation applies universally.

Glossary

gravitational constant, *G*
: a proportionality factor used in the equation for Newton's universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe

center of mass
: the point where the entire mass of an object can be thought to be concentrated

microgravity
: an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface

Newton's universal law of gravitation
: every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them

Conceptual Questions

1: Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

2: Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

3: Draw a free body diagram for a satellite in an elliptical orbit showing why its speed increases as it approaches its parent body and decreases as it moves away.

4: Newton's laws of motion and gravity were among the first to convincingly demonstrate the underlying simplicity and unity in nature. Many other examples have since been discovered, and we now expect to find such underlying order in complex situations. Is there proof that such order will always be found in new explorations?

Problems & Exercises

1: (a) Calculate Earth's mass given the acceleration due to gravity at the North Pole is 9.830 m/s^2 and the radius of the Earth is 6371 km from center to pole.

(b) Compare this with the accepted value of 5.979×10^{24} kg.

2: (a) Calculate the magnitude of the acceleration due to gravity on the surface of Earth due to the Moon.

(b) Calculate the magnitude of the acceleration due to gravity at Earth due to the Sun.

(c) Take the ratio of the Moon's acceleration to the Sun's and comment on why the tides are predominantly due to the Moon in spite of this number.

3: (a) What is the acceleration due to gravity on the surface of the Moon?

(b) On the surface of Mars? The mass of Mars is 6.418×10^{23} kg and its radius is 3.38×10^6 m.

4: (a) Calculate the acceleration due to gravity on the surface of the Sun.

(b) By what factor would your weight increase if you could stand on the Sun? (Never mind that you cannot.)

5: The Moon and Earth rotate about their common center of mass, which is located about 4700 km from the center of Earth. (This is 1690 km below the surface.)

(a) Calculate the magnitude of the acceleration due to the Moon's gravity at that point.

(b) Calculate the magnitude of the centripetal acceleration of the center of Earth as it rotates about that point once each lunar month (about 27.3 d) and compare it with the acceleration found in part (a). Comment on whether or not they are equal and why they should or should not be.

6: Solve part (b) of Example 1 using $a_c = v^2/r$.

7: Astrology, that unlikely and vague pseudoscience, makes much of the position of the planets at the moment of one's birth. The only known force a planet exerts on Earth is gravitational.

(a) Calculate the magnitude of the gravitational force exerted on a 4.20 kg baby by a 100 kg father 0.200 m away at birth (he is assisting, so he is close to the child).

(b) Calculate the magnitude of the force on the baby due to Jupiter if it is at its closest distance to Earth, some 6.29×10^{11} m away. How does the force of Jupiter on the baby compare to the force of the father on the baby? Other objects in the room and the hospital building also exert similar gravitational forces. (Of course, there could be an unknown force acting, but scientists first need to be convinced that there is even an effect, much less that an unknown force causes it.)

8: The existence of the dwarf planet Pluto was proposed based on irregularities in Neptune's orbit. Pluto was subsequently discovered near its predicted position. But it now appears that the discovery was fortuitous, because Pluto is small and the irregularities in Neptune's orbit were not well known. To illustrate that Pluto has a minor effect on the orbit of Neptune compared with the closest planet to Neptune:

(a) Calculate the acceleration due to gravity at Neptune due to Pluto when they are 4.50×10^{12} m apart, as they are at present. The mass of Pluto is 1.4×10^{22} kg.

(b) Calculate the acceleration due to gravity at Neptune due to Uranus, presently about 2.50×10^{12} m apart, and compare it with that due to Pluto. The mass of Uranus is 8.62×10^{25} kg.

9: (a) The Sun orbits the Milky Way galaxy once each 2.60×10^8 y, with a roughly circular orbit averaging 3.00×10^4 light years in radius. (A light year is the distance traveled by light in 1 y.) Calculate the centripetal acceleration of the Sun in its galactic orbit. Does your result support the contention that a nearly inertial frame of reference can be located at the Sun?

(b) Calculate the average speed of the Sun in its galactic orbit. Does the answer surprise you?

10: Unreasonable Result

A mountain 10.0 km from a person exerts a gravitational force on him equal to 2.00% of his weight.

(a) Calculate the mass of the mountain.

(b) Compare the mountain's mass with that of Earth.

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent? (Note that accurate gravitational measurements can easily detect the effect of nearby mountains and variations in local geology.)

Solutions

Problems & Exercises

1: (a) 5.979 x 10^{24} kg (b) This is identical to the best value to three significant figures.

3: (a) 1.62 m/s^2 (b) 3.75 m/s^2

5: (a) 3.42 x 10^{-5} m/s^2 (b) 3.34 x 10^{-5} m/s^2

The values are nearly identical. One would expect the gravitational force to be the same as the centripetal force at the core of the system.

7: (a) 7.01 x 10^{-7} N (b) 1.35 x 10^{-6} N, 0.521

9: (a) 1.66 x 10^{-10} m/s^2 (b) 2.17 x 10^5 m/s

10: (a) 2.937 x 10^{17} kg (b) 4.91 x 10^{-8} of the Earth's mass. (c) The mass of the mountain and its fraction of the Earth's mass are too great. (d) The gravitational force assumed to be exerted by the mountain is too great.

5.4 Optional: Satellites and Kepler's Three Laws

Summary

- State Kepler's laws of planetary motion.
- Derive the third Kepler's law for circular orbits.
- Discuss the Ptolemaic model of the universe.
- Define a planet according to the I.A.U. the International Astronomical Union
- Explain why Pluto is not a planet

Examples of gravitational orbits abound. Hundreds of artificial satellites orbit Earth together with thousands of pieces of debris. The Moon's orbit about Earth has intrigued humans from time immemorial. The orbits of planets, asteroids, meteors, and comets about the Sun are no less interesting. If we look further, we see almost unimaginable numbers of stars, galaxies, and other celestial objects orbiting one another and interacting through gravity.

All these motions are governed by gravitational force, and it is possible to describe them to various degrees of precision. Precise descriptions of complex systems must be made with large computers. However, we can describe an important class of orbits without the use of computers, and we shall find it instructive to study them. These orbits have the following characteristics:

1. *A small mass m_1 orbits a much larger mass m_2.*This allows us to view the motion as if m_2 were stationary—in fact, as if from an inertial frame of reference placed on m_2—without significant error. Mass m_1 is the satellite of m_2, if the orbit is gravitationally bound.
2. *The system is isolated from other masses*. This allows us to neglect any small effects due to outside masses.

The conditions are satisfied, to good approximation, by Earth's satellites (including the Moon), by objects orbiting the Sun, and by the satellites of other planets. Historically, planets were studied first, and there is a classical set of three laws, called Kepler's laws of planetary motion, that describe the orbits of all bodies satisfying the two previous conditions (not just planets in our solar system). These descriptive laws are named for the German astronomer Johannes Kepler (1571–1630), who devised them after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601). Such careful collection and detailed recording of methods and data are hallmarks of good science. Data constitute the evidence from which new interpretations and meanings can be constructed.

Kepler's Laws of Planetary Motion

Kepler's First Law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

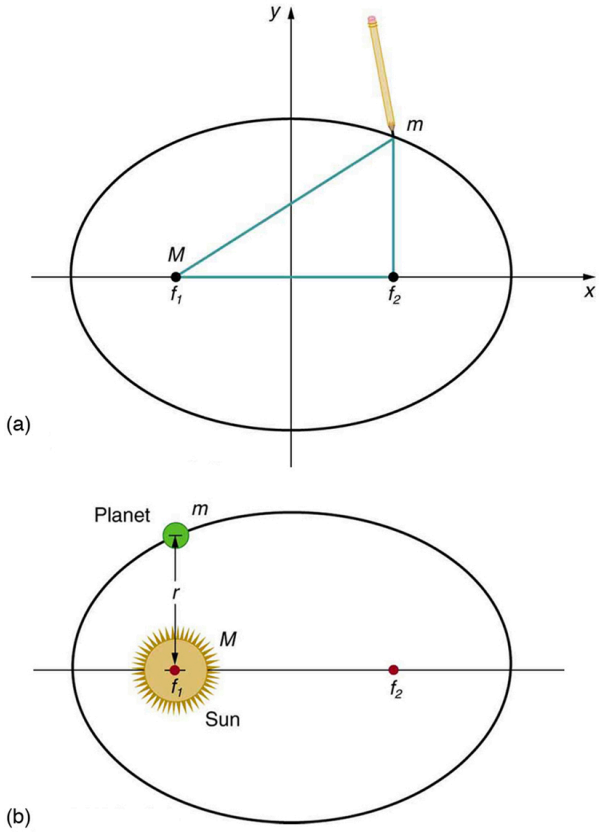


Figure 1. *(a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci (f_1 and f_2) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, m follows an elliptical path with M at one focus. Kepler's first law states this fact for planets orbiting the Sun.*

Kepler's Second Law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times (see Figure 2).

Kepler's Third Law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun. In equation form, this is

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

where T is the period (time for one orbit) and r is the average radius. This equation is valid only for comparing two small masses orbiting the same large one. Most importantly, this is a descriptive equation only, giving no information as to the cause of the equality.

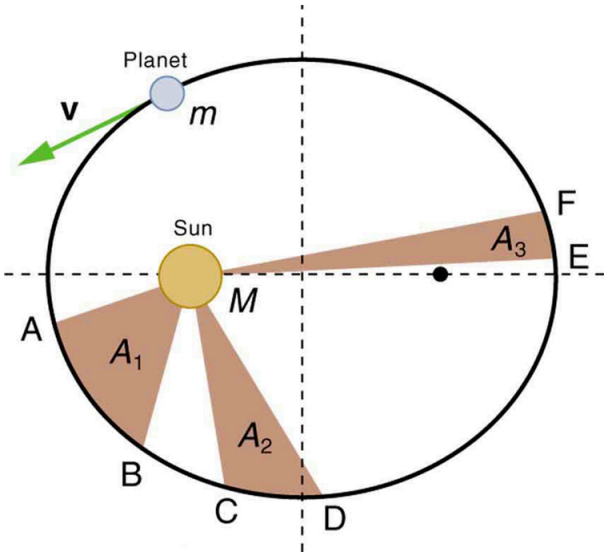


Figure 2. *The shaded regions have equal areas. It takes equal times for m to go from A to B, from C to D, and from E to F. The mass m moves fastest when it is closest to M. Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.*

Note again that while, for historical reasons, Kepler's laws are stated for planets orbiting the Sun, they are actually valid for all bodies satisfying the two previously stated conditions.

Example 1: Find the Time for One Orbit of an Earth Satellite

Given that the Moon orbits Earth each 27.3 d and that it is an average distance of 3.84×10^8 m from the center of Earth, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth's surface.

Strategy

The period, or time for one orbit, is related to the radius of the orbit by Kepler's third law, given in

mathematical form in $\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$. Let us use the subscript 1 for the Moon and the subscript 2 for the satellite. We are asked to find T_2. The given information tells us that the orbital radius of the Moon is $r_1 = 3.84 \times 10^8$ m, and that the period of the Moon is $T_1 = 27.3$ d. The height of the artificial satellite above Earth's surface is given, and so we must add the radius of Earth (6380 km) to get $r_2 = (1500 + 6380)$ km = 7880 km. Now all quantities are known, and so T_2 can be found.

Solution

Kepler's third law is

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

To solve for T_2, we cross-multiply and take the square root, yielding

$$T_2^2 = T_1^2 (\frac{r_2}{r_1})^3 T_2 = T_1 (\frac{r_2}{r_1})^{3/2}.$$

Substituting known values yields

$$\begin{aligned} T_2 &= 27.3 \text{ d} \times \frac{24.0 \text{ h}}{\text{d}} \times (\frac{7880 \text{ km}}{3.84 \times 10^5 \text{ km}})^{3/2} \\ &= 1.93 \text{ h} \end{aligned}$$

Discussion

This is a reasonable period for a satellite in a fairly low orbit. It is interesting that any satellite at this altitude will orbit in the same amount of time. This fact is related to the condition that the satellite's mass is small compared with that of Earth.

People immediately search for deeper meaning when broadly applicable laws, like Kepler's, are discovered. It was Newton who took the next giant step when he proposed the law of universal gravitation. While Kepler was able to discover *what* was happening, Newton discovered that gravitational force was the cause.

Derivation of Kepler's Third Law for Circular Orbits

We shall derive Kepler's third law, starting with Newton's laws of motion and his universal law of gravitation. The point is to demonstrate that the force of gravity is the cause for Kepler's laws (although we will only derive the third one).

Let us consider a circular orbit of a small mass m_1 around a large mass m_2, satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass m_1. Starting with Newton's second law applied to circular motion,

$$F_{\text{net}} = m_1 a_{\text{c}} = m_1 \frac{v^2}{r}.$$

The net external force on mass m_1 is gravity, and so we substitute the force of gravity for F_{net}:

$$G\frac{m_1 m_2}{r^2} = m_1 \frac{v^2}{r}.$$

The mass m_1 cancels, yielding

$$G\frac{m_2}{r} = v^2.$$

The fact that m_1 cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius *r*, all masses orbit at the same speed. (This was implied by the result of the preceding worked example.) Now, to get at Kepler's third law, we must get the period *T* into the equation. By definition, period *T* is the time for one complete orbit. Now the average speed ***v*** is the circumference divided by the period—that is,

$$v = \frac{2\pi r}{T}.$$

Substituting this into the previous equation gives

$$G\frac{m_2}{r} = \frac{4\pi^2 r^2}{T^2}.$$

Solving for T^2 yields

$$T^2 = \frac{4\pi^2}{G m_2} r^3.$$

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

This is Kepler's third law. Note that Kepler's third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body m_2 cancel.

Now consider what we get if we solve $T^2 = \frac{4\pi^2}{G m_2} r^3$. We obtain a relationship that can be used to determine the mass m_2 of a parent body from the orbits of its satellites:

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} m_2.$$

If *r* and *T* are known for a satellite, then the mass m_2 of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio

r^3/T^2 should be a constant for all satellites of the same parent body (because $r^3/T^2 = Gm_2/4\pi^2$). (See Table 2).

It is clear from Table 2 that the ratio of r^3/T^2 is constant, at least to the third digit, for all listed satellites of the Sun, and for those of Jupiter. Small variations in that ratio have two causes—uncertainties in the r and T data, and perturbations of the orbits due to other bodies. Interestingly, those perturbations can be—and have been—used to predict the location of new planets and moons. This is another verification of Newton's universal law of gravitation.

MAKING CONNECTIONS

Newton's universal law of gravitation is modified by Einstein's general theory of relativity. Newton's gravity is not seriously in error—it was and still is an extremely good approximation for most situations. Einstein's modification is most noticeable in extremely large gravitational fields, such as near black holes. However, general relativity also explains such phenomena as small but long-known deviations of the orbit of the planet Mercury from classical predictions.

The Case for Simplicity

The development of the universal law of gravitation by Newton played a pivotal role in the history of ideas. While it is beyond the scope of this text to cover that history in any detail, we note some important points. The definition of planet set in 2006 by the International Astronomical Union (IAU) states that in the solar system, a planet is a celestial body that:

1. is in orbit around the Sun,
2. has sufficient mass to assume hydrostatic equilibrium and
3. has cleared the neighborhood around its orbit.

A non-satellite body fulfilling only the first two of the above criteria is classified as "dwarf planet."

In 2006, Pluto was demoted to a 'dwarf planet' after scientists revised their definition of what constitutes a "true" planet.

Parent	Satellite	Average orbital radius r(km)	Period T(y)	r^3 / T^2 (km^3 / y^2)
Earth	Moon	3.84×10^5	0.07481	1.01×10^{19}
Sun	Mercury	5.79×10^7	0.2409	3.34×10^{24}
	Venus	1.082×10^8	0.6150	3.35×10^{24}
	Earth	1.496×10^8	1.000	3.35×10^{24}
	Mars	2.279×10^8	1.881	3.35×10^{24}
	Jupiter	7.783×10^8	11.86	3.35×10^{24}
	Saturn	1.427×10^9	29.46	3.35×10^{24}
	Neptune	4.497×10^9	164.8	3.35×10^{24}
	Pluto	5.90×10^9	248.3	3.33×10^{24}
Jupiter	Io	4.22×10^5	0.00485 (1.77 d)	3.19×10^{21}
	Europa	6.71×10^5	0.00972 (3.55 d)	3.20×10^{21}
	Ganymede	1.07×10^6	0.0196 (7.16 d)	3.19×10^{21}
	Callisto	1.88×10^6	0.0457 (16.19 d)	3.20×10^{21}

Table 2. Orbital Data and Kepler's Third Law.

The universal law of gravitation is a good example of a physical principle that is very broadly applicable. That single equation for the gravitational force describes all situations in which gravity acts. It gives a cause for a vast number of effects, such as the orbits of the planets and moons in the solar system. It epitomizes the underlying unity and simplicity of physics.

Before the discoveries of Kepler, Copernicus, Galileo, Newton, and others, the solar system was thought to revolve around Earth as shown in Figure 3(a). This is called the Ptolemaic view, for the Greek philosopher who lived in the second century AD. This model is characterized by a list of facts for the motions of planets with no cause and effect explanation. There tended to be a different rule for each heavenly body and a general lack of simplicity.

Figure 3(b) represents the modern or Copernican model. In this model, a small set of rules and a single underlying force explain not only all motions in the solar system, but all other situations involving gravity. The breadth and simplicity of the laws of physics are compelling. As our knowledge of nature has grown, the basic simplicity of its laws has become ever more evident.

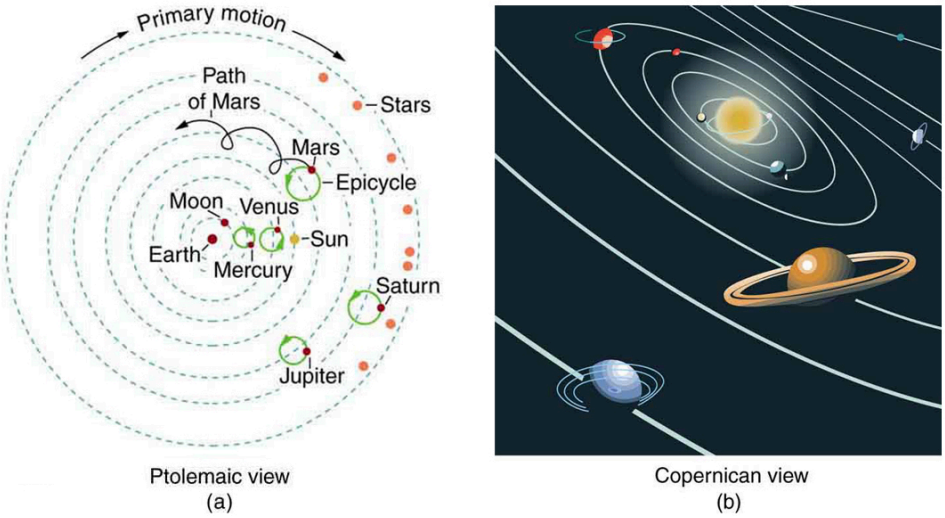


Figure 3. *(a) The Ptolemaic model of the universe has Earth at the center with the Moon, the planets, the Sun, and the stars revolving about it in complex superpositions of circular paths. This geocentric model, which can be made progressively more accurate by adding more circles, is purely descriptive, containing no hints as to what are the causes of these motions. (b) The Copernican model has the Sun at the center of the solar system. It is fully explained by a small number of laws of physics, including Newton's universal law of gravitation.*

Section Summary

- Kepler's laws are stated for a small mass m_1 orbiting a larger mass m_2 in near-isolation. Kepler's laws of planetary motion are then as follows:

  Kepler's first law

  The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

  Kepler's second law

  Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.

  Kepler's third law

  The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

  where T is the period (time for one orbit) and r is the average radius of the orbit.

- The period and radius of a satellite's orbit about a larger body m_2 are related by

$$T_2 = \frac{4\pi^2}{Gm_2} r^3$$

or

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} m_2 .$$

Conceptual Questions

1: In what frame(s) of reference are Kepler's laws valid? Are Kepler's laws purely descriptive, or do they contain causal information?

Problems & Exercises

1: A geosynchronous Earth satellite is one that has an orbital period of precisely 1 day. Such orbits are useful for communication and weather observation because the satellite remains above the same point on Earth (provided it orbits in the equatorial plane in the same direction as Earth's rotation). Calculate the radius of such an orbit based on the data for the moon in Table 2.

2: Calculate the mass of the Sun based on data for Earth's orbit and compare the value obtained with the Sun's actual mass.

3: Find the mass of Jupiter based on data for the orbit of one of its moons, and compare your result with its actual mass.

4: Find the ratio of the mass of Jupiter to that of Earth based on data in Table 2.

5: Astronomical observations of our Milky Way galaxy indicate that it has a mass of about 8.0×10^{11} solar masses. A star orbiting on the galaxy's periphery is about 6.0×10^4 light years from its center. (a) What should the orbital period of that star be? (b) If its period is 6.0×10^7 instead, what is the mass of the galaxy? Such calculations are used to imply the existence of "dark matter" in the universe and have indicated, for example, the existence of very massive black holes at the centers of some galaxies.

6: Integrated Concepts

Space debris left from old satellites and their launchers is becoming a hazard to other satellites. (a) Calculate the speed of a satellite in an orbit 900 km above Earth's surface. (b) Suppose a loose rivet is in an orbit of the same radius that intersects the satellite's orbit at an angle of 90° relative to Earth. What is the velocity of the rivet relative to the satellite just before striking it? (c) Given the rivet is 3.00 mm in size, how long will its collision with the satellite last? (d) If its mass is 0.500 g, what is the average force it exerts on the satellite? (e) How much energy in joules is generated by the collision? (The satellite's velocity does not change appreciably, because its mass is much greater than the rivet's.)

7: Unreasonable Results

(a) Based on Kepler's laws and information on the orbital characteristics of the Moon, calculate the orbital

radius for an Earth satellite having a period of 1.00 h. (b) What is unreasonable about this result? (c) What is unreasonable or inconsistent about the premise of a 1.00 h orbit?

8: Construct Your Own Problem

On February 14, 2000, the NEAR spacecraft was successfully inserted into orbit around Eros, becoming the first artificial satellite of an asteroid. Construct a problem in which you determine the orbital speed for a satellite near Eros. You will need to find the mass of the asteroid and consider such things as a safe distance for the orbit. Although Eros is not spherical, calculate the acceleration due to gravity on its surface at a point an average distance from its center of mass. Your instructor may also wish to have you calculate the escape velocity from this point on Eros.

Solutions

Problems & Exercises

2: 1.98 x 10^{30} kg

4: $\frac{M_{\text{J}}}{M_{\text{E}}} = 316$

6: (a) 7.4 x 10^{3} m/s (b) 1.05 x 10^{3} m/s (c) 2.86 x 10^{-7} s (d) 1.84 x 10^{7} N (e) 2.76 x 10^{4} J

7: (a) 5.08 x 10^{3} km (b) This radius is unreasonable because it is less than the radius of earth. (c) The premise of a one-hour orbit is inconsistent with the known radius of the earth.

5.5 Optional: Fictitious Forces and Non-Inertial Frames

Summary

- Discuss the inertial frame of reference.
- Discuss the non-inertial frame of reference.
- Describe the effects of the Coriolis force.

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may *seem* real, because the observer's frame of reference is accelerating or rotating.

When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that *you* tend to remain stationary while the *seat* pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, *forced*) toward the left relative to the car. Again, a physicist would say that *you* are going in a straight line but the *car* moves to the right, and there is no real force on you to the left. Recall Newton's first law.

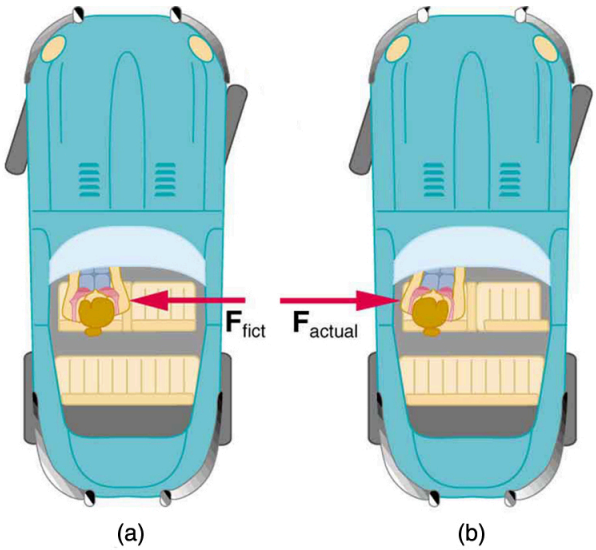


Figure 1. *(a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn.*

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). In such a frame of reference, Newton's laws of motion take the form given in Chapter 4 Dynamics: Newton's Laws of Motion The car is a **non-inertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is a **fictitious force** having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named **centrifugal force (**not to be confused with centripetal force**)**, trying to throw you off. You must hang on tightly to counteract the centrifugal force. In Earth's frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.

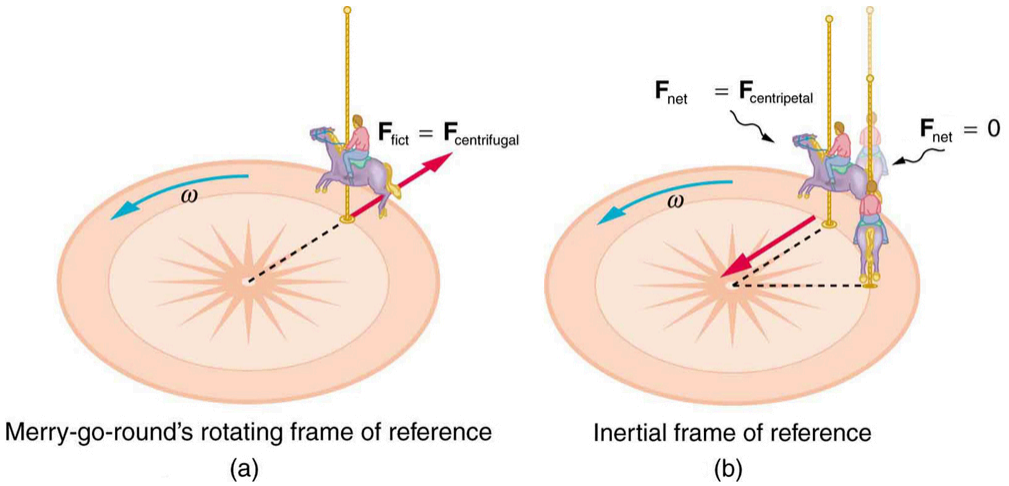


Figure 2. *(a) A rider on a merry-go-round feels as if he is being thrown off. This fictitious force is called the centrifugal force—it explains the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off and not a real force (the unshaded rider has $F_{net}=0$ and heads in a straight line). A real force, $F_{centripetal}$, is needed to cause a circular path.*

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (see Figure 3). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

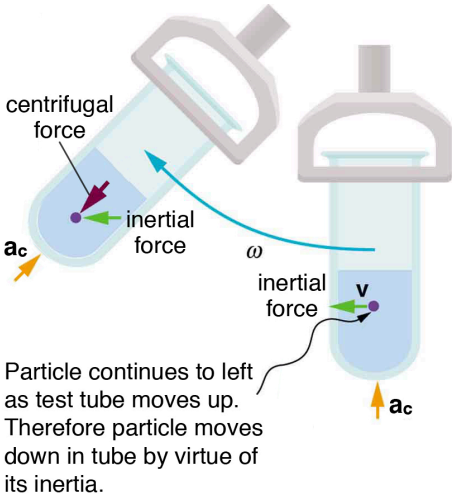


Figure 3. *Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles will come into contact with the test tube walls, which will then supply the centripetal force needed to make them move in a circle of constant radius.*

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in Figure 4? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using a fictitious force, called the **Coriolis force**, that causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's Laws in non-inertial frames of reference.

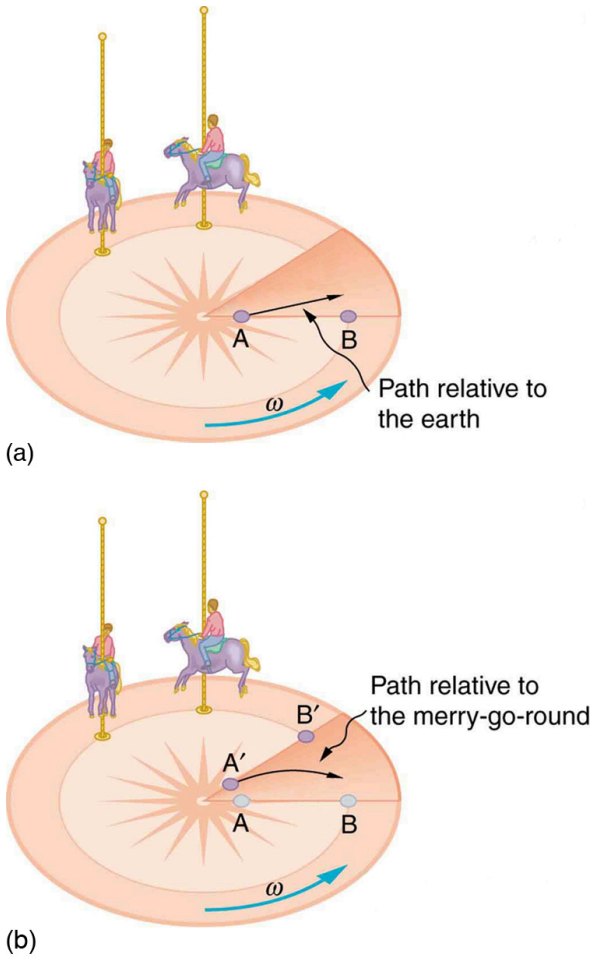


Figure 4. *Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.*

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects *do* exist—in the rotation of weather systems, for example. Most consequences of Earth's rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in Figure 4. As on the merry-go-round, any motion in Earth's northern hemisphere experiences a Coriolis force to the right. Just the opposite occurs in the southern hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the northern hemisphere to rotate in the counterclockwise direction, while the tropical cyclones (what hurricanes are called below the equator) in the southern hemisphere rotate in the clockwise direction. The terms hurricane, typhoon, and tropical storm are regionally-specific names for tropical cyclones, storm systems characterized by low pressure centers,

strong winds, and heavy rains. Figure 5 helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the northern hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the northern hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When non-inertial frames are used, fictitious forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these fictitious forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest and truest, in the sense that all forces have real origins and explanations.

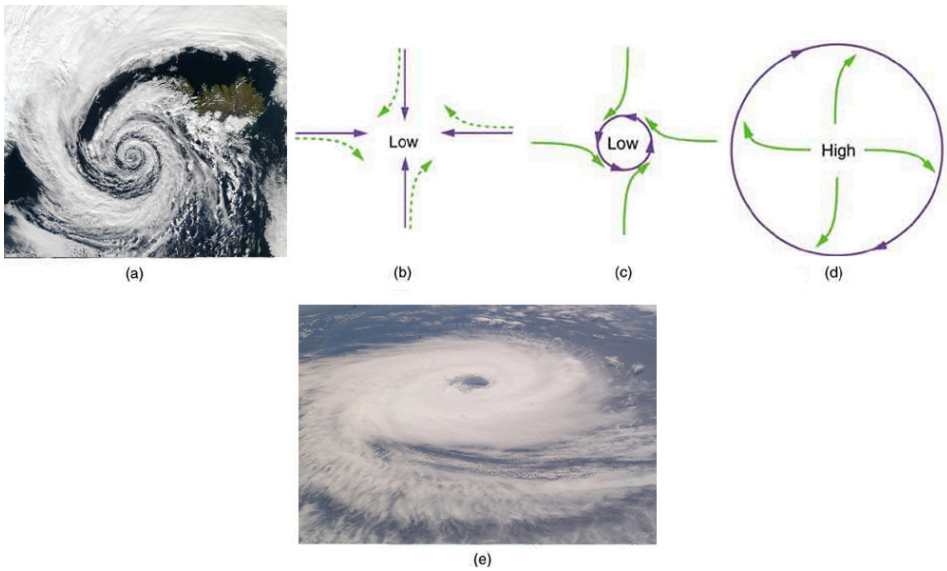


Figure 5. *(a) The counterclockwise rotation of this northern hemisphere hurricane is a major consequence of the Coriolis force. (credit: NASA) (b) Without the Coriolis force, air would flow straight into a low-pressure zone, such as that found in tropical cyclones. (c) The Coriolis force deflects the winds to the right, producing a counterclockwise rotation. (d) Wind flowing away from a high-pressure zone is also deflected to the right, producing a clockwise rotation. (e) The opposite direction of rotation is produced by the Coriolis force in the southern hemisphere, leading to tropical cyclones. (credit: NASA)*

Section Summary

- Rotating and accelerated frames of reference are non-inertial.
- Fictitious forces, such as the Coriolis force, are needed to explain motion in such frames.

Glossary

fictitious force
a force having no physical origin

centrifugal force
a fictitious force that tends to throw an object off when the object is rotating in a non-inertial frame of reference

Coriolis force
the fictitious force causing the apparent deflection of moving objects when viewed in a rotating frame of reference

non-inertial frame of reference
an accelerated frame of reference

Conceptual Questions

1: When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the northern hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

2: Is there a real force that throws water from clothes during the spin cycle of a washing machine? Explain how the water is removed.

3: In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is a fictitious force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all of the real forces acting on them.

4: Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

5: Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

6: A non-rotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

Chapter 6 Work and Energy

Figure 1. *How many forms of energy can you identify in this photograph of a wind farm in Iowa? (credit: Jürgen from Sandesneben, Germany, Wikimedia Commons)*

Energy plays an essential role both in everyday events and in scientific phenomena. You can no doubt name many forms of energy, from that provided by our foods, to the energy we use to run our cars, to the sunlight that warms us on the beach. You can also cite examples of what people call energy that may not be scientific, such as someone having an energetic personality. Not only does energy have many interesting forms, it is involved in almost all phenomena, and is one of the most important concepts of physics. What makes it even more important is that the total amount of energy in the universe is constant. Energy can change forms, but it cannot appear from nothing or disappear without a trace. Energy is thus one of a handful of physical quantities that we say is *conserved*.

Conservation of energy (as physicists like to call the principle that energy can neither be created nor destroyed) is based on experiment. Even as scientists discovered new forms of energy, conservation of energy has always been found to apply. Perhaps the most dramatic example of this was supplied by Einstein when he suggested that mass is equivalent to energy (his famous equation $E = mc^2$).

From a societal viewpoint, energy is one of the major building blocks of modern civilization. Energy resources are key limiting factors to economic growth. The world use of energy resources, especially oil, continues to grow, with ominous consequences economically, socially, politically, and environmentally. We will briefly examine the world's energy use patterns at the end of this chapter.

There is no simple, yet accurate, scientific definition for energy. Energy is characterized by its many forms and the fact that it is conserved. We can loosely define **energy** as the ability to do work, admitting that in some circumstances not all energy is available to do work. Because of the association of energy with work, we begin the chapter with a discussion of work. Work is intimately related to energy and how energy moves from one system to another or changes form.

6.1 Work: The Scientific Definition

Summary

- Explain how an object must be displaced for a force on it to do work.
- Explain how relative directions of force and displacement determine whether the work done is positive, negative, or zero.

What It Means to Do Work

The scientific definition of work differs in some ways from its everyday meaning. Certain things we think of as hard work, such as writing an exam or carrying a heavy load on level ground, are not work as defined by a scientist. The scientific definition of work reveals its relationship to energy—whenever work is done, energy is transferred.

For work, in the scientific sense, to be done, a force must be exerted and there must be displacement in the direction of the force.

Formally, the **work** done on a system by a constant force is defined to be *the product of the component of the force in the direction of motion times the distance through which the force acts*. For one-way motion in one dimension, this is expressed in equation form as

$$W = |\vec{\mathrm{F}}|(\cos\theta)|\mathrm{d}|,$$

where W is work, d is the displacement of the system, and θ is the angle between the force vector $\vec{\mathrm{F}}$ and the displacement vector d, as in Figure 1. We can also write this as

$$W = Fd\cos\theta.$$

To find the work done on a system that undergoes motion that is not one-way or that is in two or three dimensions, we divide the motion into one-way one-dimensional segments and add up the work done over each segment.

WHAT IS WORK?

The work done on a system by a constant force is *the product of the component of the force in the*

direction of motion times the distance through which the force acts. For one-way motion in one dimension, this is expressed in equation form as

$$W = Fd\cos\theta,$$

where W is work, F is the magnitude of the force on the system, d is the magnitude of the displacement of the system, and θ is the angle between the force vector $\vec{\mathrm{F}}$ and the displacement vector d.

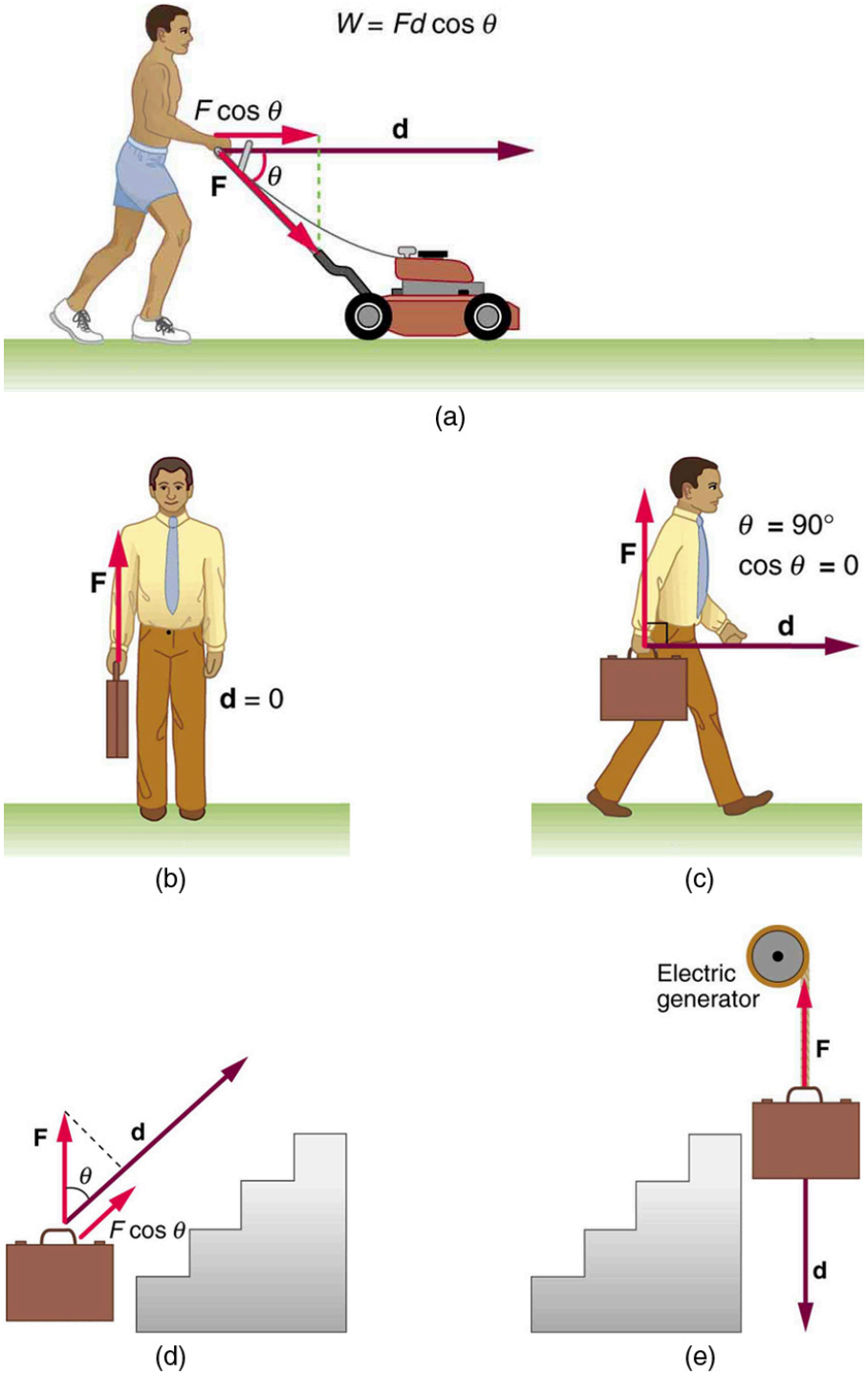


Figure 1. *Examples of work. (a) The work done by the force F on this lawn mower is Fd cosθ. Note that F cosθ is the component of the force in the direction of motion. (b) A person holding a briefcase does no work on it, because there is no displacement. No energy is transferred to or from the briefcase. (c) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (d) Work is done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force F in the direction of the motion. Energy is transferred to the briefcase and could in turn be used to do work. (e) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric*

generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because F and d are in opposite directions.

To examine what the definition of work means, let us consider the other situations shown in Figure 1. The person holding the briefcase in Figure 1(b) does no work, for example. Here $d = 0$, so $W = 0$. Why is it you get tired just holding a load? The answer is that your muscles are doing work against one another, *but they are doing no work on the system of interest* (the "briefcase-Earth system"). There must be displacement for work to be done, and there must be a component of the force in the direction of the motion. For example, the person carrying the briefcase on level ground in Figure 1(c) does no work on it, because the force is perpendicular to the motion. That is, $\cos 90° = 0$, and so $W = 0$.

In contrast, when a force exerted on the system has a component in the direction of motion, such as in Figure 1(d), work *is* done—energy is transferred to the briefcase. Finally, in Figure 1(e), energy is transferred from the briefcase to a generator. There are two good ways to interpret this energy transfer. One interpretation is that the briefcase's weight does work on the generator, giving it energy. The other interpretation is that the generator does negative work on the briefcase, thus removing energy from it. The drawing shows the latter, with the force from the generator upward on the briefcase, and the displacement downward. This makes $\theta = 180°$, and $\cos 180° = -1$; therefore, W is negative.

Calculating Work

Work and energy have the same units. From the definition of work, we see that those units are force times distance. Thus, in SI units, work and energy are measured in **newton-meters**. A newton-meter is given the special name **joule** (J), and $1\ \text{J} = 1\ \text{Nm} = 1\ \text{kgm}^2/\text{s}^2$. One joule is not a large amount of energy; it would lift a small 100-gram apple a distance of about 1 meter.

Example 1: Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn

How much work is done on the lawn mower by the person in Figure 1(a) if he exerts a constant force of 75.0 N at an angle 35° below the horizontal and pushes the mower 25.0 m on level ground? Convert the amount of work from joules to kilocalories and compare it with this person's average daily intake of 10,000 kJ (about 2400 kcal) of food energy. One *calorie* (1 cal) of heat is the amount required to warm 1 g of water by 1 °C, and is equivalent to 4.184 J, while one *food calorie* (1 kcal) is equivalent to 4184 J.

Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation $W = Fd \cos \theta$. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

$$W = Fd \cos \theta.$$

Substituting the known values gives

$$W = (75.0\ \text{N})(25.0\ \text{m})\cos\ 35.0^o$$
$$= 1536\ \text{J} = 1.54 \times 10^3\ \text{J}$$

Converting the work in joules to kilocalories yields W = (1536 J)(1 kcal/4184 J) = 0.367 kcal. The ratio of the work done to the daily consumption is

$$\frac{W}{2400\ \text{kcal}} = 1.53 \times 10^{-4}.$$

Discussion

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we "work" all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

Section Summary

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work W that a force $\vec{\text{F}}$ does on an object is the product of the magnitude F of the force, times the magnitude d of the displacement, times the cosine of the angle θ between them. In symbols,
$$W = Fd\cos\theta.$$
- The SI unit for work and energy is the joule (J), where 1 J = 1 Nm = 1 kgm^2/s^2.
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.
- The work done is positive if the force and displacement have the same direction, and negative if they have opposite direction.

Conceptual Questions

1: Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.

2: Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.

3: Describe a situation in which a force is exerted for a long time but does no work. Explain.

Problems & Exercises

1: How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N? Express your answer in joules and kilocalories.

2: A 75.0-kg person climbs stairs, gaining 2.50 meters in height. Find the work done to accomplish this task.

3: (a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?

4: Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (See Chapter 6.6 Table 1 for the energy content of gasoline.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

5: Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of 20.0° with the horizontal. (See Figure 2.) He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate *and* on his body to get up the ramp.

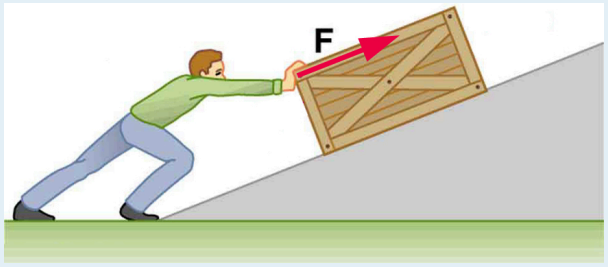


Figure 2. *A man pushes a crate up a ramp.*

6: How much work is done by the boy pulling his sister 30.0 m in a wagon as shown in Figure 3? Assume no friction acts on the wagon.

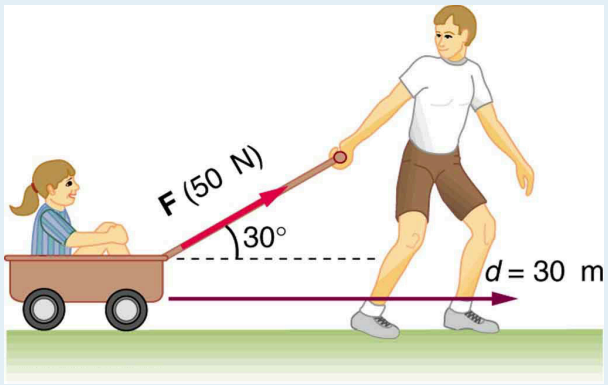


Figure 3. *The boy does work on the system of the wagon and the child when he pulls them as shown.*

7: A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force.

He pushes in a direction 25.0° below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

8: Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a 60.0° slope at constant speed, as shown in Figure 4. The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?

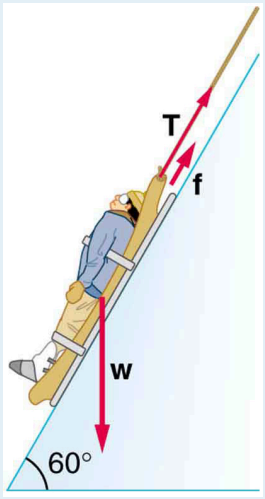


Figure 4. *A rescue sled and victim are lowered down a steep slope.*

Solutions

Problems & Exercises

1: 3.00 J = 7.17 x 10^{-4} kcal

3: (a) 5.92 x 10^{5} J (b) -5.88 x 10^{5} J (c) The net force is zero.

5: 3.14 x 10^{3} J

7: (a) -700 J (b) 0 J (c) 700 J (d) 38.6 N (e) 0 J

6.2 Kinetic Energy and the Work-Energy Theorem

Summary

- Explain work as a transfer of energy and net work as the work done by the net force.
- Explain and apply the work-energy theorem.

Work Transfers Energy

What happens to the work done on a system? Energy is transferred into the system, but in what form? Does it remain in the system or move on? The answers depend on the situation. For example, if the lawn mower in Chapter 6.1 Figure 1(a) is pushed just hard enough to keep it going at a constant speed, then energy put into the mower by the person is removed continuously by friction, and eventually leaves the system in the form of heat transfer. In contrast, work done on the briefcase by the person carrying it up stairs in Chapter 6.1 Figure 1(d) is stored in the briefcase-Earth system and can be recovered at any time, as shown in Chapter 6.1 Figure 1(e). In fact, the building of the pyramids in ancient Egypt is an example of storing energy in a system by doing work on the system. Some of the energy imparted to the stone blocks in lifting them during construction of the pyramids remains in the stone-Earth system and has the potential to do work.

In this section we begin the study of various types of work and forms of energy. We will find that some types of work leave the energy of a system constant, for example, whereas others change the system in some way, such as making it move. We will also develop definitions of important forms of energy, such as the energy of motion.

Net Work and the Work-Energy Theorem

We know from the study of Newton's laws in Chapter 4 Dynamics: Force and Newton's Laws of Motion that net force causes acceleration. We will see in this section that work done by the net force gives a system energy of motion, and in the process we will also find an expression for the energy of motion.

Let us start by considering the total, or net, work done on a system. Net work is defined to be the sum of work done by all external forces—that is, **net work** is the work done by the net external force F_{net}. In equation form, this is $W_{\text{net}} = F_{\text{net}}\, d \cos\theta$ where θ is the angle between the force vector and the displacement vector.

Figure 1(a) shows a graph of force versus displacement for the component of the force in the direction of the displacement—that is, an $F\cos\theta$ vs. d graph. In this case, $F\cos\theta$ is constant. You can see that the area under the graph is $Fd\cos\theta$, or the work done. Figure 1(b) shows a more general process where the

force varies. The area under the curve is divided into strips, each having an average force $(F \cos \theta)_{i(\text{ave})}$. The work done is $(F \cos \theta)_{i(\text{ave})} d_i$ for each strip, and the total work done is the sum of the W_i. Thus the total work done is the total area under the curve, a useful property to which we shall refer later.

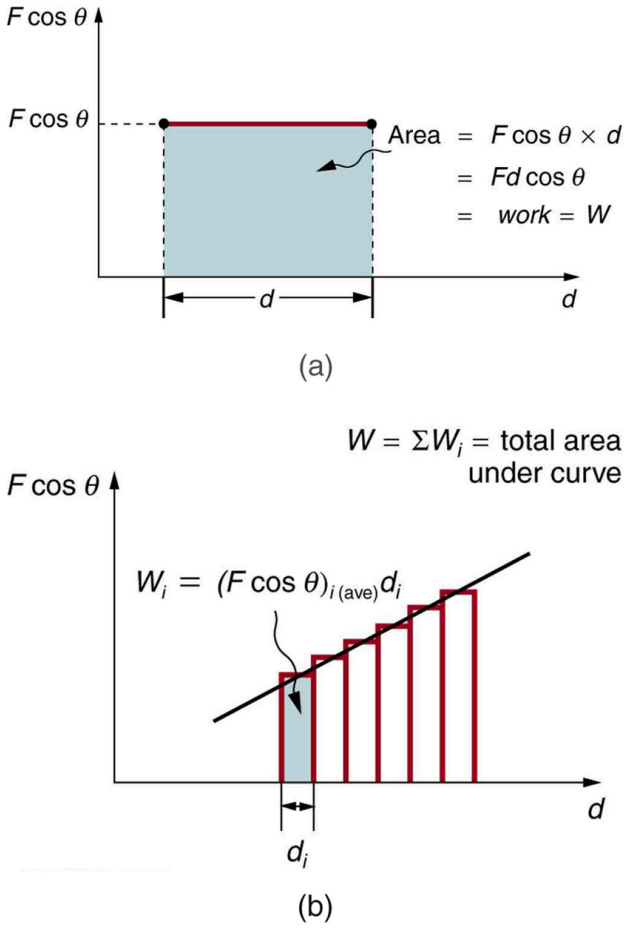


Figure 1. *(a) A graph of F cos θ vs. d, when F cos θ is constant. The area under the curve represents the work done by the force. (b) A graph of F cos θ vs. d in which the force varies. The work done for each interval is the area of each strip; thus, the total area under the curve equals the total work done.*

Net work will be simpler to examine if we consider a one-dimensional situation where a force is used to accelerate an object in a direction parallel to its initial velocity. Such a situation occurs for the package on the roller belt conveyor system shown in Figure 2**.**

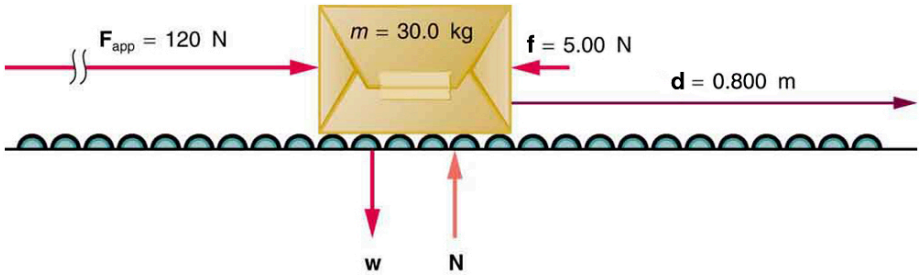


Figure 2. *A package on a roller belt is pushed horizontally through a distance d.*

The force of gravity and the normal force acting on the package are perpendicular to the displacement and do no work. Moreover, they are also equal in magnitude and opposite in direction so they cancel in calculating the net force. The net force arises solely from the horizontal applied force Fapp and the horizontal friction force *f*. Thus, as expected, the net force is parallel to the displacement, so that $\theta = 0°$ and $\cos\theta = 1$, and the net work is given by

$$W_{\text{net}} = F_{\text{net}}d.$$

The effect of the net force F_{net} is to accelerate the package from v_0 to v. The kinetic energy of the package increases, indicating that the net work done on the system is positive. (See Example 1.) By using Newton's second law, and doing some algebra, we can reach an interesting conclusion. Substituting $F_{\text{net}} = ma$ from Newton's second law gives

$$W_{\text{net}} = mad.$$

To get a relationship between net work and the speed given to a system by the net force acting on it, we take $d = x - x_0$ and use the equation studied in Chapter 2.6 Motion Equations for Constant Acceleration in One Dimension for the change in speed over a distance *d* if the acceleration has the constant value *a*; namely, $v^2 = v_0^2 + 2ad$ (note that *a* appears in the expression for the net work). Solving for acceleration gives $a = \dfrac{v^2 - v_0^2}{2d}$. When *a* is substituted into the preceding expression for W_{net}, we obtain

$$W_{\text{net}} = m\left(\frac{v^2 - v_0^2}{2d}\right)d.$$

The *d* cancels, and we rearrange this to obtain

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

This expression is called the **work-energy theorem**, and it actually applies *in general* (even for forces that vary in direction and magnitude), although we have derived it for the special case of a constant force parallel to the displacement. The theorem implies that the net work on a system equals the change in the quantity $\frac{1}{2}mv^2$. This quantity is our first example of a form of energy.

THE WORK-ENERGY THEOREM

The net work on a system equals the change in the quantity $\frac{1}{2}mv^2$.

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

The quantity $\frac{1}{2}mv^2$ in the work-energy theorem is defined to be the translational kinetic energy (KE) of a mass *m* moving at a speed *v*. (*Translational* kinetic energy is distinct from *rotational* kinetic energy, which is considered later.) In equation form, the translational kinetic energy,

$$\text{KE} = \frac{1}{2}mv^2,$$

is the energy associated with translational motion. Kinetic energy is a form of energy associated with the motion of a particle, single body, or system of objects moving together.

We are aware that it takes energy to get an object, like a car or the package in Figure 2, up to speed, but it may be a bit surprising that kinetic energy is proportional to speed squared. This proportionality means, for example, that a car traveling at 100 km/h has four times the kinetic energy it has at 50 km/h, helping to explain why high-speed collisions are so devastating. We will now consider a series of examples to illustrate various aspects of work and energy.

Example 1: Calculating the Kinetic Energy of a Package

Suppose a 30.0-kg package on the roller belt conveyor system in Figure 2 is moving at 0.500 m/s. What is its kinetic energy?

Strategy

Because the mass *m* and speed *v* are given, the kinetic energy can be calculated from its definition as given in the equation $\text{KE} = \frac{1}{2}mv^2$.

Solution

The kinetic energy is given by

$$KE = \frac{1}{2}mv^2,$$

Entering known values gives

$$\text{KE} = 0.5(30.0\ \text{kg})(0.500\ \text{m/s})^2,$$

which yields

$$\text{KE} = 3.75\ \text{k}\cdot\text{m}^2/\text{s}^2 = 3.75\ \text{J}.$$

Discussion

Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

Example 2: Determining the Work to Accelerate a Package

Suppose that you push on the 30.0-kg package in Figure 2 with a constant force of 120 N through a distance of 0.800 m, and that the opposing friction force averages 5.00 N.

(a) Calculate the net work done on the package. (b) Solve the same problem as in part (a), this time by finding the work done by each force that contributes to the net force.

Strategy and Concept for (a)

This is a motion in one dimension problem, because the downward force (from the weight of the package) and the normal force have equal magnitude and opposite direction, so that they cancel in calculating the net force, while the applied force, friction, and the displacement are all horizontal. (See Figure 2.) As expected, the net work is the net force times distance.

Solution for (a)

The net force is the push force minus friction, or F_{net} = 120 N – 5.00 N = 115 N. Thus the net work is

$$\begin{aligned} W_{\text{net}} &= F_{\text{net}}\, d = (115\ \text{N})(0.800\ \text{m}) \\ &= 92.0\ \text{N}\cdot\text{m} = 92.0\ \text{J}. \end{aligned}$$

Discussion for (a)

This value is the net work done on the package. The person actually does more work than this, because friction opposes the motion. Friction does negative work and removes some of the energy the person expends and converts it to thermal energy. The net work equals the sum of the work done by each individual force.

Strategy and Concept for (b)

The forces acting on the package are gravity, the normal force, the force of friction, and the applied force. The normal force and force of gravity are each perpendicular to the displacement, and therefore do no work.

Solution for (b)

The applied force does work.

$$\begin{aligned} W_{\text{net}} &= F_{\text{net}}\ d = (115\ \text{N})(0.800\ \text{m}) \\ &= 92.0\ \text{N}\cdot\text{m} = 92.0\ \text{J} \end{aligned}$$

So the amounts of work done by gravity, by the normal force, by the applied force, and by friction are, respectively,

$$\begin{aligned} W_{gr} &= 0, \\ W_N &= 0, \\ W_{app} &= 96.0\ \text{J}, \\ W_{fr} &= -4.00\ \text{J}. \end{aligned}$$

The total work done as the sum of the work done by each force is then seen to be

$$W_{\text{total}} = W_{\text{gr}} + W_{\text{N}} + W_{\text{app}} + W_{\text{fr}} = 92.0\ \ \text{J}.$$

Discussion for (b)

The calculated total work W_{total} as the sum of the work by each force agrees, as expected, with the work W_{net} done by the net force. The work done by a collection of forces acting on an object can be calculated by either approach.

Example 3: Determining Speed from Work and Energy

Find the speed of the package in Figure 2 at the end of the push, using work and energy concepts.

Strategy

Here the work-energy theorem can be used, because we have just calculated the net work, W_{net}, and the initial kinetic energy, $\frac{1}{2}mv_0^2$. These calculations allow us to find the final kinetic energy, $\frac{1}{2}mv^2$, and thus the final speed v.

Solution

The work-energy theorem in equation form is

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

Solving for $\frac{1}{2}mv^2$ gives

$$\frac{1}{2}mv^2 = W_{\text{net}} + \frac{1}{2}mv_0^2.$$

Thus,

$$\frac{1}{2}mv^2 = 92.0\ \text{J} + 3.75\ \text{J} = 95.75\ \text{J}$$

Solving for the final speed as requested and entering known values gives

$$v = \sqrt{\frac{2(95.75\ \text{J})}{m}} = \sqrt{\frac{191.5\ \text{kg}\cdot\text{m}^2/\text{s}^2}{30.0\ \text{kg}}}$$

Discussion

Using work and energy, we not only arrive at an answer, we see that the final kinetic energy is the sum of the initial kinetic energy and the net work done on the package. This means that the work indeed adds to the energy of the package.

Example 4: Work and Energy Can Reveal Distance, Too

How far does the package in Figure 2 coast after the push, assuming friction remains constant? Use work and energy considerations.

Strategy

We know that once the person stops pushing, friction will bring the package to rest. In terms of energy, friction does negative work until it has removed all of the package's kinetic energy. The work done by friction is the force of friction times the distance traveled times the cosine of the angle between the friction force and displacement; hence, this gives us a way of finding the distance traveled after the person stops pushing.

Solution

The normal force and force of gravity cancel in calculating the net force. The horizontal friction force is then the net force, and it acts opposite to the displacement, so $\theta = 180°$. To reduce the kinetic energy of the package to zero, the work W_{fr} by friction must be minus the kinetic energy that the package started with plus what the package accumulated due to the pushing. Thus $W_{\text{fr}} = -95.75$ J. Furthermore, $W_{\text{fr}} = fd' \cos\theta = -fd'$, where d' is the distance it takes to stop. Thus,

$$d' = -\frac{W_{\text{fr}}}{f} = -\frac{-95.75 \text{ J}}{5.00 \text{ N}},$$

and so

$$d' = 19.2 \text{ m}.$$

Discussion

This is a reasonable distance for a package to coast on a relatively friction-free conveyor system. Note that the work done by friction is negative (the force is in the opposite direction of motion), so it removes the kinetic energy.

Some of the examples in this section can be solved without considering energy, but at the expense of missing out on gaining insights about what work and energy are doing in this situation. On the whole, solutions involving energy are generally shorter and easier than those using kinematics and dynamics alone.

Section Summary

- The net work W_{net} is the work done by the net force acting on an object.
- Work done on an object transfers energy to the object.
- The translational kinetic energy of an object of mass m moving at speed v is $\text{KE} = \frac{1}{2}mv^2$.
- The work-energy theorem states that the net work W_{net} on a system changes its kinetic

energy, $W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$.

Glossary

net work
work done by the net force, or vector sum of all the forces, acting on an object

work-energy theorem
the result, based on Newton's laws, that the net work done on an object is equal to its change in kinetic energy

kinetic energy
the energy an object has by reason of its motion, equal to $\frac{1}{2}mv^2$ for the translational (i.e., non-rotational) motion of an object of mass *m* moving at speed *v*

Conceptual Questions

1: The person in Figure 3 does work on the lawn mower. Under what conditions would the mower gain energy? Under what conditions would it lose energy?

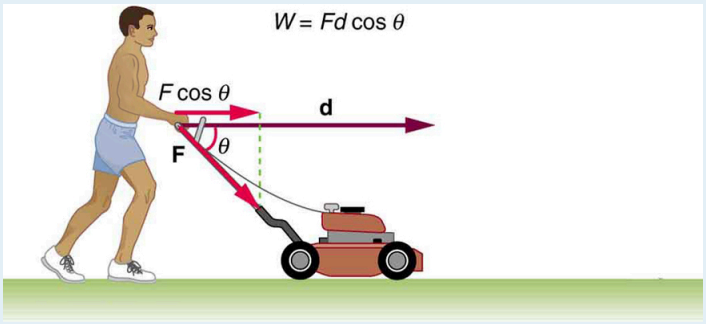


Figure 3.

2: Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.

3: When solving for speed in Example 3, we kept only the positive root. Why?

Problems & Exercises

1: Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.

2: (a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running

at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.

3: Confirm the value given for the kinetic energy of an aircraft carrier in Chapter 7.6 Table 1. You will need to look up the definition of a nautical mile (1 knot = 1 nautical mile/h).

4: (a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).

5: A car's bumper is designed to withstand a 4.0-km/h (1.1-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.1 m/s.

6: Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent's face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the gory old days when no gloves were used and the knuckles and face would compress only 2.00 cm. (c) Discuss the magnitude of the force with glove on. Does it seem high enough to cause damage even though it is lower than the force with no glove?

7: Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.

Solutions

Problems & Exercises

1: 1/250

3: 1.1×10^{10} J

5: 2.8×10^{3} N

7: 102 N

6.3 Gravitational Potential Energy

Summary

- Explain gravitational potential energy in terms of work done against gravity.
- Show that the gravitational potential energy of an object of mass m at height h on Earth is given by $PE_g = mgh$.
- Show how knowledge of the potential energy as a function of position can be used to simplify calculations and explain physical phenomena.

Work Done Against Gravity

Climbing stairs and lifting objects is work in both the scientific and everyday sense—it is work done against the gravitational force. When there is work, there is a transformation of energy. The work done against the gravitational force goes into an important form of stored energy that we will explore in this section.

Let us calculate the work done in lifting an object of mass m through a height h, such as in Figure 1. If the object is lifted straight up at constant speed, then the force needed to lift it is equal to its weight mg. The work done on the mass is then $W = Fd = mgh$. We define this to be the gravitational potential energy (PE_g)put into (or gained by) the object-Earth system. This energy is associated with the state of separation between two objects that attract each other by the gravitational force. For convenience, we refer to this as the PE_g gained by the object, recognizing that this is energy stored in the gravitational field of Earth. Why do we use the word “system”? Potential energy is a property of a system rather than of a single object—due to its physical position. An object’s gravitational potential is due to its position relative to the surroundings within the Earth-object system. The force applied to the object is an external force, from outside the system. When it does positive work it increases the gravitational potential energy of the system. Because gravitational potential energy depends on relative position, we need a reference level at which to set the potential energy equal to 0. We usually choose this point to be Earth’s surface, but this point is arbitrary; what is important is the *difference* in gravitational potential energy, because this difference is what relates to the work done. The difference in gravitational potential energy of an object (in the Earth-object system) between two rungs of a ladder will be the same for the first two rungs as for the last two rungs.

Converting Between Potential Energy and Kinetic Energy

Gravitational potential energy may be converted to other forms of energy, such as kinetic energy. If we release the mass, gravitational force will do an amount of work equal to mgh on it, thereby increasing

its kinetic energy by that same amount (by the work-energy theorem). We will find it more useful to consider just the conversion of PE_g to KE without explicitly considering the intermediate step of work. (See Example 2.) This shortcut makes it is easier to solve problems using energy (if possible) rather than explicitly using forces.

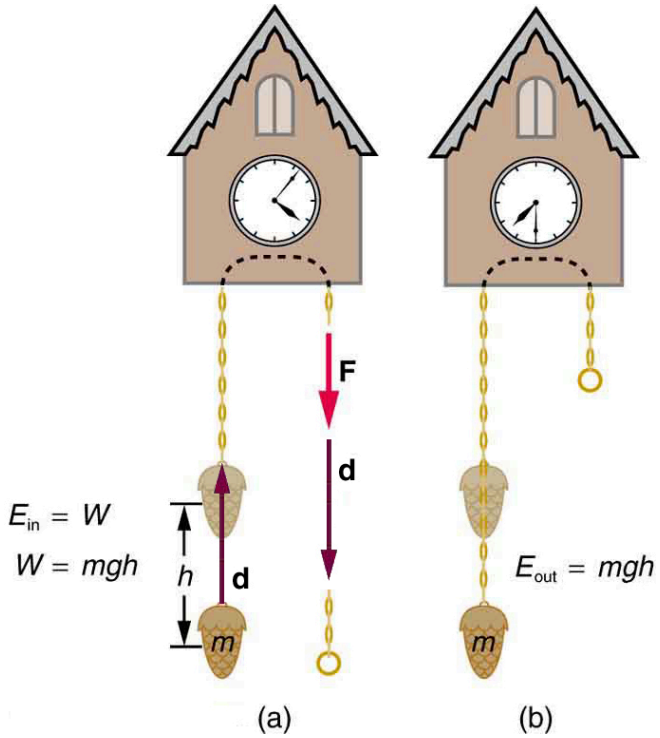


Figure 1. *(a) The work done to lift the weight is stored in the mass-Earth system as gravitational potential energy. (b) As the weight moves downward, this gravitational potential energy is transferred to the cuckoo clock.*

More precisely, we define the *change* in gravitational potential energy ΔPE_g to be

$$\Delta \mathrm{PE_g} = mgh,$$

where, for simplicity, we denote the change in height by h rather than the usual Δh. Note that h is positive when the final height is greater than the initial height, and vice versa. For example, if a 0.500-kg mass hung from a cuckoo clock is raised 1.00 m, then its change in gravitational potential energy is

$$\begin{aligned} mgh &= (0.500\ \mathrm{kg})(9.80\ \mathrm{m/s})(21.00\ \mathrm{m}) \\ &= 4.90\ \mathrm{kg \cdot m^2/s^2} = 4.90\ \mathrm{J} \end{aligned}$$

Note that the units of gravitational potential energy turn out to be joules, the same as for work and other forms of energy. As the clock runs, the mass is lowered. We can think of the mass as gradually giving up its 4.90 J of gravitational potential energy, *without directly considering the force of gravity that does the work*.

Using Potential Energy to Simplify Calculations

The equation $\Delta PE_g = mgh$ applies for any path that has a change in height of h, not just when the mass is lifted straight up. (See Figure 2.) It is much easier to calculate mgh (a simple multiplication) than it is to calculate the work done along a complicated path. The idea of gravitational potential energy has the double advantage that it is very broadly applicable and it makes calculations easier. From now on, we will consider that any change in vertical position h of a mass m is accompanied by a change in gravitational potential energy mgh.

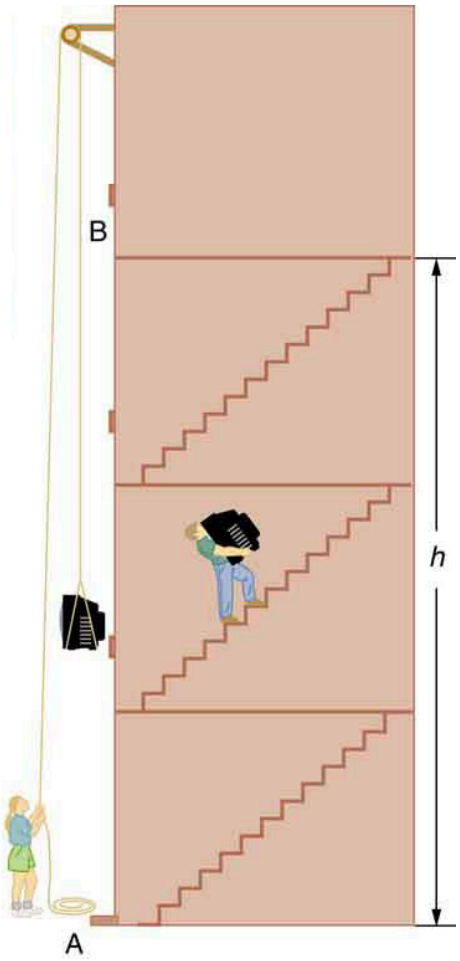


Figure 2. *The change in gravitational potential energy (ΔPE_g) between points A and B is independent of the path. $\Delta PE_g = mgh$ for any path between the two points. Gravity is one of a small class of forces where the work done by or against the force depends only on the starting and ending points, not on the path between them.*

Work Done By The Gravitational Force

The concept of potential energy (and the change in potential energy) can also be used to quickly determine the **work** done by the gravitational force. For example, in Figure 1a, when the mass moves upwards the gravitational force acts in the opposite direction to motion. This means that even though the gravitational potential energy increased, the work done by the gravitational force was negative. That is, the work done by the gravitational force is the negative of the change in potential energy,

$$W_{\text{g}} = -\Delta U_{\text{g}}$$

In Figure 1b, when the mass moves downwards, the gravitational force acts in the same direction as the motion. In this case, the change in potential energy is negative but the work down by gravity is positive (which follows from $W_{\text{g}} = -\Delta U_{\text{g}}$).

It turns out that this concept holds for *any* conservative force, not just the gravitational force (you will see another conservative force – the spring force – in the next section). In general, the work done by a conservative force (W_{cons}) is equal to the negative of the change in potential energy related to that force (U),

$$W_{\text{cons}} = -\Delta U$$

Example 1: The Force to Stop Falling

A 60.0-kg person jumps onto the floor from a height of 3.00 m. If he lands stiffly (with his knee joints compressing by 0.500 cm), calculate the force on the knee joints.

Strategy

This person's energy is brought to zero in this situation by the work done on him by the floor as he stops. The initial PE_{g} is transformed into KE as he falls. The work done by the floor reduces this kinetic energy to zero.

Solution

The work done on the person by the floor as he stops is given by

$$W = Fd\cos\theta = -Fd,$$

with a minus sign because the displacement while stopping and the force from floor are in opposite directions ($\cos\theta = \cos 180° = 1$). The floor removes energy from the system, so it does negative work.

The kinetic energy the person has upon reaching the floor is the amount of potential energy lost by falling through height h:

$$\text{KE} = -\Delta\text{PE}_{\text{g}} = -mgh,$$

The distance d that the person's knees bend is much smaller than the height h of the fall, so the additional change in gravitational potential energy during the knee bend is ignored.

The work *W* done by the floor on the person stops the person and brings the person's kinetic energy to zero:

$$W = -\text{KE} = mgh.$$

Combining this equation with the expression for *W* gives

$$-Fd = mgh.$$

Recalling that *h* is negative because the person fell *down*, the force on the knee joints is given by

$$F = -\frac{mgh}{d} = -\frac{(60.0\ \text{kg})(9.80\ \text{m/s}^2)(-3.00\ \text{m})}{5.00 \times 10^{-3}\ \text{m}} = 3.53 \times 10^5\ \text{N}.$$

Discussion

Such a large force (500 times more than the person's weight) over the short impact time is enough to break bones. A much better way to cushion the shock is by bending the legs or rolling on the ground, increasing the time over which the force acts. A bending motion of 0.5 m this way yields a force 100 times smaller than in the example. A kangaroo's hopping shows this method in action. The kangaroo is the only large animal to use hopping for locomotion, but the shock in hopping is cushioned by the bending of its hind legs in each jump.(See Figure 3.)

Example 2: Finding the Speed of a Roller Coaster from its Height

(a) What is the final speed of the roller coaster shown in Figure 3 if it starts from rest at the top of the 20.0 m hill and work done by frictional forces is negligible? (b) What is its final speed (again assuming negligible friction) if its initial speed is 5.00 m/s?

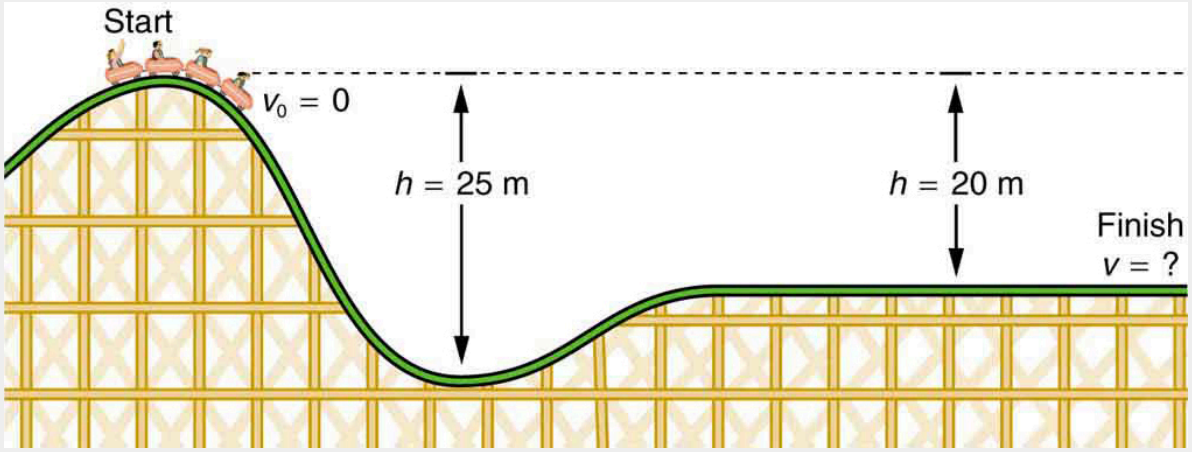


Figure 3. *The speed of a roller coaster increases as gravity pulls it downhill and is greatest at its lowest point. Viewed in terms of energy, the roller-coaster-Earth system's gravitational potential energy is converted to kinetic energy. If work done by friction is negligible, all* ΔPE_g *is converted to KE.*

Strategy

The roller coaster loses potential energy as it goes downhill. We neglect friction, so that the remaining force exerted by the track is the normal force, which is perpendicular to the direction of motion and does no work. The net work on the roller coaster is then done by gravity alone. The *loss* of gravitational potential energy from moving *downward* through a distance *h* equals the *gain* in kinetic energy. This can be written in equation form as -ΔPEg = ΔKE. Using the equations for PE_g and KE, we can solve for the final speed ***v***, which is the desired quantity.

Solution for (a)

Here the initial kinetic energy is zero, so that $\Delta \text{KE} = \frac{1}{2}mv^2$. The equation for change in potential energy states that $\Delta PE_g = mgh$. Since *h* is negative in this case, we will rewrite this as $\Delta PE_g = –mg|h|$ to show the minus sign clearly. Thus,

$$-\Delta \text{PE}_\text{g} = \Delta \text{KE}$$

becomes

$$mg|h| = \frac{1}{2}mv^2.$$

Solving for *v*, we find that mass cancels and that

$$v = \sqrt{2g|h|}.$$

Substituting known values,

$$\begin{aligned} v &= \sqrt{2(9.80\ \text{m/s})(20.0\ \text{m})} \\ &= 19.8\ \text{m/s}. \end{aligned}$$

Solution for (b)

Again -ΔPE_g = ΔKE. In this case there is initial kinetic energy, so $\Delta \text{KE} = \frac{1}{2}mv^2 - \frac{1}{2}m{v_0}^2$. Thus,

$$mg|h| = \frac{1}{2}mv^2 - \frac{1}{2}m{v_0}^2.$$

Rearranging gives

$$\frac{1}{2}mv^2 = mg|h| + \frac{1}{2}m{v_0}^2.$$

This means that the final kinetic energy is the sum of the initial kinetic energy and the gravitational potential energy. Mass again cancels, and

$$v = \sqrt{2g|h| + {v_0}^2}.$$

This equation is very similar to the kinematics equation $v = \sqrt{{v_0}^2 + 2ad}$, but it is more general—the kinematics equation is valid only for constant acceleration, whereas our equation above

is valid for any path regardless of whether the object moves with a constant acceleration. Now, substituting known values gives

$$\begin{aligned} v &= \sqrt{2(9.80\ \text{m/s}^2)(20.0\ \text{m}) + (5.00\ \text{m/s})^2} \\ &= 20.4\ \ \text{m/s}. \end{aligned}$$

Discussion and Implications

First, note that mass cancels. This is quite consistent with observations made in Chapter 2.8 Falling Objects that all objects fall at the same rate if friction is negligible. Second, only the speed of the roller coaster is considered; there is no information about its direction at any point. This reveals another general truth. When friction is negligible, the speed of a falling body depends only on its initial speed and height, and not on its mass or the path taken. For example, the roller coaster will have the same final speed whether it falls 20.0 m straight down or takes a more complicated path like the one in the figure. Third, and perhaps unexpectedly, the final speed in part (b) is greater than in part (a), but by far less than 5.00 m/s. Finally, note that speed can be found at *any* height along the way by simply using the appropriate value of *h* at the point of interest.

We have seen that work done by or against the gravitational force depends only on the starting and ending points, and not on the path between, allowing us to define the simplifying concept of gravitational potential energy. We can do the same thing for a few other forces, and we will see that this leads to a formal definition of the law of conservation of energy.

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION— CONVERTING POTENTIAL TO KINETIC ENERGY

One can study the conversion of gravitational potential energy into kinetic energy in this experiment. On a smooth, level surface, use a ruler of the kind that has a groove running along its length and a book to make an incline (see Figure 4). Place a marble at the 10-cm position on the ruler and let it roll down the ruler. When it hits the level surface, measure the time it takes to roll one meter. Now place the marble at the 20-cm and the 30-cm positions and again measure the times it takes to roll 1 m on the level surface. Find the velocity of the marble on the level surface for all three positions. Plot velocity squared versus the distance traveled by the marble. What is the shape of each plot? If the shape is a straight line, the plot shows that the marble's kinetic energy at the bottom is proportional to its potential energy at the release point.

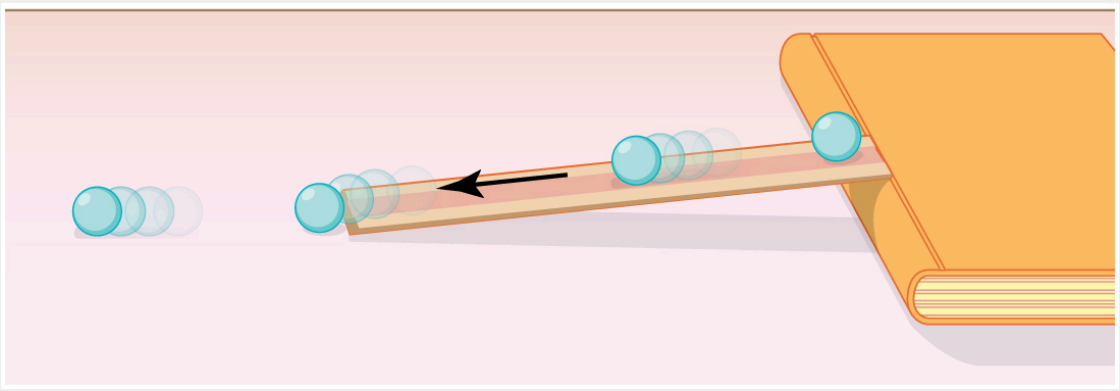


Figure 4. *A marble rolls down a ruler, and its speed on the level surface is measured.*

Section Summary

- Work done against gravity in lifting an object becomes potential energy of the object-Earth system.
- The change in gravitational potential energy, ΔPE_g, is $\Delta PEg = mgh$, with h being the increase in height and g the acceleration due to gravity.
- The gravitational potential energy of an object near Earth's surface is due to its position in the mass-Earth system. Only differences in gravitational potential energy, ΔPE_g, have physical significance.
- As an object descends without friction, its gravitational potential energy changes into kinetic energy corresponding to increasing speed, so that $\Delta KE = -\Delta PE_g$.

Glossary

gravitational potential energy
the energy an object has due to its position in a gravitational field

Conceptual Questions

1: In Example 2, we calculated the final speed of a roller coaster that descended 20 m in height and had an initial speed of 5 m/s downhill. Suppose the roller coaster had had an initial speed of 5 m/s *uphill* instead, and it coasted uphill, stopped, and then rolled back down to a final point 20 m below the start. We would find in that case that it had the same final speed. Explain in terms of conservation of energy.

2: Does the work you do on a book when you lift it onto a shelf depend on the path taken? On the time taken? On the height of the shelf? On the mass of the book?

Problems & Exercises

1: A hydroelectric power facility (see Figure 65) converts the gravitational potential energy of water behind a dam to electric energy. (a) What is the gravitational potential energy relative to the generators of a lake of volume 50.0 km^3 mass = 5.00×10^{13} kg, given that the lake has an average height of 40.0 m above the generators? (b) Compare this with the energy stored in a 9-megaton fusion bomb.

Figure 5. *Hydroelectric facility (credit: Denis Belevich, Wikimedia Commons)*

2: (a) How much gravitational potential energy (relative to the ground on which it is built) is stored in the Great Pyramid of Cheops, given that its mass is about 7×10^9 kg and its center of mass is 36.5 m above the surrounding ground? (b) How does this energy compare with the daily food intake of a person?

3: Suppose a 350-g kookaburra (a large kingfisher bird) picks up a 75-g snake and raises it 2.5 m from the ground to a branch. (a) How much work did the bird do on the snake? (b) How much work did it do to raise its own center of mass to the branch?

4: In Example 2, we found that the speed of a roller coaster that had descended 20.0 m was only slightly greater when it had an initial speed of 5.00 m/s than when it started from rest. This implies that $\Delta PE >> KE_i$. Confirm this statement by taking the ratio of ΔPE to KE_i. (Note that mass cancels.)

5: A 100-g toy car is propelled by a compressed spring that starts it moving. Show that the final speed of the toy car is 0.687 m/s if its initial speed is 2.00 m/s and it coasts up a frictionless track, gaining 0.180 m in altitude.

6: In a downhill ski race, surprisingly, little advantage is gained by getting a running start. (This is because the initial kinetic energy is small compared with the gain in gravitational potential energy on even small hills.) To demonstrate this, find the final speed and the time taken for a skier who skies 70.0 m along a 30° slope neglecting friction: (a) Starting from rest. (b) Starting with an initial speed of 2.50 m/s. (c) Does the answer surprise you? Discuss why it is still advantageous to get a running start in very competitive events.

Solutions

Problems & Exercises

1: (a) 1.96 x 10^{16} J (b) The ratio of gravitational potential energy in the lake to the energy stored in the bomb is 0.52. That is, the energy stored in the lake is approximately half that in a 9-megaton fusion bomb.

3: (a) 1.8 J (b) 8.6 J

5:

$$v_f = \sqrt{2gh + {v_0}^2} = \sqrt{2(9.80\ \text{m/s}^2)(-0.180\ \text{m}) + (2.00\ \text{m/s})^2} = 0.687\ \text{m/s}$$

6.4 Conservative Forces and Potential Energy

Summary

- Define conservative force, potential energy, and mechanical energy.
- Explain the potential energy of a spring in terms of its compression when Hooke's law applies.
- Use the work-energy theorem to show how having only conservative forces implies conservation of mechanical energy.

Potential Energy and Conservative Forces

Work is done by a force, and some forces, such as weight, have special characteristics. A **conservative force** is one, like the gravitational force, for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken. We can define a **potential energy** (PE) for any conservative force, just as we did for the gravitational force. For example, when you wind up a toy, an egg timer, or an old-fashioned watch, you do work against its spring and store energy in it. (We treat these springs as ideal, in that we assume there is no friction and no production of thermal energy.) This stored energy is recoverable as work, and it is useful to think of it as potential energy contained in the spring. Indeed, the reason that the spring has this characteristic is that its force is *conservative*. That is, a conservative force results in stored or potential energy. Gravitational potential energy is one example, as is the energy stored in a spring. We will also see how conservative forces are related to the conservation of energy.

POTENTIAL ENERGY AND CONSERVATIVE FORCES

Potential energy is the energy a system has due to position, shape, or configuration. It is stored energy that is completely recoverable.

A conservative force is one for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken.

We can define a potential energy (PE) for any conservative force. The work done against a conservative force to reach a final configuration depends on the configuration, not the path followed, and is the potential energy added.

Potential Energy of a Spring

First, let us obtain an expression for the potential energy stored in a spring (PE_s). We calculate the work done to stretch or compress a spring that obeys Hooke's law, which states that the magnitude of force F on the spring and the resulting deformation ΔL are proportional, $F = k\Delta L$ (see Figure 1). For our spring, we will replace ΔL (the amount of deformation produced by a force F) by the distance x that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude $F = kx$, where k is the spring's force constant. The force increases linearly from 0 at the start to kx in the fully stretched position. The average force is $kx/2$. Thus the work done in stretching or compressing the spring is $W_s = Fd = (\frac{kx}{2})x = \frac{1}{2}kx^2$. Alternatively, we noted in Chapter 6.2 Kinetic Energy and the Work-Energy Theorem that the area under a graph of F vs. x is the work done by the force. In Figure 1(c) we see that this area is also $\frac{1}{2}kx^2$. We therefore define the **potential energy of a spring**, PE_s, to be

$$\mathrm{PE_s} = \frac{1}{2}kx^2,$$

where k is the spring's force constant and x is the displacement from its undeformed position. The potential energy represents the work done *on* the spring and the energy stored in it as a result of stretching or compressing it a distance x. The potential energy of the spring PE_s does not depend on the path taken; it depends only on the stretch or squeeze x in the final configuration.

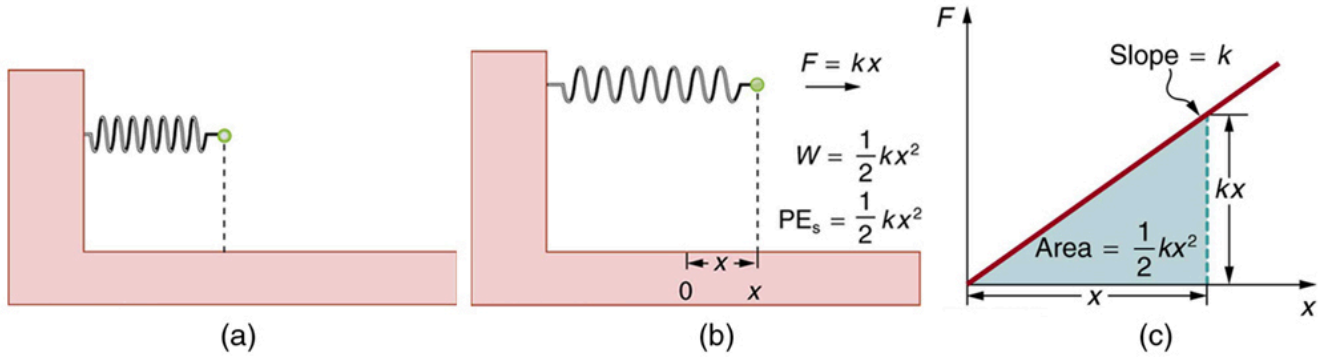


Figure 1. *(a) An undeformed spring has no PE_s stored in it. (b) The force needed to stretch (or compress) the spring a distance x has a magnitude F=kx, and the work done to stretch (or compress) it is 1/2 kx^2. Because the force is conservative, this work is stored as potential energy (PE_s) in the spring, and it can be fully recovered. (c) A graph of F vs. x has a slope of k, and the area under the graph is 1/2 kx^2. Thus the work done or potential energy stored is 1/2 kx^2.*

The equation $\mathrm{PE_s} = \frac{1}{2}kx^2$ has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of **potential energy** is energy due to position, shape, or configuration. For shape or position deformations, stored energy is $\mathrm{PE_s} = \frac{1}{2}kx^2$, where k is the force constant of the particular system and x is its deformation.

Conservation of Mechanical Energy

Let us now consider what form the work-energy theorem takes when only conservative forces are involved. This will lead us to the conservation of energy principle. The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy. In equation form, this is

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = \Delta\text{KE}.$$

If only conservative forces act, then

$$W_{\text{net}} = W_{\text{c}},$$

where W_{c} is the total work done by all conservative forces. Thus,

$$W_{\text{c}} = \Delta\text{KE}.$$

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy. That is, W_{c} = -ΔPE. Therefore,

$$-\Delta\text{PE} = \Delta\text{KE}$$

or

$$\Delta\text{KE} + \Delta\text{PE} = 0.$$

This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

$$\begin{aligned} \text{KE}+\text{PE}&=\text{constant} \\ \text{KE}_{\text{i}}+\text{PE}_{\text{i}}&=\text{KE}_{\text{f}}+\text{PE}_{\text{f}} \end{aligned}$$

where i and f denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the **conservation of mechanical energy** principle. Remember that this applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its **mechanical energy**, (KE + PE). In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and the various types of PE, with the total energy remaining constant.

Example 1: Using Conservation of Mechanical Energy to Calculate the Speed of a Toy Car

A 0.100-kg toy car is propelled by a compressed spring, as shown in Figure 2. The car follows a track that rises 0.180 m above the starting point. The spring is compressed 4.00 cm and has a force constant of 250.0 N/m. Assuming work done by friction to be negligible, find (a) how fast the car is going before it starts up the slope and (b) how fast it is going at the top of the slope.

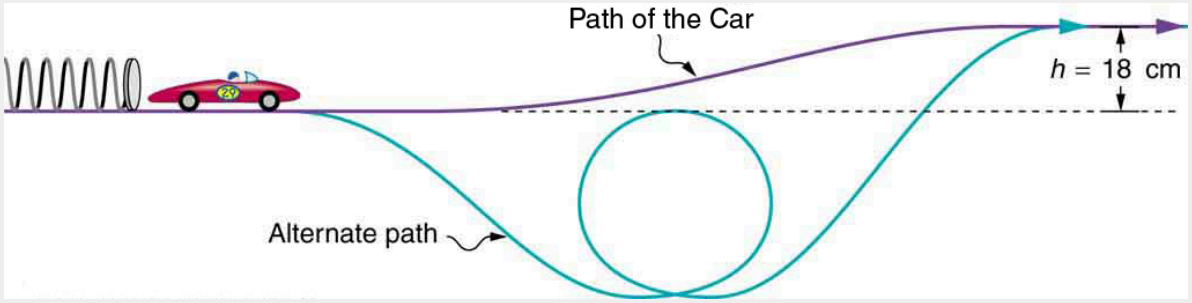


Figure 2. *A toy car is pushed by a compressed spring and coasts up a slope. Assuming negligible friction, the potential energy in the spring is first completely converted to kinetic energy, and then to a combination of kinetic and gravitational potential energy as the car rises. The details of the path are unimportant because all forces are conservative—the car would have the same final speed if it took the alternate path shown.*

Strategy

The spring force and the gravitational force are conservative forces, so conservation of mechanical energy can be used. Thus,

$$\mathrm{KE_i} + \mathrm{PE_i} = \mathrm{KE_f} + \mathrm{PE_f}$$

or

$$\frac{1}{2}m{v_\mathrm{i}}^2 + mgh_\mathrm{i} + \frac{1}{2}k{x_\mathrm{i}}^2 = \frac{1}{2}m{v_\mathrm{f}}^2 + mgh_\mathrm{f} + \frac{1}{2}kx_\mathrm{f}^2,$$

where h is the height (vertical position) and x is the compression of the spring. This general statement looks complex but becomes much simpler when we start considering specific situations. First, we must identify the initial and final conditions in a problem; then, we enter them into the last equation to solve for an unknown.

Solution for (a)

This part of the problem is limited to conditions just before the car is released and just after it leaves the spring. Take the initial height to be zero, so that both h_i and h_f are zero. Furthermore, the initial speed v_i is zero and the final compression of the spring x_f is zero, and so several terms in the conservation of mechanical energy equation are zero and it simplifies to

$$\frac{1}{2}k{x_\mathrm{i}}^2 = \frac{1}{2}m{v_\mathrm{f}}^2.$$

In other words, the initial potential energy in the spring is converted completely to kinetic energy in the absence of friction. Solving for the final speed and entering known values yields

$$\begin{aligned} v_f &= \sqrt{\frac{k}{m}}x_i \\ &= \sqrt{\frac{250.0\ \mathrm{N/m}}{0.100\ \mathrm{kg}}}(0.0400\ \mathrm{m}) \qquad = 2.00\ \mathrm{m/s}. \end{aligned}$$

Solution for (b)

One method of finding the speed at the top of the slope is to consider conditions just before the car is released and just after it reaches the top of the slope, completely ignoring everything in between.

Doing the same type of analysis to find which terms are zero, the conservation of mechanical energy becomes

$$\frac{1}{2}k{x_\mathrm{i}}^2 = \frac{1}{2}m{v_\mathrm{f}}^2 + mgh_\mathrm{f}.$$

This form of the equation means that the spring's initial potential energy is converted partly to gravitational potential energy and partly to kinetic energy. The final speed at the top of the slope will be less than at the bottom. Solving for v_f and substituting known values gives

$$\begin{aligned} v_\mathrm{f} &= \sqrt{\frac{k{x_\mathrm{i}}^2}{m} - 2gh_\mathrm{f}} \\ &= \sqrt{(\frac{250.0\ \mathrm{N/m}}{0.100\ \mathrm{kg}})(0.0400\ \mathrm{m})^2 - 2(9.80\ \mathrm{m/s^2})(0.180\ \ \mathrm{m})} \\ &= 0.687\ \mathrm{m/s} \end{aligned}$$

Discussion

Another way to solve this problem is to realize that the car's kinetic energy before it goes up the slope is converted partly to potential energy—that is, to take the final conditions in part (a) to be the initial conditions in part (b).

Note that, for conservative forces, we do not directly calculate the work they do; rather, we consider their effects through their corresponding potential energies, just as we did in Example 1. Note also that we do not consider details of the path taken—only the starting and ending points are important (as long as the path is not impossible). This assumption is usually a tremendous simplification, because the path may be complicated and forces may vary along the way.

PHET EXPLORATIONS: ENERGY SKATE PARK

Learn about conservation of energy with a skater dude! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

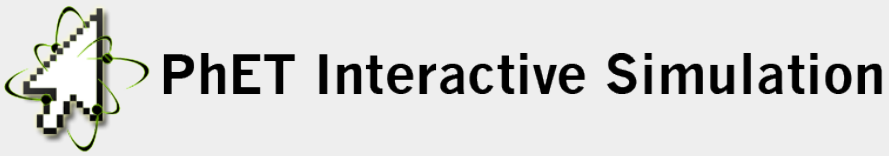


Figure 4. *Energy Skate Park*

Section Summary

- A conservative force is one for which work depends only on the starting and ending points of a motion, not on the path taken.
- We can define potential energy (PE) for any conservative force, just as we defined PE_g for the gravitational force.
- The potential energy of a spring is $\text{PE}_\text{s} = \frac{1}{2}kx^2$, where *k* is the spring's force constant and *x* is the displacement from its undeformed position.
- Mechanical energy is defined to be KE + PE for a conservative force.
- When only conservative forces act on and within a system, the total mechanical energy is constant. In equation form,

$$\begin{aligned} \text{KE}+\text{PE}&=\text{constant} \\ \text{KE}_\text{i}+\text{PE}_\text{i}&=\text{KE}_\text{f}+\text{PE}_\text{f} \end{aligned}$$

where i and f denote initial and final values. This is known as the conservation of mechanical energy.

Glossary

conservative force
: a force that does the same work for any given initial and final configuration, regardless of the path followed

potential energy
: energy due to position, shape, or configuration

potential energy of a spring
: the stored energy of a spring as a function of its displacement; when Hooke's law applies, it is given by the expression $\frac{1}{2}kx^2$ where *x* is the distance the spring is compressed or extended and *k* is the spring constant

conservation of mechanical energy
: the rule that the sum of the kinetic energies and potential energies remains constant if only conservative forces act on and within a system

mechanical energy
: the sum of kinetic energy and potential energy

Conceptual Questions

1: What is a conservative force?

2: The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer dives from it, starting just before the swimmer steps on the board until just after his feet leave it.

3: Define mechanical energy. What is the relationship of mechanical energy to nonconservative forces? What happens to mechanical energy if only conservative forces act?

4: What is the relationship of potential energy to conservative force?

Problems & Exercises

1: A 5.00×10^5-kg subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the force constant k of the spring?

2: A pogo stick has a spring with a force constant of 2.50×10^4 N/m, which can be compressed 12.0 cm. To what maximum height can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40.0 kg? Explicitly show how you follow the steps in the Problem-Solving Strategies for Energy.

Solutions

Problems & Exercises

1: 7.81 x 10^5 N/m

6.5 Nonconservative Forces

Summary

- Define nonconservative forces and explain how they affect mechanical energy.
- Show how the principle of conservation of energy can be applied by treating the conservative forces in terms of their potential energies and any nonconservative forces in terms of the work they do.

Nonconservative Forces and Friction

Forces are either conservative or nonconservative. Conservative forces were discussed in Chapter 6.4 Conservative Forces and Potential Energy. A **nonconservative force** is one for which work depends on the path taken. Friction is a good example of a nonconservative force. As illustrated in Figure 1, work done against friction depends on the length of the path between the starting and ending points. Because of this dependence on path, there is no potential energy associated with nonconservative forces. An important characteristic is that the work done by a nonconservative force *adds or removes mechanical energy from a system*. **Friction**, for example, creates **thermal energy** that dissipates, removing energy from the system. Furthermore, even if the thermal energy is retained or captured, it cannot be fully converted back to work, so it is lost or not recoverable in that sense as well.

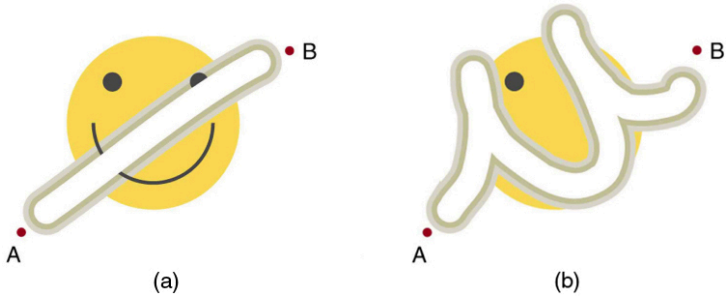


Figure 1. *The amount of the happy face erased depends on the path taken by the eraser between points A and B, as does the work done against friction. Less work is done and less of the face is erased for the path in (a) than for the path in (b). The force here is friction, and most of the work goes into thermal energy that subsequently leaves the system (the happy face plus the eraser). The energy expended cannot be fully recovered.*

How Nonconservative Forces Affect Mechanical Energy

Mechanical energy *may* not be conserved when nonconservative forces act. For example, when a

car is brought to a stop by friction on level ground, it loses kinetic energy, which is dissipated as thermal energy, reducing its mechanical energy. Figure 2 compares the effects of conservative and nonconservative forces. We often choose to understand simpler systems such as that described in Figure 2(a) first before studying more complicated systems as in Figure 2(b).

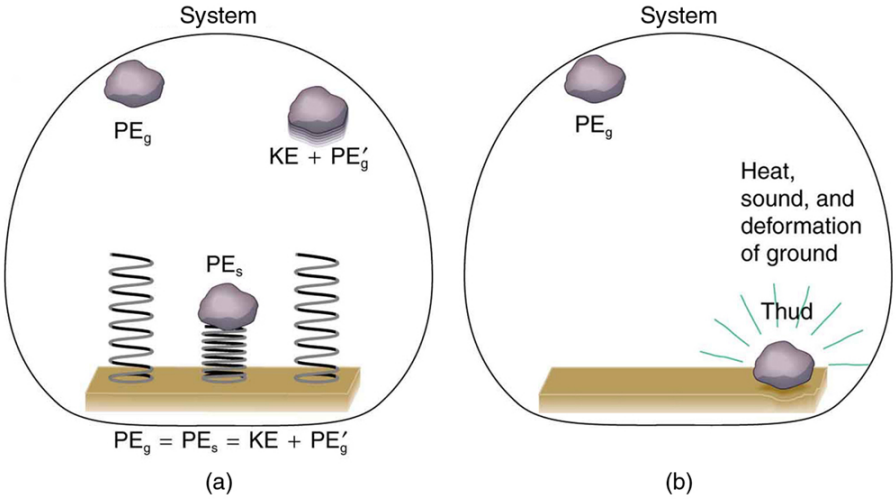


Figure 2. *Comparison of the effects of conservative and nonconservative forces on the mechanical energy of a system. (a) A system with only conservative forces. When a rock is dropped onto a spring, its mechanical energy remains constant (neglecting air resistance) because the force in the spring is conservative. The spring can propel the rock back to its original height, where it once again has only potential energy due to gravity. (b) A system with nonconservative forces. When the same rock is dropped onto the ground, it is stopped by nonconservative forces that dissipate its mechanical energy as thermal energy, sound, and surface distortion. The rock has lost mechanical energy.*

How the Work-Energy Theorem Applies

Now let us consider what form the work-energy theorem takes when both conservative and nonconservative forces act. We will see that the work done by nonconservative forces equals the change in the mechanical energy of a system. As noted in Chapter 6.2 Kinetic Energy and the Work-Energy Theorem, the work-energy theorem states that the net work on a system equals the change in its kinetic energy, or $W_{\text{net}} = \Delta\text{KE}$. The net work is the sum of the work by nonconservative forces plus the work by conservative forces. That is,

$$W_{\text{net}} = W_{\text{nc}} + W_{\text{c}},$$

so that

$$W_{\text{nc}} + W_{\text{c}} = \Delta\text{KE},$$

where W_{nc} is the total work done by all nonconservative forces and W_{c} is the total work done by all conservative forces.

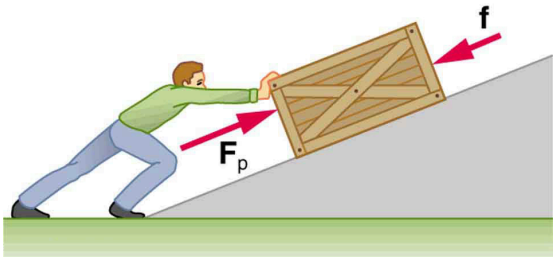


Figure 3. *A person pushes a crate up a ramp, doing work on the crate. Friction and gravitational force (not shown) also do work on the crate; both forces oppose the person's push. As the crate is pushed up the ramp, it gains mechanical energy, implying that the work done by the person is greater than the work done by friction.*

Consider Figure 3, in which a person pushes a crate up a ramp and is opposed by friction. As in the previous section, we note that work done by a conservative force comes from a loss of gravitational potential energy, so that $W_c = -\Delta PE$. Substituting this equation into the previous one and solving for W_{nc} gives

$$W_{nc} = \Delta KE + \Delta PE.$$

This equation means that the total mechanical energy (KE + PE) changes by exactly the amount of work done by nonconservative forces. In Figure 3, this is the work done by the person minus the work done by friction. So even if energy is not conserved for the system of interest (such as the crate), we know that an equal amount of work was done to cause the change in total mechanical energy.

We rearrange $W_{nc} = \Delta KE + \Delta PE$ to obtain

$$KE_i + PE_i + W_{nc} = KE_f + PE_f.$$

This means that the amount of work done by nonconservative forces adds to the mechanical energy of a system. If W_{nc} is positive, then mechanical energy is increased, such as when the person pushes the crate up the ramp in Figure 3. If W_{nc} is negative, then mechanical energy is decreased, such as when the rock hits the ground in Figure 2(b). If W_{nc} is zero, then mechanical energy is conserved, and nonconservative forces are balanced. For example, when you push a lawn mower at constant speed on level ground, your work done is removed by the work of friction, and the mower has a constant energy.

Applying Energy Conservation with Nonconservative Forces

When no change in potential energy occurs, applying $KE_i + PE_i + W_{nc} = KE_f + PE_f$ amounts to applying the work-energy theorem by setting the change in kinetic energy to be equal to the net work done on the system, which in the most general case includes both conservative and nonconservative forces. But when seeking instead to find a change in total mechanical energy in situations that involve changes in both potential and kinetic energy, the previous equation $KE_i + PE_i + W_{nc} = KE_f + PE_f$ says that you can start by finding the change in mechanical energy that would have resulted from just the conservative

forces, including the potential energy changes, and add to it the work done, with the proper sign, by any nonconservative forces involved.

Example 1: Calculating Distance Traveled: How Far a Baseball Player Slides

Consider the situation shown in Figure 4, where a baseball player slides to a stop on level ground. Using energy considerations, calculate the distance the 65.0-kg baseball player slides, given that his initial speed is 6.00 m/s and the force of friction against him is a constant 450 N.

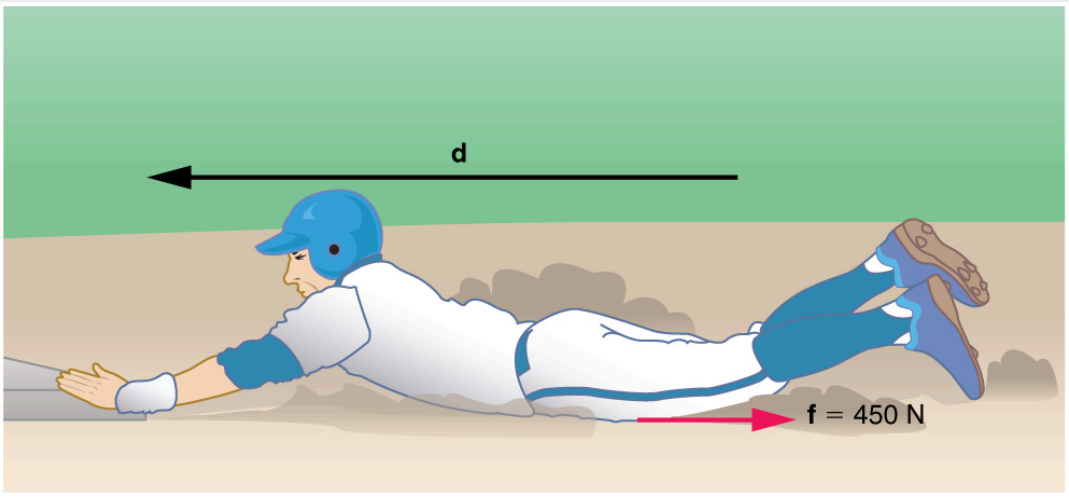


Figure 4. *The baseball player slides to a stop in a distance d. In the process, friction removes the player's kinetic energy by doing an amount of work fd equal to the initial kinetic energy.*

Strategy

Friction stops the player by converting his kinetic energy into other forms, including thermal energy. In terms of the work-energy theorem, the work done by friction, which is negative, is added to the initial kinetic energy to reduce it to zero. The work done by friction is negative, because f is in the opposite direction of the motion (that is, $\theta = 180°$, and so $\cos\theta = -1$). Thus $W_{nc} = -fd$. The equation simplifies to

$$\frac{1}{2}mv_i^2 - fd = 0$$

or

$$fd = \frac{1}{2}mv_i^2.$$

This equation can now be solved for the distance d.

Solution

Solving the previous equation for d and substituting known values yields

$$d = \frac{m{v_{\text{i}}}^2}{2f}$$
$$= \frac{(65.0\ \text{kg})(6.00\ \text{m/s})^2}{(2)(450\ \text{N})}$$
$$= 2.60\ \text{m}.$$

Discussion

The most important point of this example is that the amount of nonconservative work equals the change in mechanical energy. For example, you must work harder to stop a truck, with its large mechanical energy, than to stop a mosquito.

Example 2: Calculating Distance Traveled: Sliding Up an Incline

Suppose that the player from Example 1 is running up a hill having a 5.00° incline upward with a surface similar to that in the baseball stadium. The player slides with the same initial speed. Determine how far he slides.

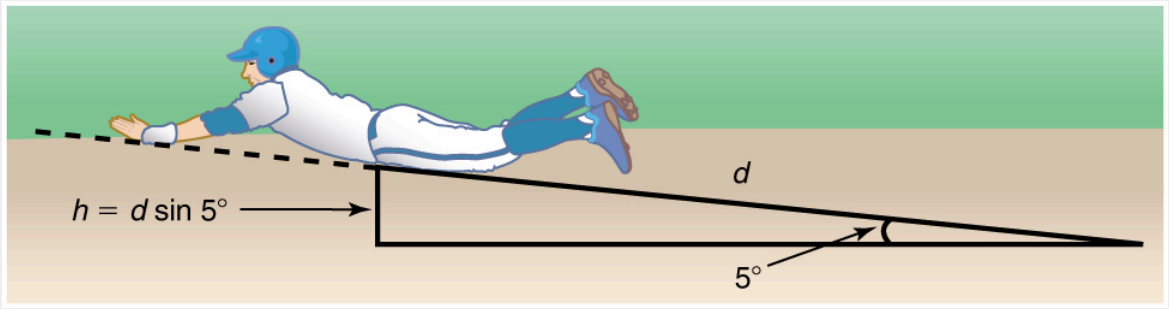


Figure 5. *The same baseball player slides to a stop on a 5.00^0 slope.*

Strategy

In this case, the work done by the nonconservative friction force on the player reduces the mechanical energy he has from his kinetic energy at zero height, to the final mechanical energy he has by moving through distance *d* to reach height *h* along the hill, with *h* = *d* sin 5.00^0. This is expressed by the equation

$$\text{KE}_\text{i} + \text{PE}_\text{i} + W_\text{nc} = \text{KE}_\text{f} + \text{PE}_\text{f}.$$

Solution

The work done by friction is again $W_\text{nc} = -fd$. Initially the potential energy is zero because we define the initial height h = 0 m. $\text{PE}_\text{i} = mg \times 0 = 0$ and the kinetic energy is $\text{KE}_\text{i} = \frac{1}{2}mv_\text{i}^2$; the final

energy contributions are $KE_f = 0$ for the kinetic energy and $PE_f = mgh = mg\,d \sin\theta$ for the potential energy.

Substituting these values gives

$$\frac{1}{2}m{v_i}^2 + 0 + (-fd) = 0 + mgd\sin\theta.$$

Solve this for d to obtain

Formula does not parse

Discussion

As might have been expected, the player slides a shorter distance by sliding uphill. Note that the problem could also have been solved in terms of the forces directly and the work energy theorem, instead of using the potential energy. This method would have required combining the normal force and force of gravity vectors, which no longer cancel each other because they point in different directions, and friction, to find the net force. You could then use the net force and the net work to find the distance d that reduces the kinetic energy to zero. By applying conservation of energy and using the potential energy instead, we need only consider the gravitational potential energy mgh, without combining and resolving force vectors. This simplifies the solution considerably.

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION—DETERMINING FRICTION FROM STOPPING DISTANCE

This experiment involves the conversion of gravitational potential energy into thermal energy. You will need a ruler, book, and marble. In addition, you will need a foam cup with a small hole in the side, as shown in Figure 6. From the 10-cm position on the ruler, let the marble roll into the cup positioned at the bottom of the ruler. Measure the distance d the cup moves before stopping. What forces caused it to stop? What happened to the kinetic energy of the marble at the bottom of the ruler? Next, place the marble at the 20-cm and the 30-cm positions and again measure the distance the cup moves after the marble enters it. Plot the distance the cup moves versus the initial marble position on the ruler. Is this relationship linear?

With some simple assumptions, you can use these data to find the coefficient of kinetic friction μ_k of the cup on the table. The force of friction f on the cup is $\mu_k N$, where the normal force N is just the weight of the cup plus the marble. The normal force and force of gravity do no work because they are perpendicular to the displacement of the cup, which moves horizontally. The work done by friction is fd. You will need the mass of the marble as well to calculate its initial kinetic energy.

It is interesting to do the above experiment also with a steel marble (or ball bearing). Releasing it from the same positions on the ruler as you did with the glass marble, is the velocity of this steel marble the same as the velocity of the marble at the bottom of the ruler? Is the distance the cup moves proportional to the mass of the steel and glass marbles?

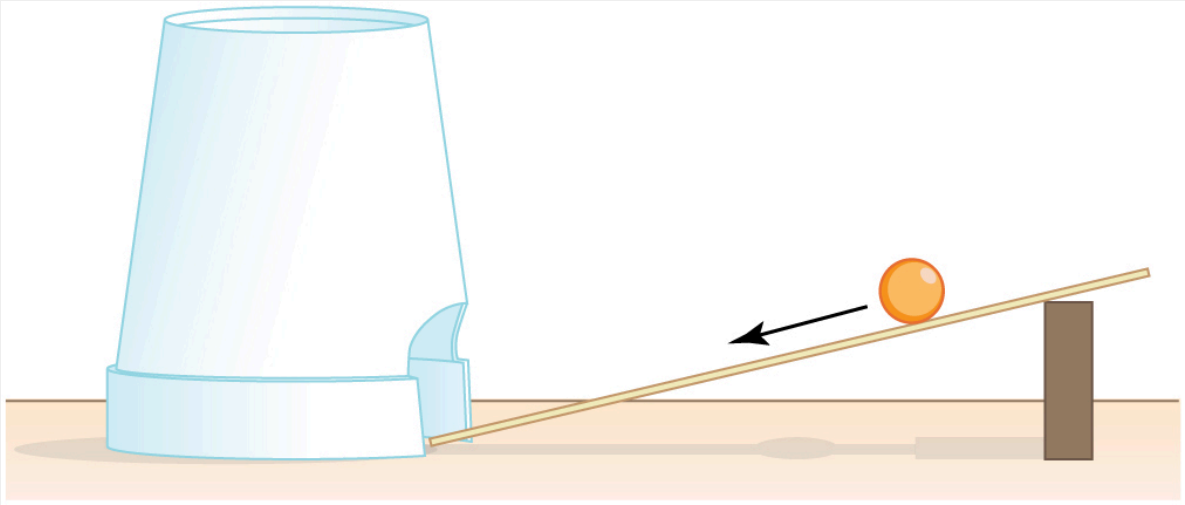


Figure 6. *Rolling a marble down a ruler into a foam cup.*

PHET EXPLORATIONS: THE RAMP

Explore forces, energy and work as you push household objects up and down a ramp. Lower and raise the ramp to see how the angle of inclination affects the parallel forces acting on the file cabinet. Graphs show forces, energy and work.

PhET Interactive Simulation

Figure 7. The Ramp

Section Summary

- A nonconservative force is one for which work depends on the path.
- Friction is an example of a nonconservative force that changes mechanical energy into thermal energy.
- Work W_{nc} done by a nonconservative force changes the mechanical energy of a system. In equation form, $W_{nc} = \Delta KE + \Delta PE$ or, equivalently, $KE_i + PE_i + W_{nc} = KE_f + PE_f$.
- When both conservative and nonconservative forces act, energy conservation can be applied

and used to calculate motion in terms of the known potential energies of the conservative forces and the work done by nonconservative forces, instead of finding the net work from the net force, or having to directly apply Newton's laws.

Glossary

nonconservative force
a force whose work depends on the path followed between the given initial and final configurations

friction
the force between surfaces that opposes one sliding on the other; friction changes mechanical energy into thermal energy

Problems & Exercises

1: A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m-high rise as shown in Figure 8. Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.0800. (Hint: Find the distance traveled up the incline assuming a straight-line path as shown in the figure.)

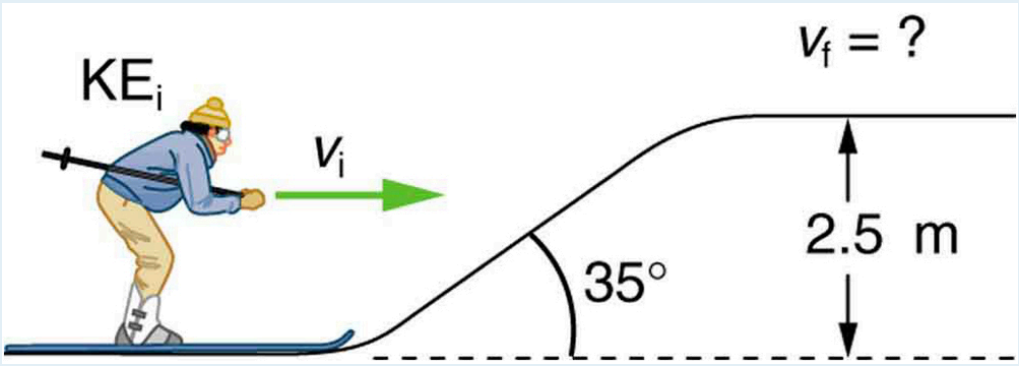


Figure 8. *The skier's initial kinetic energy is partially used in coasting to the top of a rise.*

2: (a) How high a hill can a car coast up (engine disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope 2.5° above the horizontal?

Solutions

Problems & Exercises

1: 9.46 m/s

6.6 Conservation of Energy

Summary

- Explain the law of the conservation of energy.
- Describe some of the many forms of energy.
- Define efficiency of an energy conversion process as the fraction left as useful energy or work, rather than being transformed, for example, into thermal energy.

Law of Conservation of Energy

Energy, as we have noted, is conserved, making it one of the most important physical quantities in nature. The **law of conservation of energy** can be stated as follows:

Total energy is constant in any process. It may change in form or be transferred from one system to another, but the total remains the same.

We have explored some forms of energy and some ways it can be transferred from one system to another. This exploration led to the definition of two major types of energy—mechanical energy (KE + PE) and energy transferred via work done by nonconservative forces (W_{nc}). But energy takes *many* other forms, manifesting itself in *many* different ways, and we need to be able to deal with all of these before we can write an equation for the above general statement of the conservation of energy.

Other Forms of Energy than Mechanical Energy

At this point, we deal with all other forms of energy by lumping them into a single group called **other energy** (OE). Then we can state the conservation of energy in equation form as

$$\text{KE}_{\text{i}} + \text{PE}_{\text{i}} + W_{\text{nc}} + \text{OE}_{\text{i}} = \text{KE}_{\text{f}} + \text{PE}_{\text{f}} + \text{OE}_{\text{f}}.$$

All types of energy and work can be included in this very general statement of conservation of energy. Kinetic energy is KE, work done by a conservative force is represented by PE, work done by nonconservative forces is W_{nc}, and all other energies are included as OE. This equation applies to all previous examples; in those situations OE was constant, and so it subtracted out and was not directly considered.

MAKING CONNECTIONS: USEFULNESS OF THE ENERGY CONSERVATION PRINCIPLE

The fact that energy is conserved and has many forms makes it very important. You will find that energy is discussed in many contexts, because it is involved in all processes. It will also become apparent that many situations are best understood in terms of energy and that problems are often most easily conceptualized and solved by considering energy.

When does OE play a role? One example occurs when a person eats. Food is oxidized with the release of carbon dioxide, water, and energy. Some of this chemical energy is converted to kinetic energy when the person moves, to potential energy when the person changes altitude, and to thermal energy (another form of OE).

Some of the Many Forms of Energy

What are some other forms of energy? You can probably name a number of forms of energy not yet discussed. Many of these will be covered in later chapters, but let us detail a few here. **Electrical energy** is a common form that is converted to many other forms and does work in a wide range of practical situations. Fuels, such as gasoline and food, carry **chemical energy** that can be transferred to a system through oxidation. Chemical fuel can also produce electrical energy, such as in batteries. Batteries can in turn produce light, which is a very pure form of energy. Most energy sources on Earth are in fact stored energy from the energy we receive from the Sun. We sometimes refer to this as **radiant energy**, or electromagnetic radiation, which includes visible light, infrared, and ultraviolet radiation. **Nuclear energy** comes from processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into the energy of sunlight, into electrical energy in power plants, and into the energy of the heat transfer and blast in weapons. Atoms and molecules inside all objects are in random motion. This internal mechanical energy from the random motions is called **thermal energy**, because it is related to the temperature of the object. These and all other forms of energy can be converted into one another and can do work.

Table 1 gives the amount of energy stored, used, or released from various objects and in various phenomena. The range of energies and the variety of types and situations is impressive.

PROBLEM SOLVING STRATEGIES FOR ENERGY

You will find the following problem-solving strategies useful whenever you deal with energy. The strategies help in organizing and reinforcing energy concepts. In fact, they are used in the examples presented in this chapter. The familiar general problem-solving strategies presented earlier—involving

identifying physical principles, knowns, and unknowns, checking units, and so on—continue to be relevant here.

Step 1. Determine the system of interest and identify what information is given and what quantity is to be calculated. A sketch will help.

Step 2. Examine all the forces involved and determine whether you know or are given the potential energy from the work done by the forces. Then use step 3 or step 4.

Step 3. If you know the potential energies for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of energy is

$$\mathrm{KE_i} + \mathrm{PE_i} = \mathrm{KE_f} + \mathrm{PE_f}.$$

Step 4. If you know the potential energy for only some of the forces, possibly because some of them are nonconservative and do not have a potential energy, or if there are other energies that are not easily treated in terms of force and work, then the conservation of energy law in its most general form must be used.

$$\mathrm{KE_i} + \mathrm{PE_i} + W_{\mathrm{nc}} + \mathrm{OE_i} = \mathrm{KE_f} + \mathrm{PE_f} + \mathrm{OE_f}.$$

In most problems, one or more of the terms is zero, simplifying its solution. Do not calculate W_{c}, the work done by conservative forces; it is already incorporated in the PE terms.

Step 5. You have already identified the types of work and energy involved (in step 2). Before solving for the unknown, *eliminate terms wherever possible* to simplify the algebra. For example, choose $h = 0$ at either the initial or final point, so that $\mathrm{PE_g}$ is zero there. Then solve for the unknown in the customary manner.

Step 6. *Check the answer to see if it is reasonable*. Once you have solved a problem, reexamine the forms of work and energy to see if you have set up the conservation of energy equation correctly. For example, work done against friction should be negative, potential energy at the bottom of a hill should be less than that at the top, and so on. Also check to see that the numerical value obtained is reasonable. For example, the final speed of a skateboarder who coasts down a 3-m-high ramp could reasonably be 20 km/h, but *not* 80 km/h.

Transformation of Energy

The transformation of energy from one form into others is happening all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. In a larger example, the chemical energy contained in coal is converted into thermal energy as it burns to turn water into steam in a boiler. This thermal energy in the steam in turn is converted to mechanical energy as it spins a turbine, which is connected to a generator to produce electrical energy. (In all of these examples, not all of the initial energy is converted into the forms mentioned. This important point is discussed later in this section.)

Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell (see Figure 1) produces electricity, which in turn can be used to run an electric motor. Energy is converted from the primary source of solar energy into electrical energy and then into mechanical energy.

Figure 1. *Solar energy is converted into electrical energy by solar cells, which is used to run a motor in this solar-power aircraft. (credit: NASA)*

Object/phenomenon	Energy in joules
Big Bang	10^{68}
Energy released in a supernova	10^{44}
Fusion of all the hydrogen in Earth's oceans	10^{34}
Annual world energy use	4×10^{20}
Large fusion bomb (9 megaton)	3.8×10^{16}
1 kg hydrogen (fusion to helium)	6.4×10^{14}
1 kg uranium (nuclear fission)	8.0×10^{13}
Hiroshima-size fission bomb (10 kiloton)	4.2×10^{13}
90,000-ton aircraft carrier at 30 knots	1.1×10^{10}
1 barrel crude oil	5.9×10^{9}
1 ton TNT	4.2×10^{9}
1 gallon of gasoline	1.2×10^{8}
Daily home electricity use (developed countries)	7×10^{7}
Daily adult food intake (recommended)	1.2×10^{7}
1000-kg car at 90 km/h	3.1×10^{5}
1 g fat (9.3 kcal)	3.9×10^{4}
ATP hydrolysis reaction	3.2×10^{4}
1 g carbohydrate (4.1 kcal)	1.7×10^{4}
1 g protein (4.1 kcal)	1.7×10^{4}
Tennis ball at 100 km/h	22
Mosquito $\left(10^{-2}\ \text{g at } 0.5\ \text{m/s}\right)$	1.3×10^{-6}
Single electron in a TV tube beam	4.0×10^{-15}
Energy to break one DNA strand	10^{-19}

Table 1. Energy of Various Objects and Phenomena.

Efficiency

Even though energy is conserved in an energy conversion process, the output of *useful energy* or work will be less than the energy input. The **efficiency *Eff*** of an energy conversion process is defined as

$$\text{Efficiency}(Eff) = \frac{\text{useful energy or work output}}{\text{total energy input}} = \frac{W_{\text{out}}}{E_{\text{in}}}.$$

Table 2 lists some efficiencies of mechanical devices and human activities. In a coal-fired power plant, for example, about 40% of the chemical energy in the coal becomes useful electrical energy. The other 60% transforms into other (perhaps less useful) energy forms, such as thermal energy, which is then released to the environment through combustion gases and cooling towers.

Activity/device	**Efficiency (%)**[1]
Cycling and climbing	20
Swimming, surface	2
Swimming, submerged	4
Shoveling	3
Weightlifting	9
Steam engine	17
Gasoline engine	30
Diesel engine	35
Nuclear power plant	35
Coal power plant	42
Electric motor	98
Compact fluorescent light	20
Gas heater (residential)	90
Solar cell	10

Table 2. Efficiency of the Human Body and Mechanical Devices.

PHET EXPLOTATIONS: MASSES AND SPRINGS

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and

damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energies for each spring.

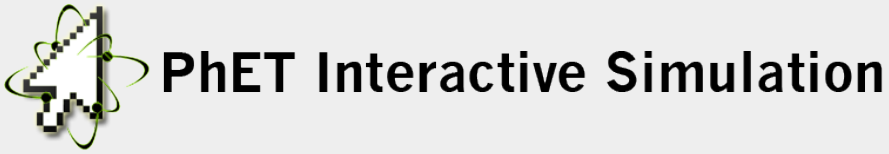


Figure 2. *Masses and Springs*

Section Summary

- The law of conservation of energy states that the total energy is constant in any process. Energy may change in form or be transferred from one system to another, but the total remains the same.
- When all forms of energy are considered, conservation of energy is written in equation form as $KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f$, where OE is all **other forms of energy** besides mechanical energy.
- Commonly encountered forms of energy include electric energy, chemical energy, radiant energy, nuclear energy, and thermal energy.
- Energy is often utilized to do work, but it is not possible to convert all the energy of a system to work.
- The efficiency *Eff* of a machine or human is defined to be $Eff = \frac{W_{\text{out}}}{E_{\text{in}}}$, where W_{out} is useful work output and E_{in} is the energy consumed.

Glossary

law of conservation of energy
: the general law that total energy is constant in any process; energy may change in form or be transferred from one system to another, but the total remains the same

electrical energy
: the energy carried by a flow of charge

chemical energy
: the energy in a substance stored in the bonds between atoms and molecules that can be released in a chemical reaction

radiant energy
: the energy carried by electromagnetic waves

nuclear energy
: energy released by changes within atomic nuclei, such as the fusion of two light nuclei or the fission of a heavy nucleus

thermal energy
: the energy within an object due to the random motion of its atoms and molecules that accounts for the object's temperature

efficiency
: a measure of the effectiveness of the input of energy to do work; useful energy or work divided by the total input of energy

Conceptual Questions

1: Consider the following scenario. A car for which friction is *not* negligible accelerates from rest down a hill, running out of gasoline after a short distance. The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events. (See Figure 3.)

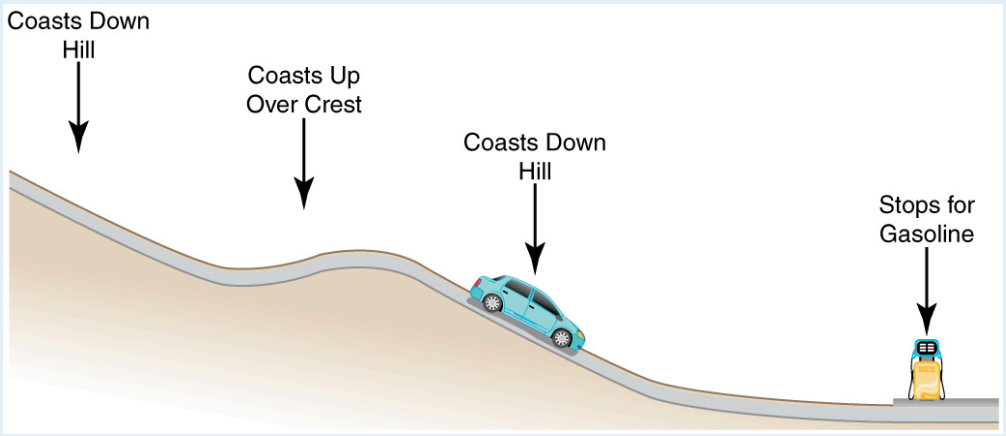


Figure 3. *A car experiencing non-negligible friction coasts down a hill, over a small crest, then downhill again, and comes to a stop at a gas station.*

2: Describe the energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.

3: Do devices with efficiencies of less than one violate the law of conservation of energy? Explain.

4: List four different forms or types of energy. Give one example of a conversion from each of these forms to another form.

5: List the energy conversions that occur when riding a bicycle.

Problems & Exercises

1: Using values from Table 1, how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous x rays. Later model tube TVs had shielding that absorbed x rays before they escaped and exposed viewers.)

2: Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown.

3: If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year's supply of energy (using data from Table 1)? This is not as far-fetched as it may sound—there are thousands of nuclear bombs, and their energy can be trapped in underground explosions and converted to electricity, as natural geothermal energy is.

4: (a) Use of hydrogen fusion to supply energy is a dream that may be realized in the next century. Fusion would be a relatively clean and almost limitless supply of energy, as can be seen from Table 1. To illustrate this, calculate how many years the present energy needs of the world could be supplied by one millionth of the oceans' hydrogen fusion energy. (b) How does this time compare with historically significant events, such as the duration of stable economic systems?

Footnotes

1. 1 Representative values

Solutions

Problems & Exercises

1: 4 x 10^4 molecules

2: Equating $\Delta \text{PE}_\text{g}$ and ΔKE, we obtain

$$v = \sqrt{2gh + v_0^2} = \sqrt{2(9.80\ \text{m/s}^2)(20.0\ \text{m}) + (15.0\ \text{m/s})^2} = 24.8\ \text{m/s}$$

4: (a) 25 x 10^6 years (b) This is much, much longer than human time scales.

6.7 Power

Summary

- Calculate power by calculating changes in energy over time.
- Examine power consumption and calculations of the cost of energy consumed.

What is Power?

Power—the word conjures up many images: a professional football player muscling aside his opponent, a dragster roaring away from the starting line, a volcano blowing its lava into the atmosphere, or a rocket blasting off, as in Figure 1.

Figure 1. *This powerful rocket on the Space Shuttle Endeavor did work and consumed energy at a very high rate. (credit: NASA)*

These images of power have in common the rapid performance of work, consistent with the scientific definition of **power** (*P*) as the rate at which work is done.

POWER

Power is the rate at which work is done.

$$P = \frac{W}{t}$$

The SI unit for power is the **watt** (W), where 1 watt equals 1 joule/second (1 W = 1 J/s).

Because work is energy transfer, power is also the rate at which energy is expended. A 60-W light bulb, for example, expends 60 J of energy per second. Great power means a large amount of work or energy developed in a short time. For example, when a powerful car accelerates rapidly, it does a large amount of work and consumes a large amount of fuel in a short time.

Calculating Power from Energy

Example 1: Calculating the Power to Climb Stairs

What is the power output for a 60.0-kg woman who runs up a 3.00 m high flight of stairs in 3.50 s, starting from rest but having a final speed of 2.00 m/s? (See Figure 2.)

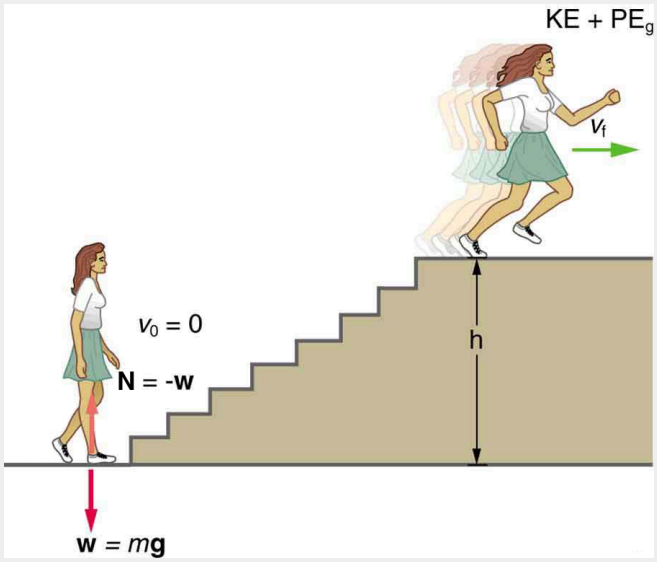


Figure 2. *When this woman runs upstairs starting from rest, she converts the chemical energy originally from food into kinetic energy and gravitational potential energy. Her power output depends on how fast she does this.*

Strategy and Concept

The work going into mechanical energy is W = KE + PE. At the bottom of the stairs, we take both KE

and PE_{gas} initially zero; thus, $W = \mathrm{KE_f} + \mathrm{PE_g} = \frac{1}{2}{mv_f}^2 + mgh$, where *h* is the vertical height of the stairs. Because all terms are given, we can calculate *W* and then divide it by time to get power.

Solution

Substituting the expression for *W* into the definition of power given in the previous equation, *P* = *W*/*t* yields

$$P = \frac{W}{t} = \frac{\frac{1}{2}{mv_f}^2 + mgh}{t}.$$

Entering known values yields

$$\begin{aligned} P &= \frac{0.5(60.0\ \mathrm{kg})(2.00\ \mathrm{m/s})^2 + (60.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2})(3.00\ \mathrm{m})}{3.50\ \mathrm{s}} \\ &= \frac{120\ \mathrm{J} + 1764\ \mathrm{J}}{3.50\ \mathrm{s}} \\ &= 538\ \mathrm{W}. \end{aligned}$$

Discussion

The woman does 1764 J of work to move up the stairs compared with only 120 J to increase her kinetic energy; thus, most of her power output is required for climbing rather than accelerating.

It is impressive that this woman's useful power output is slightly less than 1 horsepower (1 hp = 746 W)! People can generate more than a horsepower with their leg muscles for short periods of time by rapidly converting available blood sugar and oxygen into work output. (A horse can put out 1 hp for hours on end.) Once oxygen is depleted, power output decreases and the person begins to breathe rapidly to obtain oxygen to metabolize more food—this is known as the *aerobic* stage of exercise. If the woman climbed the stairs slowly, then her power output would be much less, although the amount of work done would be the same.

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION—MEASURE YOUR POWER RATING

Determine your own power rating by measuring the time it takes you to climb a flight of stairs. We will ignore the gain in kinetic energy, as the above example showed that it was a small portion of the energy gain. Don't expect that your output will be more than about 0.5 hp.

Examples of Power

Examples of power are limited only by the imagination, because there are as many types as there are forms of work and energy. (See Table 3 for some examples.) Sunlight reaching Earth's surface carries a maximum power of about 1.3 kilowatts per square meter (kW/m^2). A tiny fraction of this is retained by Earth over the long term. Our consumption rate of fossil fuels is far greater than the rate at which they are stored, so it is inevitable that they will be depleted. Power implies that energy is transferred, perhaps changing form. It is never possible to change one form completely into another without losing some of it as thermal energy. For example, a 60-W incandescent bulb converts only 5 W of electrical power to light, with 55 W dissipating into thermal energy. Furthermore, the typical electric power plant converts only 35 to 40% of its fuel into electricity. The remainder becomes a huge amount of thermal energy that must be dispersed as heat transfer, as rapidly as it is created. A coal-fired power plant may produce 1000 megawatts; 1 megawatt (MW) is 10^6 W of electric power. But the power plant consumes chemical energy at a rate of about 2500 MW, creating heat transfer to the surroundings at a rate of 1500 MW. (See Figure 3.)

Figure 3. *Tremendous amounts of electric power are generated by coal-fired power plants such as this one in China, but an even larger amount of power goes into heat transfer to the surroundings. The large cooling towers here are needed to transfer heat as rapidly as it is produced. The transfer of heat is not unique to coal plants but is an unavoidable consequence of generating electric power from any fuel—nuclear, coal, oil, natural gas, or the like. (credit: Kleinolive, Wikimedia Commons)*

Object or Phenomenon	Power in Watts
Supernova (at peak)	5×10^{37}
Milky Way galaxy	10^{37}
Crab Nebula pulsar	10^{28}
The Sun	4×10^{26}
Volcanic eruption (maximum)	4×10^{15}
Lightning bolt	2×10^{12}
Nuclear power plant (total electric and heat transfer)	3×10^{9}
Aircraft carrier (total useful and heat transfer)	10^{8}
Dragster (total useful and heat transfer)	2×10^{6}
Car (total useful and heat transfer)	8×10^{4}
Football player (total useful and heat transfer)	5×10^{3}
Clothes dryer	4×10^{3}
Person at rest (all heat transfer)	100
Typical incandescent light bulb (total useful and heat transfer)	60
Heart, person at rest (total useful and heat transfer)	8
Electric clock	3
Pocket calculator	10^{-3}

Table 3. Power Output or Consumption.

Power and Energy Consumption

We usually have to pay for the energy we use. It is interesting and easy to estimate the cost of energy for an electrical appliance if its power consumption rate and time used are known. The higher the power consumption rate and the longer the appliance is used, the greater the cost of that appliance. The power consumption rate is $P = W/t = E/t$, where E is the energy supplied by the electricity company. So the energy consumed over a time t is

$$E = Pt$$

Electricity bills state the energy used in units of **kilowatt-hours** (**kW h**), which is the product of power in kilowatts and time in hours. This unit is convenient because electrical power consumption at the kilowatt level for hours at a time is typical.

Example 2: Calculating Energy Costs

What is the cost of running a 0.200-kW computer 6.00 h per day for 30.0 d if the cost of electricity is \$0.120 per kW h?

Strategy

Cost is based on energy consumed; thus, we must find E from $E = Pt$ and then calculate the cost. Because electrical energy is expressed in kW h, at the start of a problem such as this it is convenient to convert the units into kW and hours.

Solution

The energy consumed in kW h is

$$\begin{aligned} E = Pt &= (0.200\ \text{kW})(6.00\ \text{h/d})(30.0\ \text{d}) \\ &= 36.0\ \text{kW}\cdot\text{h}, \end{aligned}$$

and the cost is simply given by

$$\text{Cost} = (36.0\ \text{kW}\cdot\text{h})(\$0.120\ \text{per kW}\cdot\text{h}) = \$4.32\ \text{per month}.$$

Discussion

The cost of using the computer in this example is neither exorbitant nor negligible. It is clear that the cost is a combination of power and time. When both are high, such as for an air conditioner in the summer, the cost is high.

The motivation to save energy has become more compelling with its ever-increasing price. Armed with the knowledge that energy consumed is the product of power and time, you can estimate costs for yourself and make the necessary value judgments about where to save energy. Either power or time must be reduced. It is most cost-effective to limit the use of high-power devices that normally operate for long periods of time, such as water heaters and air conditioners. This would not include relatively high power devices like toasters, because they are on only a few minutes per day. It would also not include electric clocks, in spite of their 24-hour-per-day usage, because they are very low power devices. It is sometimes possible to use devices that have greater efficiencies—that is, devices that consume less power to accomplish the same task. One example is the compact fluorescent light bulb, which produces over four times more light per watt of power consumed than its incandescent cousin.

Modern civilization depends on energy, but current levels of energy consumption and production are not sustainable. The likelihood of a link between global warming and fossil fuel use (with its concomitant production of carbon dioxide), has made reduction in energy use as well as a shift to non-fossil fuels of the utmost importance. Even though energy in an isolated system is a conserved quantity, the final result of most energy transformations is waste heat transfer to the environment, which is no longer useful for doing work.

Section Summary

- Power is the rate at which work is done, or in equation form, for the average power ***P*** for work *W* done over a time *t*, $P = W/t$.
- The SI unit for power is the watt (W), where 1 W = 1 J/s.
- The power of many devices such as electric motors is also often expressed in horsepower (hp), where 1 hp = 746 W.

Glossary

power
the rate at which work is done
watt
(W) SI unit of power, with 1 W = 1 J/s
horsepower
an older non-SI unit of power, with 1 hp = 746 W
kilowatt-hour
(kW • h) unit used primarily for electrical energy provided by electric utility companies

Conceptual Questions

1: Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.

2: Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?

3: A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.

Problems & Exercises

1: The Crab Nebula (see Figure 4) pulsar is the remnant of a supernova that occurred in A.D. 1054. Using data from Table 3, calculate the approximate factor by which the power output of this astronomical object has declined since its explosion.

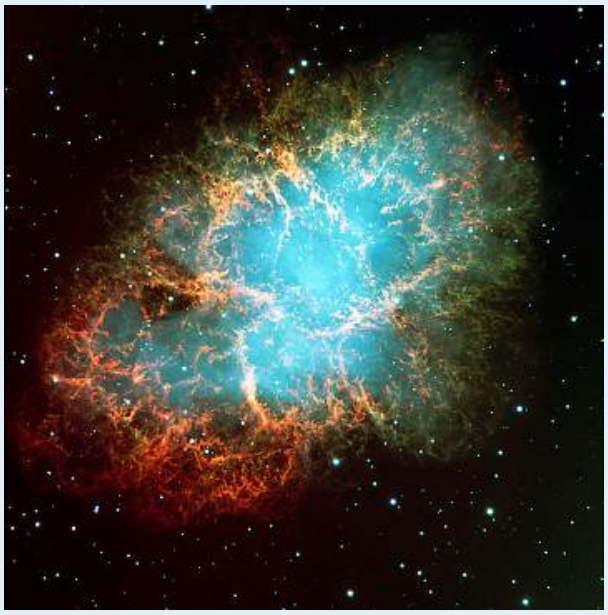


Figure 4. *Crab Nebula (credit: ESO, via Wikimedia Commons)*

2: Suppose a star 1000 times brighter than our Sun (that is, emitting 1000 times the power) suddenly goes supernova. Using data from Table 3: (a) By what factor does its power output increase? (b) How many times brighter than our entire Milky Way galaxy is the supernova? (c) Based on your answers, discuss whether it should be possible to observe supernovas in distant galaxies. Note that there are on the order of 10^{11} observable galaxies, the average brightness of which is somewhat less than our own galaxy.

3: A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?

4: What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is \$0.0900 per kW h?

5: A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is \$0.110 per kW h?

6: (a) What is the average power consumption in watts of an appliance that uses 5.00 kWh of energy per day? (b) How many joules of energy does this appliance consume in a year?

7: (a) What is the average useful power output of a person who does 6.00×10^6 J of useful work in 8.00 h? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)

8: A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N. What is its average power output in watts and horsepower if this takes 7.30 s?

9: (a) How long will it take an 850-kg car with a useful power output of 40.0 hp (1 hp = 746 W) to reach a speed of 15.0 m/s, neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m-high hill in the process?

10: (a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is \$0.0900 per kW • h?

11: (a) What is the available energy content, in joules, of a battery that operates a 2.00-W electric clock for 18 months? (b) How long can a battery that can supply 8.00×10^4 J run a pocket calculator that consumes energy at the rate of 1.00×10^{-3} W?

12: (a) How long would it take a 1.50×10^5-kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (Hint: You must find the distance the plane travels in 1200 s assuming constant acceleration.)

13: Calculate the power output needed for a 950-kg car to climb a 2.00° slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N. Explicitly show how you follow the steps in the Problem-Solving Strategies for Energy.

14: (a) Calculate the power per square meter reaching Earth's upper atmosphere from the Sun. (Take the power output of the Sun to be 4.00×10^{26} W.) (b) Part of this is absorbed and reflected by the atmosphere, so that a maximum of 1.30 kW/m^2 reaches Earth's surface. Calculate the area in km^2 of solar energy collectors needed to replace an electric power plant that generates 750 MW if the collectors convert an average of 2.00% of the maximum power into electricity. (This small conversion efficiency is due to the devices themselves, and the fact that the sun is directly overhead only briefly.) With the same assumptions, what area would be needed to meet the United States' energy needs (1.05×10^{20} J)? Australia's energy needs (5.4×10^{18} J)? China's energy needs (6.3×10^{19} J)? (These energy consumption values are from 2006.)

Solutions

Problems & Exercises

1: 2×10^{-10}

3: (a) 40 (b) 8 million

5: 149

7: (a) 208 W (b) 144 s

9: (a) 3.20 s (b) 4.04 s

11: (a) 9.46×10^7 J (b) 2.54 y

13: Identify knowns: m = 950 kg, Slope angle θ = 2.00°, v = 3.00 m/s, f = 600 N

Identify unknowns: power P of the car, force F that car applies to road

Solve for unknown:

$$P = \frac{W}{t} = \frac{Fd}{t} = F(\frac{d}{t}) = Fv,$$

where F is parallel to the incline and must oppose the resistive forces and the force of gravity:

$$F = f + w = 600 \text{ N} + mg \sin \theta$$

Insert this into the expression for power and solve:

$$\begin{aligned} P &= (f + mg \sin \theta)v \\ &= [600 \text{ N} + (950 \text{ kg})(9.80 \text{ m/s}^2) \sin 2^o](30.0 \text{ m/s}) \\ &= 2.77x10^4 \text{ W} \end{aligned}$$

About 28 kW (or about 37 hp) is reasonable for a car to climb a gentle incline.

6.8 Optional: Work, Energy, and Power in Humans

Summary

- Explain the human body's consumption of energy when at rest vs. when engaged in activities that do useful work.
- Calculate the conversion of chemical energy in food into useful work.

Energy Conversion in Humans

Our own bodies, like all living organisms, are energy conversion machines. Conservation of energy implies that the chemical energy stored in food is converted into work, thermal energy, and/or stored as chemical energy in fatty tissue. (See Figure 1.) The fraction going into each form depends both on how much we eat and on our level of physical activity. If we eat more than is needed to do work and stay warm, the remainder goes into body fat.

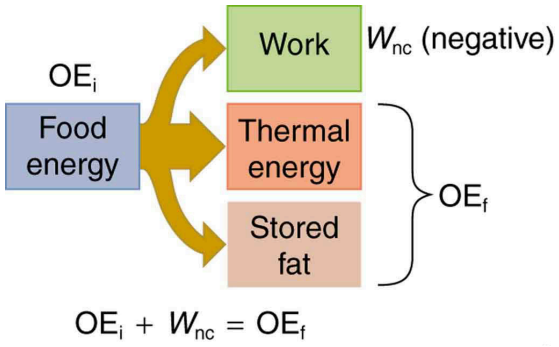


Figure 1. *Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity.*

Power Consumed at Rest

The *rate* at which the body uses food energy to sustain life and to do different activities is called the **metabolic rate**. The total energy conversion rate of a person *at rest* is called the **basal metabolic rate** (BMR) and is divided among various systems in the body, as shown in Table 4. The largest fraction goes to the liver and spleen, with the brain coming next. Of course, during vigorous exercise, the energy consumption of the skeletal muscles and heart increase markedly. About 75% of the calories burned in a day go into these basic functions. The BMR is a function of age, gender, total body weight, and amount

of muscle mass (which burns more calories than body fat). Athletes have a greater BMR due to this last factor.

Organ	Power consumed at rest (W)	Oxygen consumption (mL/min)	Percent of BMR
Liver & spleen	23	67	27
Brain	16	47	19
Skeletal muscle	15	45	18
Kidney	9	26	10
Heart	6	17	7
Other	16	48	19
Totals	85 W	250 mL/min	100%

Table 4. Basal Metabolic Rates (BMR).

Energy consumption is directly proportional to oxygen consumption because the digestive process is basically one of oxidizing food. We can measure the energy people use during various activities by measuring their oxygen use. (See Figure 2.) Approximately 20 kJ of energy are produced for each liter of oxygen consumed, independent of the type of food. Table 5 shows energy and oxygen consumption rates (power expended) for a variety of activities.

Power of Doing Useful Work

Work done by a person is sometimes called **useful work**, which is *work done on the outside world*, such as lifting weights. Useful work requires a force exerted through a distance on the outside world, and so it excludes internal work, such as that done by the heart when pumping blood. Useful work does include that done in climbing stairs or accelerating to a full run, because these are accomplished by exerting forces on the outside world. Forces exerted by the body are nonconservative, so that they can change the mechanical energy (KE + PE) of the system worked upon, and this is often the goal. A baseball player throwing a ball, for example, increases both the ball's kinetic and potential energy.

If a person needs more energy than they consume, such as when doing vigorous work, the body must draw upon the chemical energy stored in fat. So exercise can be helpful in losing fat. However, the amount of exercise needed to produce a loss in fat, or to burn off extra calories consumed that day, can be large, as Example 1 illustrates.

Example 1: Calculating Weight Loss from Exercising

If a person who normally requires an average of 12,000 kJ (3000 kcal) of food energy per day

consumes 13,000 kJ per day, he will steadily gain weight. How much bicycling per day is required to work off this extra 1000 kJ?

Solution

Table 5 states that 400 W are used when cycling at a moderate speed. The time required to work off 1000 kJ at this rate is then

$$\text{Time} = \frac{Energy}{(\frac{Energy}{Time})} = \frac{1000 \text{ kJ}}{400 \text{ W}} = 2500 \text{ s} = 42 \text{ min}.$$

Discussion

If this person uses more energy than he or she consumes, the person's body will obtain the needed energy by metabolizing body fat. If the person uses 13,000 kJ but consumes only 12,000 kJ, then the amount of fat loss will be

$$\text{Fat loss} = (1000 \text{ kJ})(\frac{1.0 \text{ g fat}}{39 \text{ kJ}}) = 26 \text{ g},$$

assuming the energy content of fat to be 39 kJ/g.

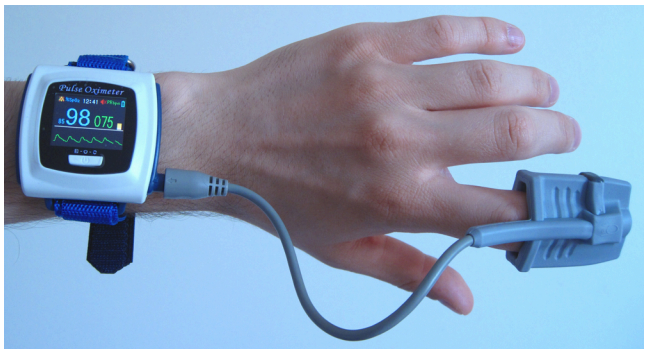


Figure 2. *A pulse oximeter is an apparatus that measures the amount of oxygen in blood. Oxymeters can be used to determine a person's metabolic rate, which is the rate at which food energy is converted to another form. Such measurements can indicate the level of athletic conditioning as well as certain medical problems. (credit: UusiAjaja, Wikimedia Commons)*

Activity	Energy consumption in watts	Oxygen consumption in liters O_2/min
Sleeping	83	0.24
Sitting at rest	120	0.34
Standing relaxed	125	0.36
Sitting in class	210	0.60
Walking (5 km/h)	280	0.80
Cycling (13–18 km/h)	400	1.14
Shivering	425	1.21
Playing tennis	440	1.26
Swimming breaststroke	475	1.36
Ice skating (14.5 km/h)	545	1.56
Climbing stairs (116/min)	685	1.96
Cycling (21 km/h)	700	2.00
Running cross-country	740	2.12
Playing basketball	800	2.28
Cycling, professional racer	1855	5.30
Sprinting	2415	6.90

Table 5. Energy and Oxygen Consumption Rates[1] (Power).

All bodily functions, from thinking to lifting weights, require energy. (See Figure 3.) The many small muscle actions accompanying all quiet activity, from sleeping to head scratching, ultimately become thermal energy, as do less visible muscle actions by the heart, lungs, and digestive tract. Shivering, in fact, is an involuntary response to low body temperature that pits muscles against one another to produce thermal energy in the body (and do no work). The kidneys and liver consume a surprising amount of energy, but the biggest surprise of all it that a full 25% of all energy consumed by the body is used to maintain electrical potentials in all living cells. (Nerve cells use this electrical potential in nerve impulses.) This bioelectrical energy ultimately becomes mostly thermal energy, but some is utilized to power chemical processes such as in the kidneys and liver, and in fat production.

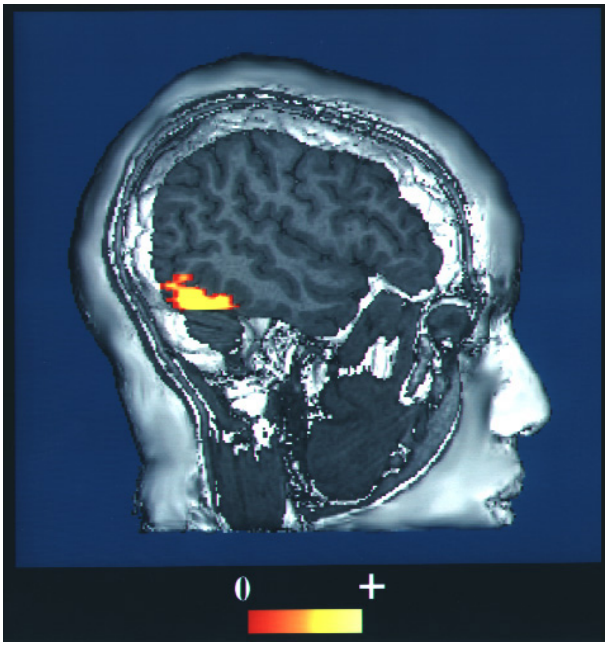


Figure 3. *This fMRI scan shows an increased level of energy consumption in the vision center of the brain. Here, the patient was being asked to recognize faces. (credit: NIH via Wikimedia Commons)*

Section Summary

- The human body converts energy stored in food into work, thermal energy, and/or chemical energy that is stored in fatty tissue.
- The *rate* at which the body uses food energy to sustain life and to do different activities is called the metabolic rate, and the corresponding rate when at rest is called the basal metabolic rate (BMR)
- The energy included in the basal metabolic rate is divided among various systems in the body, with the largest fraction going to the liver and spleen, and the brain coming next.
- About 75% of food calories are used to sustain basic body functions included in the basal metabolic rate.
- The energy consumption of people during various activities can be determined by measuring their oxygen use, because the digestive process is basically one of oxidizing food.

Glossary

metabolic rate
: the rate at which the body uses food energy to sustain life and to do different activities

basal metabolic rate
: the total energy conversion rate of a person at rest

useful work
work done on an external system

Conceptual Questions

1: Explain why it is easier to climb a mountain on a zigzag path rather than one straight up the side. Is your increase in gravitational potential energy the same in both cases? Is your energy consumption the same in both?

2: Do you do work on the outside world when you rub your hands together to warm them? What is the efficiency of this activity?

3: Shivering is an involuntary response to lowered body temperature. What is the efficiency of the body when shivering, and is this a desirable value?

4: Discuss the relative effectiveness of dieting and exercise in losing weight, noting that most athletic activities consume food energy at a rate of 400 to 500 W, while a single cup of yogurt can contain 1360 kJ (325 kcal). Specifically, is it likely that exercise alone will be sufficient to lose weight? You may wish to consider that regular exercise may increase the metabolic rate, whereas protracted dieting may reduce it.

Problems & Exercises

1: (a) How long can you rapidly climb stairs (116/min) on the 93.0 kcal of energy in a 10.0-g pat of butter? (b) How many flights is this if each flight has 16 stairs?

2: (a) What is the power output in watts and horsepower of a 70.0-kg sprinter who accelerates from rest to 10.0 m/s in 3.00 s? (b) Considering the amount of power generated, do you think a well-trained athlete could do this repetitively for long periods of time?

3: Calculate the power output in watts and horsepower of a shot-putter who takes 1.20 s to accelerate the 7.27-kg shot from rest to 14.0 m/s, while raising it 0.800 m. (Do not include the power produced to accelerate his body.)

Figure 4. *Shot putter at the Dornoch Highland Gathering in 2007. (credit: John Haslam, Flickr)*

4: (a) What is the efficiency of an out-of-condition professor who does 2.10×10^5 J of useful work while metabolizing 500 kcal of food energy? (b) How many food calories would a well-conditioned athlete metabolize in doing the same work with an efficiency of 20%?

5: Energy that is not utilized for work or heat transfer is converted to the chemical energy of body fat containing about 39 kJ/g. How many grams of fat will you gain if you eat 10,000 kJ (about 2500 kcal) one day and do nothing but sit relaxed for 16.0 h and sleep for the other 8.00 h? Use data from Table 5 for the energy consumption rates of these activities.

6: Using data from Table 5, calculate the daily energy needs of a person who sleeps for 7.00 h, walks for 2.00 h, attends classes for 4.00 h, cycles for 2.00 h, sits relaxed for 3.00 h, and studies for 6.00 h. (Studying consumes energy at the same rate as sitting in class.)

7: What is the efficiency of a subject on a treadmill who puts out work at the rate of 100 W while consuming oxygen at the rate of 2.00 L/min? (Hint: See Table 5.)

8: Shoveling snow can be extremely taxing because the arms have such a low efficiency in this activity. Suppose a person shoveling a footpath metabolizes food at the rate of 800 W. (a) What is her useful power output? (b) How long will it take her to lift 3000 kg of snow 1.20 m? (This could be the amount of heavy snow on 20 m of footpath.) (c) How much waste heat transfer in kilojoules will she generate in the process?

9: Very large forces are produced in joints when a person jumps from some height to the ground. (a) Calculate the magnitude of the force produced if an 80.0-kg person jumps from a 0.600–m-high ledge and lands stiffly, compressing joint material 1.50 cm as a result. (Be certain to include the weight of the person.) (b) In practice the knees bend almost involuntarily to help extend the distance over which you stop. Calculate the magnitude of the force produced if the stopping distance is 0.300 m. (c) Compare both forces with the weight of the person.

10: Jogging on hard surfaces with insufficiently padded shoes produces large forces in the feet and legs. (a) Calculate the magnitude of the force needed to stop the downward motion of a jogger’s leg, if his leg has a mass of 13.0 kg, a speed of 6.00 m/s, and stops in a distance of 1.50 cm. (Be certain to include the weight of the 75.0-kg jogger’s body.) (b) Compare this force with the weight of the jogger.

11: (a) Calculate the energy in kJ used by a 55.0-kg woman who does 50 deep knee bends in which her center

of mass is lowered and raised 0.400 m. (She does work in both directions.) You may assume her efficiency is 20%. (b) What is the average power consumption rate in watts if she does this in 3.00 min?

12: Kanellos Kanellopoulos flew 119 km from Crete to Santorini, Greece, on April 23, 1988, in the *Daedalus 88*, an aircraft powered by a bicycle-type drive mechanism (see Figure 5). His useful power output for the 234-min trip was about 350 W. Using the efficiency for cycling from Table 2, calculate the food energy in kilojoules he metabolized during the flight.

Figure 5. *The Daedalus 88 in flight. (credit: NASA photo by Beasley)*

13: The swimmer shown in Figure 6 exerts an average horizontal backward force of 80.0 N with his arm during each 1.80 m long stroke. (a) What is his work output in each stroke? (b) Calculate the power output of his arms if he does 120 strokes per minute.

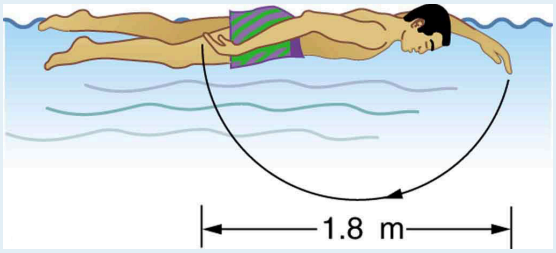


Figure 6.

14: Mountain climbers carry bottled oxygen when at very high altitudes. (a) Assuming that a mountain climber uses oxygen at twice the rate for climbing 116 stairs per minute (because of low air temperature and winds), calculate how many liters of oxygen a climber would need for 10.0 h of climbing. (These are liters at sea level.) Note that only 40% of the inhaled oxygen is utilized; the rest is exhaled. (b) How much useful work does the climber do if he and his equipment have a mass of 90.0 kg and he gains 1000 m of altitude? (c) What is his efficiency for the 10.0-h climb?

15: The awe-inspiring Great Pyramid of Cheops was built more than 4500 years ago. Its square base, originally 230 m on a side, covered 13.1 acres, and it was 146 m high, with a mass of about 7×10^9 kg. (The pyramid's dimensions are slightly different today due to quarrying and some sagging.) Historians estimate that 20,000 workers spent 20 years to construct it, working 12-hour days, 330 days per year. (a) Calculate the gravitational potential energy stored in the pyramid, given its center of mass is at one-fourth its height. (b)

Only a fraction of the workers lifted blocks; most were involved in support services such as building ramps (see Figure 7), bringing food and water, and hauling blocks to the site. Calculate the efficiency of the workers who did the lifting, assuming there were 1000 of them and they consumed food energy at the rate of 300 kcal/h. What does your answer imply about how much of their work went into block-lifting, versus how much work went into friction and lifting and lowering their own bodies? (c) Calculate the mass of food that had to be supplied each day, assuming that the average worker required 3600 kcal per day and that their diet was 5% protein, 60% carbohydrate, and 35% fat. (These proportions neglect the mass of bulk and nondigestible materials consumed.)

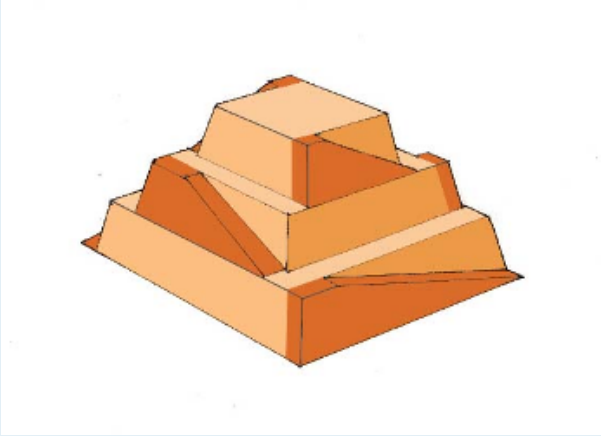


Figure 7. *Ancient pyramids were probably constructed using ramps as simple machines. (credit: Franck Monnier, Wikimedia Commons)*

16: (a) How long can you play tennis on the 800 kJ (about 200 kcal) of energy in a candy bar? (b) Does this seem like a long time? Discuss why exercise is necessary but may not be sufficient to cause a person to lose weight.

Footnotes

1. 1 for an average 76-kg male

Solutions

Problems & Exercises

1: (a) 9.5 min (b) 69 flights of stairs

3: 641 W or 0.860 hp

5: 31 g

7: 14.3%

9: (a) 3.21×10^4 N (b) 2.35×10^3 N (c) Ratio of net force to weight of person is 41.0 in part (a); 3.00 in part (b)

11: 108 kJ (b) 599 W

13: (a) 144 J (b) 288 W

15: (a) 2.50×10^{12} J (b) 2.52% (c) 1.4×10^4 kg (14 metric tons)

6.9 Optional: World Energy Use

Summary

- Describe the distinction between renewable and nonrenewable energy sources.
- Explain why the inevitable conversion of energy to less useful forms makes it necessary to conserve energy resources.

Energy is an important ingredient in all phases of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. But current levels of energy consumption and production are not sustainable. About 40% of the world's energy comes from oil, and much of that goes to transportation uses. Oil prices are dependent as much upon new (or foreseen) discoveries as they are upon political events and situations around the world. The U.S., with 4.5% of the world's population, consumes 24% of the world's oil production per year; 66% of that oil is imported!

Renewable and Nonrenewable Energy Sources

The principal energy resources used in the world are shown in Figure 1. The fuel mix has changed over the years but now is dominated by oil, although natural gas and solar contributions are increasing. **Renewable forms of energy** are those sources that cannot be used up, such as water, wind, solar, and biomass. About 85% of our energy comes from nonrenewable **fossil fuels**—oil, natural gas, coal. The likelihood of a link between global warming and fossil fuel use, with its production of carbon dioxide through combustion, has made, in the eyes of many scientists, a shift to non-fossil fuels of utmost importance—but it will not be easy.

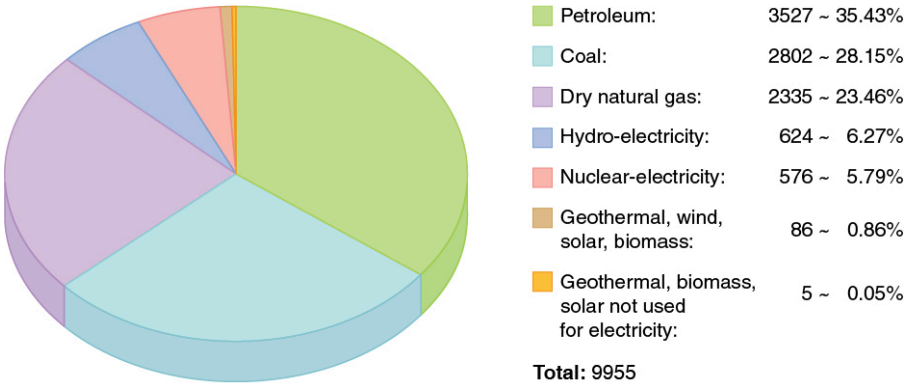


Figure 1. *World energy consumption by source, in billions of kilowatt-hours: 2006. (credit: KVDP)*

The World's Growing Energy Needs

World energy consumption continues to rise, especially in the developing countries. (See Figure 2.) Global demand for energy has tripled in the past 50 years and might triple again in the next 30 years. While much of this growth will come from the rapidly booming economies of China and India, many of the developed countries, especially those in Europe, are hoping to meet their energy needs by expanding the use of renewable sources. Although presently only a small percentage, renewable energy is growing very fast, especially wind energy. For example, Germany plans to meet 20% of its electricity and 10% of its overall energy needs with renewable resources by the year 2020. (See Figure 3.) Energy is a key constraint in the rapid economic growth of China and India. In 2003, China surpassed Japan as the world's second largest consumer of oil. However, over 1/3 of this is imported. Unlike most Western countries, coal dominates the commercial energy resources of China, accounting for 2/3 of its energy consumption. In 2009 China surpassed the United States as the largest generator of CO_2. In India, the main energy resources are biomass (wood and dung) and coal. Half of India's oil is imported. About 70% of India's electricity is generated by highly polluting coal. Yet there are sizeable strides being made in renewable energy. India has a rapidly growing wind energy base, and it has the largest solar cooking program in the world.

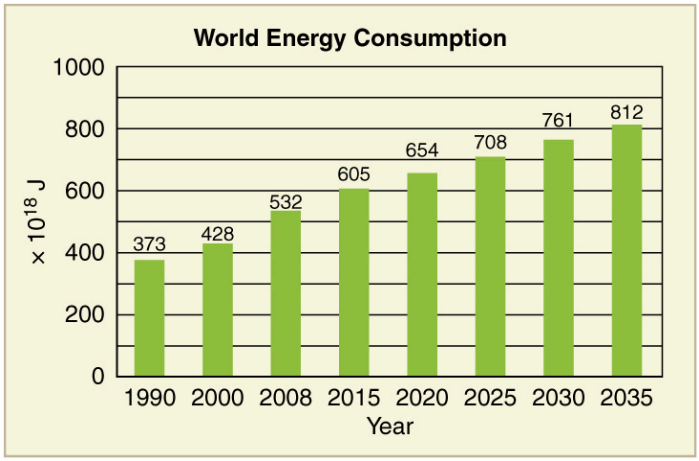


Figure 2. *Past and projected world energy use (source: Based on data from U.S. Energy Information Administration, 2011)*

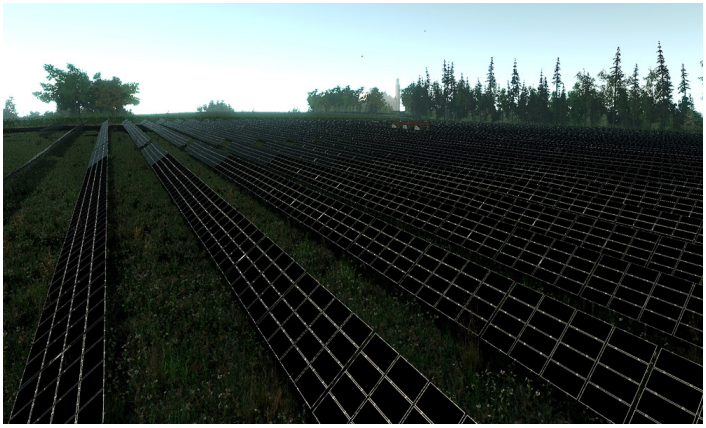


Figure 3. *Solar cell arrays at a power plant in Steindorf, Germany (credit: Michael Betke, Flickr)*

Table 6 displays the 2006 commercial energy mix by country for some of the prime energy users in the world. While non-renewable sources dominate, some countries get a sizeable percentage of their electricity from renewable resources. For example, about 67% of New Zealand's electricity demand is met by hydroelectric. Only 10% of the U.S. electricity is generated by renewable resources, primarily hydroelectric. It is difficult to determine total contributions of renewable energy in some countries with a large rural population, so these percentages in this table are left blank.

Country	Consumption, in EJ (10^{18} J)	Oil	Natural Gas	Coal	Nuclear	Hydro	Other Renewables	Electricity Use per capita (kWh/yr)	Energy Use per capita (GJ/yr)
Australia	5.4	34%	17%	44%	0%	3%	1%	10000	260
Brazil	9.6	48%	7%	5%	1%	35%	2%	2000	50
China	63	22%	3%	69%	1%	6%		1500	35
Egypt	2.4	50%	41%	1%	0%	6%		990	32
Germany	16	37%	24%	24%	11%	1%	3%	6400	173
India	15	34%	7%	52%	1%	5%		470	13
Indonesia	4.9	51%	26%	16%	0%	2%	3%	420	22
Japan	24	48%	14%	21%	12%	4%	1%	7100	176
New Zealand	0.44	32%	26%	6%	0%	11%	19%	8500	102
Russia	31	19%	53%	16%	5%	6%		5700	202
U.S.	105	40%	23%	22%	8%	3%	1%	12500	340
World	**432**	**39%**	**23%**	**24%**	**6%**	**6%**	**2%**	**2600**	**71**

Table 6. Energy Consumption—Selected Countries (2006).

Energy and Economic Well-being

The last two columns in this table examine the energy and electricity use per capita. Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (gross domestic product) per capita, are matched by higher levels of energy consumption per capita. This is borne out in Figure 4. Increased efficiency of energy use will change this dependency. A global problem is balancing energy resource development against the harmful effects upon the environment in its extraction and use.

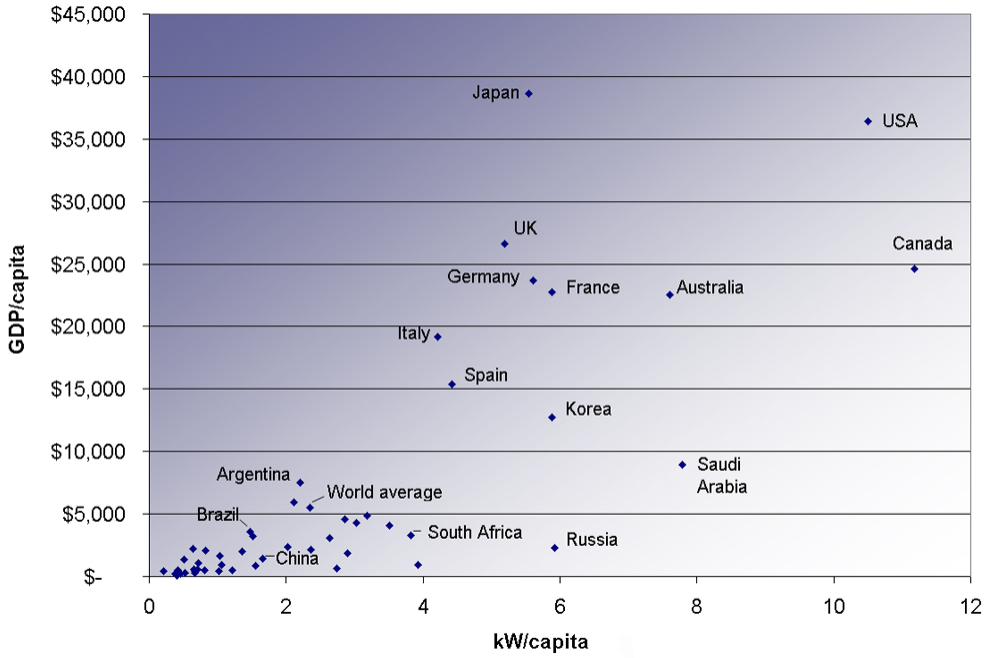


Figure 4. *Power consumption per capita versus GDP per capita for various countries. Note the increase in energy usage with increasing GDP. (2007, credit: Frank van Mierlo, Wikimedia Commons)*

Conserving Energy

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As has been mentioned elsewhere, the "law of the conservation of energy" is a very useful principle in analyzing physical processes. It is a statement that cannot be proven from basic principles, but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system will always remain constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through (1) reduced activities (e.g., turning down thermostats, driving fewer kilometers) and/or (2) increasing conversion efficiencies in the performance of a particular task—such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed or created or generated, one might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat transfer to the environment and conversion to energy forms no longer useful for doing work. To state it in another way, the potential for energy to produce useful work has been "degraded" in the energy transformation.

Section Summary

- The relative use of different fuels to provide energy has changed over the years, but fuel use is currently dominated by oil, although natural gas and solar contributions are increasing.
- Although non-renewable sources dominate, some countries meet a sizeable percentage of their electricity needs from renewable resources.
- The United States obtains only about 10% of its energy from renewable sources, mostly hydroelectric power.
- Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (Gross Domestic Product) per capita, are matched by higher levels of energy consumption per capita.
- Even though, in accordance with the law of conservation of energy, energy can never be created or destroyed, energy that can be used to do work is always partly converted to less useful forms, such as waste heat to the environment, in all of our uses of energy for practical purposes.

Glossary

renewable forms of energy
: those sources that cannot be used up, such as water, wind, solar, and biomass

fossil fuels
: oil, natural gas, and coal

Conceptual Questions

1: What is the difference between energy conservation and the law of conservation of energy? Give some examples of each.

2: If the efficiency of a coal-fired electrical generating plant is 35%, then what do we mean when we say that energy is a conserved quantity?

Problems & Exercises

1: Integrated Concepts

(a) Calculate the force the woman in Figure 5 exerts to do a push-up at constant speed, taking all data to be known to three digits. (b) How much work does she do if her center of mass rises 0.240 m? (c) What is her useful power output if she does 25 push-ups in 1 min? (Should work done lowering her body be included? See the discussion of useful work in Chapter 7.8 Work, Energy, and Power in Humans.

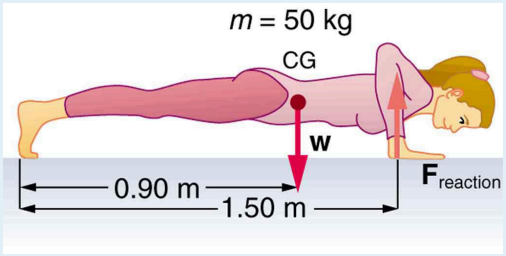


Figure 5. *Forces involved in doing push-ups. The woman's weight acts as a force exerted downward on her center of gravity (CG).*

2: Integrated Concepts

A 75.0-kg cross-country skier is climbing a 3.0° slope at a constant speed of 2.00 m/s and encounters air resistance of 25.0 N. Find his power output for work done against the gravitational force and air resistance. (b) What average force does he exert backward on the snow to accomplish this? (c) If he continues to exert this force and to experience the same air resistance when he reaches a level area, how long will it take him to reach a velocity of 10.0 m/s?

3: Integrated Concepts

The 70.0-kg swimmer in Chapter 7.8 Work, Energy, and Power in Humans starts a race with an initial velocity of 1.25 m/s and exerts an average force of 80.0 N backward with his arms during each 1.80 m long stroke. (a) What is his initial acceleration if water resistance is 45.0 N? (b) What is the subsequent average resistance force from the water during the 5.00 s it takes him to reach his top velocity of 2.50 m/s? (c) Discuss whether water resistance seems to increase linearly with velocity.

4: Integrated Concepts

A toy gun uses a spring with a force constant of 300 N/m to propel a 10.0-g steel ball. If the spring is compressed 7.00 cm and friction is negligible: (a) How much force is needed to compress the spring? (b) To what maximum height can the ball be shot? (c) At what angles above the horizontal may a child aim to hit a target 3.00 m away at the same height as the gun? (d) What is the gun's maximum range on level ground?

5: Integrated Concepts

(a) What force must be supplied by an elevator cable to produce an acceleration of 0.800 m/s^2 against a 200-N frictional force, if the mass of the loaded elevator is 1500 kg? (b) How much work is done by the cable in lifting the elevator 20.0 m? (c) What is the final speed of the elevator if it starts from rest? (d) How much work went into thermal energy?

6: Unreasonable Results

A car advertisement claims that its 900-kg car accelerated from rest to 30.0 m/s and drove 100 km, gaining 3.00 km in altitude, on 1.0 gal of gasoline. The average force of friction including air resistance was 700 N. Assume all values are known to three significant figures. (a) Calculate the car's efficiency. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

7: Unreasonable Results

Body fat is metabolized, supplying 9.30 kcal/g, when dietary intake is less than needed to fuel metabolism. The manufacturers of an exercise bicycle claim that you can lose 0.500 kg of fat per day by vigorously exercising for 2.00 h per day on their machine. (a) How many kcal are supplied by the metabolization of

0.500 kg of fat? (b) Calculate the kcal/min that you would have to utilize to metabolize fat at the rate of 0.500 kg in 2.00 h. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

8: Construct Your Own Problem

Consider a person climbing and descending stairs. Construct a problem in which you calculate the long-term rate at which stairs can be climbed considering the mass of the person, his ability to generate power with his legs, and the height of a single stair step. Also consider why the same person can descend stairs at a faster rate for a nearly unlimited time in spite of the fact that very similar forces are exerted going down as going up. (This points to a fundamentally different process for descending versus climbing stairs.)

9: Construct Your Own Problem

Consider humans generating electricity by pedaling a device similar to a stationary bicycle. Construct a problem in which you determine the number of people it would take to replace a large electrical generation facility. Among the things to consider are the power output that is reasonable using the legs, rest time, and the need for electricity 24 hours per day. Discuss the practical implications of your results.

10: Integrated Concepts

A 105-kg basketball player crouches down 0.400 m while waiting to jump. After exerting a force on the floor through this 0.400 m, his feet leave the floor and his center of gravity rises 0.950 m above its normal standing erect position. (a) Using energy considerations, calculate his velocity when he leaves the floor. (b) What average force did he exert on the floor? (Do not neglect the force to support his weight as well as that to accelerate him.) (c) What was his power output during the acceleration phase?

Solutions

Problems & Exercises

1: (a) 294 N (b) 118 J (c) 49.0 W

3: (a) 0.500 m/s^2 (b) 62.5 N (c) Assuming the acceleration of the swimmer decreases linearly with time over the 5.00 s interval, the frictional force must therefore be increasing linearly with time, since $f = F - ma$. If the acceleration decreases linearly with time, the velocity will contain a term dependent on time squared (t^2). Therefore, the water resistance will not depend linearly on the velocity.

5: (a) 16.1×10^3 N (b) 3.22×10^5 J (c) 5.66 m/s (d) 4.00 kJ

7: (a) 4.65×10^3 kcal (b) 39.9 kcal/min

(c) This power output is higher than the highest value on Table 5, which is about 35 kcal/min (corresponding to 2415 watts) for sprinting.

(d) It would be impossible to maintain this power output for 2 hours (imagine sprinting for 2 hours!).

10: (a) 4.32 m/s (b) 3.47×10^3 N (c) 8.93 kW

Chapter 7 Linear Momentum and Collisions

Figure 1. *Each rugby player has great momentum, which will affect the outcome of their collisions with each other and the ground. (credit: ozzzie, Flickr)*

We use the term momentum in various ways in everyday language, and most of these ways are consistent with its precise scientific definition. We speak of sports teams or politicians gaining and maintaining the momentum to win. We also recognize that momentum has something to do with collisions. For example, looking at the rugby players in the photograph colliding and falling to the ground, we expect their momenta to have great effects in the resulting collisions. Generally, momentum implies a tendency to continue on course—to move in the same direction—and is associated with great mass and speed.

Momentum, like energy, is important because it is conserved. Only a few physical quantities are conserved in nature, and studying them yields fundamental insight into how nature works, as we shall see in our study of momentum.

7.1 Linear Momentum and Force

Summary

- Define linear momentum.
- Explain the relationship between momentum and force.
- State Newton's second law of motion in terms of momentum.
- Calculate momentum given mass and velocity.

Linear Momentum

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as

$$\vec{\mathrm{p}} = m\vec{\mathrm{v}}.$$

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum $\vec{\mathbf{p}}$ is a vector having the same direction as the velocity $\vec{\mathrm{v}}$. The SI unit for momentum is kg · m/s.

LINEAR MOMENTUM

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

$$\vec{\mathrm{p}} = m\vec{\mathrm{v}}.$$

Example 1: Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, *p*. (As usual, a symbol that is in italics is a magnitude, whereas one that has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

$$p = mv$$

when only magnitudes are considered.

Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

$$p_{\text{player}} = (110\ \text{kg})(8.00\ \text{m/s}) = 880\ \text{kg}\cdot\text{m/s}$$

Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

$$p_{\text{ball}} = (0.410\ \text{kg})(25.0\ \text{m/s}) = 10.3\ \text{kg}\cdot\text{m/s}$$

The ratio of the player's momentum to that of the ball is

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9.$$

Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player's motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the "quantity of motion." Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

$$\vec{\text{F}}_{\text{net}} = \frac{\Delta\vec{\text{p}}}{\Delta t},$$

where $\vec{\text{F}}_{\text{net}}$ is the net external force, $\Delta\vec{\text{p}}$ is the change in momentum, and Δt is the change in time.

NEWTON'S SECOND LAW OF MOTION IN TERMS OF MOMENTUM

The net external force equals the change in momentum of a system divided by the time over which it changes.

$$\vec{\mathrm{F}}_{\mathrm{net}} = \frac{\Delta \vec{\mathrm{p}}}{\Delta t}$$

MAKING CONNECTIONS: FORCE AND MOMENTUM

Force and momentum are intimately related. Force acting over time can change momentum, and Newton's second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton's second law of motion includes the more familiar $\vec{\mathrm{F}}_{\mathrm{net}} = m\vec{\mathrm{a}}$ as a special case. We can derive this form as follows. First, note that the change in momentum $\Delta \vec{\mathrm{p}}$ is given by

$$\Delta \vec{\mathrm{p}} = \Delta(m\vec{\mathrm{v}}).$$

If the mass of the system is constant, then

$$\Delta(m\vec{\mathrm{v}}) = m\Delta\vec{\mathrm{v}}.$$

So that for constant mass, Newton's second law of motion becomes

$$\vec{\mathrm{F}}_{\mathrm{net}} = \frac{\Delta \vec{\mathrm{p}}}{\Delta t} = \frac{m\Delta\vec{\mathrm{v}}}{\Delta t}.$$

Because $\dfrac{\Delta \vec{\mathrm{v}}}{\Delta t} = \vec{\mathrm{a}}$, we get the familiar equation

$$\vec{\mathrm{F}}_{\mathrm{net}} = m\vec{\mathrm{a}}$$

when the mass of the system is constant.

Newton's second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail**;** however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

Example 2: Calculating Force: Venus Williams' Racquet

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

$$F_{\text{net}} = \frac{\Delta p}{\Delta t}.$$

As noted above, when mass is constant, the change in momentum is given by

$$\Delta p = m\Delta v = m(v_{\text{f}} - v_{\text{i}}).$$

In this example, the velocity just after impact and the change in time are given; thus, once Δp is calculated, $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ can be used to find the force.

Solution

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

$$\begin{aligned} \Delta p &= m(v_f - v_i) \\ &= (0.057\ \text{kg})(58\ \text{m/s} - 0\ \text{m/s}) \\ &= 3.306\ \text{kg}\cdot\text{m/s} \approx 3.3\ \text{kg}\cdot\text{m/s} \end{aligned}$$

Now the magnitude of the net external force can determined by using $F_{\text{net}} = \frac{\Delta p}{\Delta t}$:

$$\begin{aligned} F_{\text{net}} = \frac{\Delta p}{\Delta t} &= \frac{3.306\ \text{kg}\cdot\text{m/s}}{5.0 \times 10^{-3}\ \text{s}} \\ &= 661\ \text{N} \approx 660\ \ \text{N}, \end{aligned}$$

where we have retained only two significant figures in the final step.

Discussion

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{\text{net}} = ma$, but one additional step would be required compared with the strategy used in this example.

Section Summary

- Linear momentum (*momentum* for brevity) is defined as the product of a system's mass multiplied by its velocity.
- In symbols, linear momentum $\vec{\mathrm{p}}$ is defined to be

$$\vec{\mathrm{p}} = m\vec{\mathrm{v}},$$

  where m is the mass of the system and $\vec{\mathrm{v}}$ is its velocity.
- The SI unit for momentum is kg m/s.
- Newton's second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
- In symbols, Newton's second law of motion is defined to be

$$\vec{\mathrm{F}}_{\mathrm{net}} = \frac{\Delta\vec{\mathrm{p}}}{\Delta t},$$

  $\vec{\mathrm{F}}_{\mathrm{net}}$ is the net external force, $\Delta\vec{\mathrm{p}}$ is the change in momentum, and Δt is the change time.

Glossary

linear momentum
: the product of mass and velocity

second law of motion
: physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes

Conceptual Questions

1: An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

2: An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

3: Professional Application

Football coaches advise players to block, hit, and tackle with their feet on the ground rather than by leaping through the air. Using the concepts of momentum, work, and energy, explain how a football player can be more effective with his feet on the ground.

4: How can a small force impart the same momentum to an object as a large force?

Problems & Exercises

1: (a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

2: (a) What is the mass of a large ship that has a momentum of 1.60×10^9 kg m/s, when the ship is moving at a speed of 48.0 km/h? (b) Compare the ship's momentum to the momentum of a 1100-kg artillery shell fired at a speed of 1200 m/s.

3: (a) At what speed would a 2.00×10^4-kg airplane have to fly to have a momentum of 1.60×10^9 kg m/s (the same as the ship's momentum in the problem above)? (b) What is the plane's momentum when it is taking off at a speed of 60.0 m/s? (c) If the ship is an aircraft carrier that launches these airplanes with a catapult, discuss the implications of your answer to (b) as it relates to recoil effects of the catapult on the ship.

4: (a) What is the momentum of a garbage truck that is 1.20×10^4 kg and is moving at 10.0 m/s? (b) At what speed would an 8.00-kg trash can have the same momentum as the truck?

5: A runaway train car that has a mass of 15,000 kg travels at a speed of 5.4 m/s down a track. Compute the time required for a force of 1500 N to bring the car to rest.

6: The mass of Earth is 5.972×10^{24} kg and its orbital radius is an average of 1.496×10^{11} m. Calculate its linear momentum.

Solutions

Problems & Exercises

1: (a) 1.50 x 10^4 kg m/s (b) 625:1 (c) 6.66 x 10^2 kg m/s

3: (a) 8.00 x 10^4 m/s (b) 1.20 x 10^6 kg m/s (c) Because the momentum of the airplane is 3 orders of magnitude smaller than of the ship, the ship will not recoil very much. The recoil would be $-0.0100\ \mathrm{m/s}$, which is probably not noticeable.

5: 54 s

7.2 Impulse

Summary

- Define impulse.
- Describe effects of impulses in everyday life.
- Determine the average effective force using graphical representation.
- Calculate average force and impulse given mass, velocity, and time.

The effect of a force on an object depends on how long it acts, as well as how great the force is. A very large force acting for a short time can have a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum $\Delta\vec{\mathrm{p}}$.

By rearranging the equation $\vec{\mathrm{F}}_{\mathrm{net}} = \dfrac{\Delta\vec{\mathrm{p}}}{\Delta t}$ to be

$$\Delta\vec{\mathrm{p}} = \vec{\mathrm{F}}_{\mathrm{net}}\,\Delta t,$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $\vec{\mathrm{F}}_{\mathrm{net}}\,\Delta t$ is given the name **impulse**. Impulse is the same as the change in momentum.

IMPULSE: CHANGE IN MOMENTUM

Change in momentum equals the average net external force multiplied by the time this force acts.

$$\Delta\vec{\mathrm{p}} = \vec{\mathrm{F}}_{\mathrm{net}}\,\Delta t$$

The quantity $\vec{\mathrm{F}}_{\mathrm{net}}\,\Delta t$ is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a sudden stop. The momentum change is the same for an occupant, whether an air bag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of

plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

Example 1: Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30° from the perpendicular, and bounces off at an angle of 30° from perpendicular to the wall.

(a) Determine the direction of the force on the wall due to each ball.

(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Strategy for (a)

In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the *x*-axis to be normal to the wall and to be positive in the initial direction of motion. Choose the *y*-axis to be along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

Solution for (a)

The first ball bounces directly into the wall and exerts a force on it in the $+x$ direction. Therefore the wall exerts a force on the ball in the $\mathbf{-x}$ direction. The second ball continues with the same momentum component in the $\mathbf{y}$ direction, but reverses its *x*-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the $-x$ direction, so the force of the wall on each ball is along the $-x$ direction.

Strategy for (b)

Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for (b)

Let μ be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the *x*-axis and *y*-axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

$$p_{\text{xi}} = \mu;\ p_{\text{yi}} = 0$$
$$p_{\text{xf}} = -\mu;\ p_{\text{yf}} = 0$$

Impulse is the change in momentum vector. Therefore the *x*-component of impulse is equal to -2*mu* and the *y*-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

$$p_{\text{xi}} = \mu \cos 30^0;\ p_{\text{yi}} = -\mu \sin 30^0$$
$$p_{\text{xf}} = -\mu \cos 30^0;\ p_{\text{yf}} = -\mu \sin 30^0$$

It should be noted here that while p_{x} changes sign after the collision, p_{y} does not. Therefore the *x*-component of impulse is equal to -2*mu* cos 30° and the *y*-component of impulse is equal to zero.

The ratio of the magnitudes of the impulse imparted to the balls is

$$\frac{2\mu}{2\mu \cos 30^0} = \frac{2}{\sqrt{3}} = 1.155.$$

Discussion

The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative *x*-direction. Making use of Newton's third law, the force on the wall due to each ball is normal to the wall along the positive *x*-direction.

Our definition of impulse includes an assumption that the force is constant over the time interval Δt. *Forces are usually not constant*. Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force F_{eff} that produces the same result as the corresponding time-varying force. Figure 1 shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times t_1 and t_2. That area is equal to the area inside the rectangle bounded by F_{eff}, t_1, and t_2. Thus the impulses and their effects are the same for both the actual and effective forces.

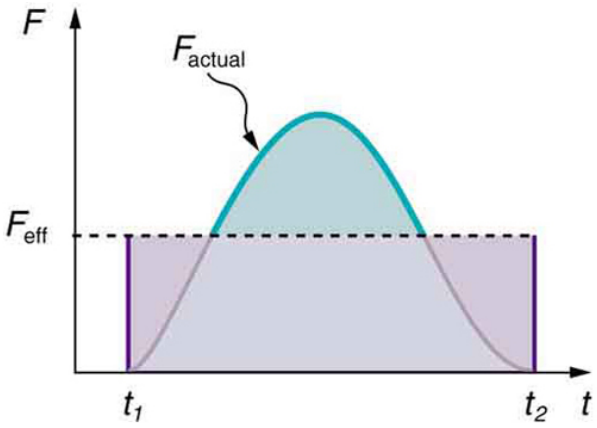


Figure 1. *A graph of force versus time with time along the* ***x****-axis and force along the* ***y****-axis for an actual force and an equivalent effective force. The areas under the two curves are equal.*

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION—HAND MOVEMENT AND IMPULSE

Try catching a ball while "giving" with the ball, pulling your hands toward your body. Then, try catching a ball while keeping your hands still. Hit water in a tub with your full palm. After the water has settled, hit the water again by diving your hand with your fingers first into the water. (Your full palm represents a swimmer doing a belly flop and your diving hand represents a swimmer doing a dive.) Explain what happens in each case and why. Which orientations would you advise people to avoid and why?

MAKING CONNECTIONS: CONSTANT FORCE AND CONSTANT ACCELERATION

The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

Section Summary

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts:

$$\Delta\vec{\mathrm{p}} = \vec{\mathrm{F}}_{\mathrm{net}}\,\Delta t.$$

- Forces are usually not constant over a period of time.

Glossary

change in momentum
the difference between the final and initial momentum; the mass times the change in velocity

impulse
the average net external force times the time it acts; equal to the change in momentum

Conceptual Questions

1: Professional Application

Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.

2: While jumping on a trampoline, sometimes you land on your back and other times on your feet. In which case can you reach a greater height and why?

3: Professional Application

Tennis racquets have “sweet spots.” If the ball hits a sweet spot then the player’s arm is not jarred as much as it would be otherwise. Explain why this is the case.

Problems & Exercises

1: A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600 m/s in a time of 2.00 ms (milliseconds)?

2: Professional Application

A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

3: A person slaps her leg with her hand, bringing her hand to rest in 2.50 milliseconds from an initial speed of 4.00 m/s. (a) What is the average force exerted on the leg, taking the effective mass of the hand and forearm to be 1.50 kg? (b) Would the force be any different if the woman clapped her hands together at the same speed and brought them to rest in the same time? Explain why or why not.

4: Professional Application

A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent’s final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent’s 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant momentum to the boxer’s body. (d) Discuss the implications of your answers for parts (b) and (c).

5: Professional Application

Suppose a child drives a bumper car head on into the side rail, which exerts a force of 4000 N on the car for 0.200 s. (a) What impulse is imparted by this force? (b) Find the final velocity of the bumper car if its initial velocity was 2.80 m/s and the car plus driver have a mass of 200 kg. You may neglect friction between the car and floor.

6: Professional Application

One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of 4.00×10^3 m/s, given the collision lasts 6.00×10^{-8} s.

7: Professional Application

A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

8: Professional Application

Military rifles have a mechanism for reducing the recoil forces of the gun on the person firing it. An internal part recoils over a relatively large distance and is stopped by damping mechanisms in the gun. The larger distance reduces the average force needed to stop the internal part. (a) Calculate the recoil velocity of a 1.00-kg plunger that directly interacts with a 0.0200-kg bullet fired at 600 m/s from the gun. (b) If this part is stopped over a distance of 20.0 cm, what average force is exerted upon it by the gun? (c) Compare this to the force exerted on the gun if the bullet is accelerated to its velocity in 10.0 ms (milliseconds).

9: A cruise ship with a mass of 1.00×10^7 kg strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

10: Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76×10^4 N.

11: Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the magnitude of the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.

12: A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?

13: Starting with the definitions of momentum and kinetic energy, derive an equation for the kinetic energy of a particle expressed as a function of its momentum.

14: A ball with an initial velocity of 10 m/s moves at an angle 60° above the $+x$-direction. The ball hits a vertical wall and bounces off so that it is moving 60° above the $-x$-direction with the same speed. What is the impulse delivered by the wall?

15: When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a

vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

16: A punter drops a ball from rest vertically 1 meter down onto his foot. The ball leaves the foot with a speed of 18 m/s at an angle 55° above the horizontal. What is the impulse delivered by the foot (magnitude and direction)?

Solutions

Problems & Exercises

1: 9.00 x 10 ^{3} N

3: (a) $2.40 \times 10^3\ \mathrm{N}$ toward the leg (b) The force on each hand would have the same magnitude as that found in part (a) (but in opposite directions by Newton's third law) because the change in momentum and the time interval are the same.

5: (a) $800\ \mathrm{kg \cdot m/s}$ away from the wall (b) $1.20\ \mathrm{m/s}$ away from the wall

7: (a) 1.50 x 10 ^{6} N away from the dashboard (b)1.00 x 10 4 N away from the dashboard

9: 4.69 x 10^{5} N in the boat's original direction of motion

11: 2.10 x 10^{3} N away from the wall

13:

$$\mathrm{p} = m\mathrm{v} \Rightarrow p^2 = m^2 v^2 \Rightarrow \frac{p^2}{m} = mv^2$$

$$\Rightarrow \frac{p^2}{2m} = \frac{1}{2}mv^2 = \mathrm{KE}$$

$$\mathrm{KE} = \frac{p^2}{2m}$$

15: 60.0 grams

7.3 Conservation of Momentum

Summary

- Describe the principle of conservation of momentum.
- Derive an expression for the conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the principle of conservation of momentum as it relates to atomic and subatomic particles.

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in Chapter 7.2 Impulse and Chapter 7.1 Linear Momentum and Force, where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in Figure 1. Both cars are coasting in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.

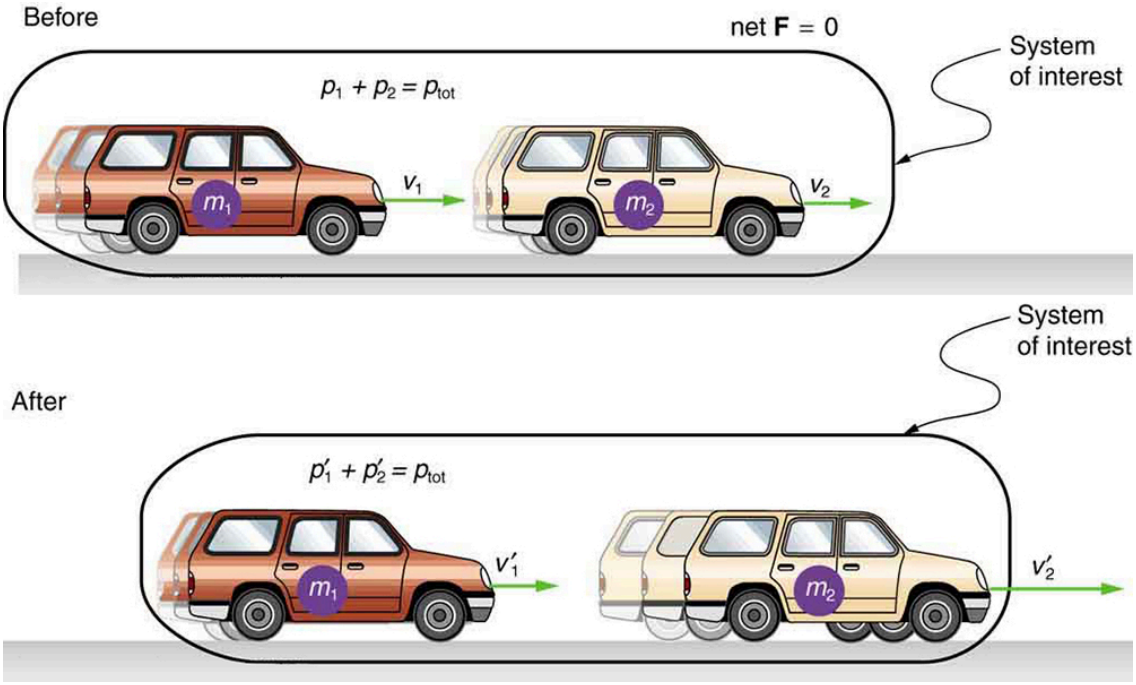


Figure 1. *A car of mass m_1 moving with a velocity of v_1 bumps into another car of mass $m_2$and velocity v_2 that it is following. As a result, the first car slows down to a velocity of $v'_1$and the second speeds up to a velocity of v'_2. The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction is negligible).*

Using the definition of impulse, the change in momentum of car 1 is given by

$$\Delta\vec{p}_1 = \vec{F}_1 \Delta t,$$

where $\vec{F}_1$ is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

$$\Delta\vec{p}_2 = \vec{F}_2 \Delta t,$$

where $\vec{F}_2$ is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $\vec{F}_1 = -\vec{F}_2$, and so

$$\Delta\vec{p}_2 = -\vec{F}_1 \Delta t = -\Delta\vec{p}_1 .$$

Thus, the changes in momentum are equal and opposite, and

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = 0.$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

$$\vec{p}_1 + \vec{p}_2 = \text{constant},$$
$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2',$$

where $\vec{p}_1'$ and $\vec{p}_2'$ are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

$$\vec{p}_{\text{tot}} = \text{constant},$$

or

$$\vec{p}_{\text{tot}} = \vec{p}_{\text{tot}}',$$

where $\vec{p}_{\text{tot}}$ is the total momentum (the sum of the momenta of the individual objects in the system) and $\vec{p}_{\text{tot}}'$ is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero $(\vec{F}_{\text{net}} = 0)$.

CONSERVATION OF MOMENTUM PRINCIPLE

$$\vec{P}_{\text{tot}} = \text{constant}$$
$$\vec{P}_{\text{tot}} = \vec{P'}_{\text{tot}} \text{ (isolated system)}$$

ISOLATED SYSTEM

An isolated system is defined to be one for which the net external force is zero $(\vec{F}_{\text{net}} = 0)$.

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton's second law in terms of momentum, $\vec{F}_{\text{net}} = \dfrac{\Delta \vec{p}_{\text{tot}}}{\Delta t}$. For an isolated system, $\vec{F}_{\text{net}} = 0$; thus, $\Delta \vec{p}_{\text{tot}} = 0$, and $\vec{p}_{\text{tot}}$ is constant.

We have noted that the three length dimensions in nature—*x*, *y*, and *z*—are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See Figure 2.) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.

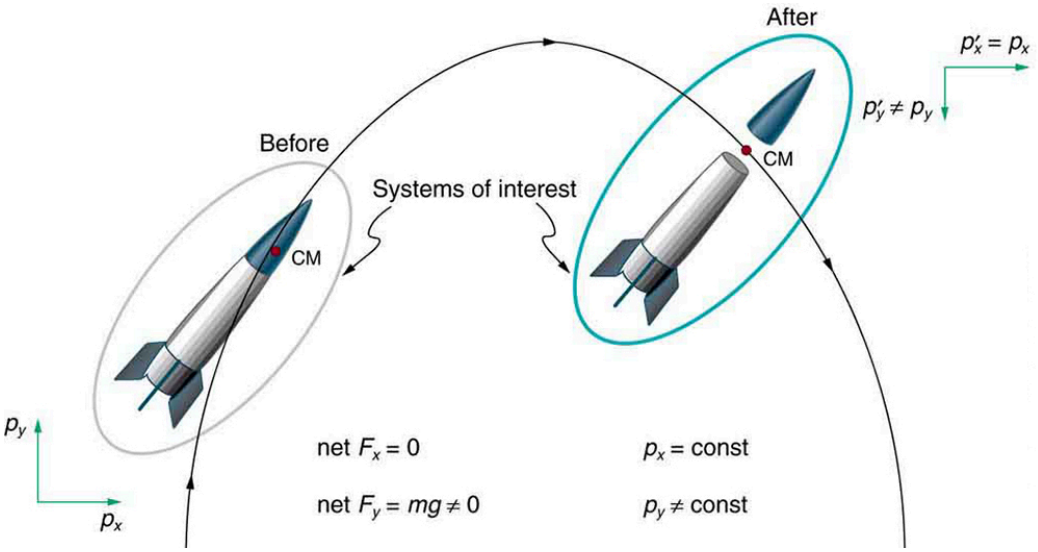


Figure 2. *The horizontal component of a projectile's momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force* $F_{\text{x-net}}$ *is still zero. The vertical component of the momentum is not conserved, because the net vertical force* $F_{\text{y-net}}$ *is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the space probe takes the same path it would if the separation did not occur.*

The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION—DROP OF TENNIS BALL AND A BASEBALL

Hold a tennis ball side by side and in contact with a basketball. Drop the balls together. (Be careful!) What happens? Explain your observations. Now hold the tennis ball above and in contact with the basketball. What happened? Explain your observations. What do you think will happen if the basketball ball is held above and in contact with the tennis ball?

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION—TWO TENNIS BALLS IN A BALLISTIC TRAJECTORY

Tie two tennis balls together with a string about a foot long. Hold one ball and let the other hang down and throw it in a ballistic trajectory. Explain your observations. Now mark the center of the string with bright ink or attach a brightly colored sticker to it and throw again. What happened? Explain your observations.

Some aquatic animals such as jellyfish move around based on the principles of conservation of momentum. A jellyfish fills its umbrella section with water and then pushes the water out resulting in motion in the opposite direction to that of the jet of water. Squids propel themselves in a similar manner but, in contrast with jellyfish, are able to control the direction in which they move by aiming their nozzle forward or backward. Typical squids can move at speeds of 8 to 12 km/h.

The ballistocardiograph (BCG) was a diagnostic tool used in the second half of the 20th century to study the strength of the heart. About once a second, your heart beats, forcing blood into the aorta. A force in the opposite direction is exerted on the rest of your body (recall Newton's third law). A ballistocardiograph is a device that can measure this reaction force. This measurement is done by using a sensor (resting on the person) or by using a moving table suspended from the ceiling. This technique can gather information on the strength of the heart beat and the volume of blood passing from the heart. However, the electrocardiogram (ECG or EKG) and the echocardiogram (cardiac ECHO or ECHO; a technique that uses ultrasound to see an image of the heart) are more widely used in the practice of cardiology.

MAKING CONNECTIONS: CONSERVATION OF MOMENTUM AND COLLISION

Conservation of momentum is quite useful in describing collisions. Momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

Subatomic Collisions and Momentum

The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results. Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may

have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements. Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. Figure 3 below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that **quarks** make up protons (one type of particle that makes up nuclei) scattered high-energy electrons off of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis was based partly on the same conservation of momentum principle that works so well on the large scale.

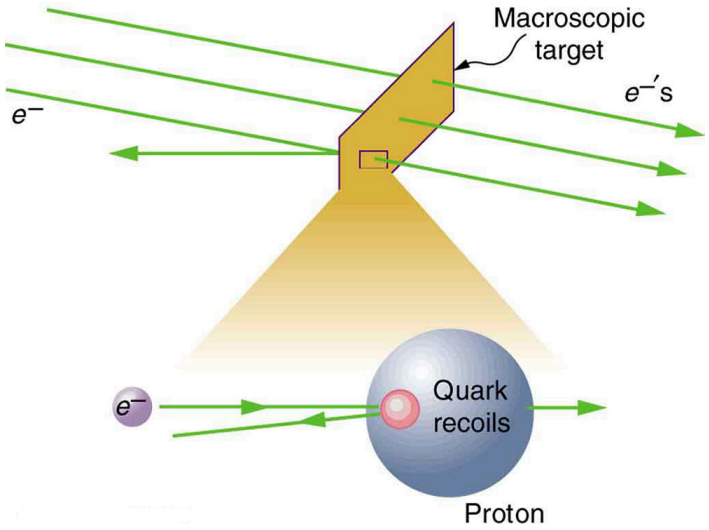


Figure 3. *A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for quarks, electrons were observed to occasionally scatter straight backward from a proton.*

Section Summary

- The conservation of momentum principle is written

$$\vec{\mathrm{p}}_{\mathrm{tot}} = \text{constant}$$

  or

$$\vec{\mathrm{p}}_{\mathrm{tot}} = \vec{\mathrm{p}}'_{\mathrm{tot}} \ (\text{isolated system}),$$

  $\vec{\mathrm{p}}_{\mathrm{tot}}$ is the initial total momentum and $\vec{\mathrm{p}}'_{\mathrm{tot}}$ is the total momentum some time later.

- An isolated system is defined to be one for which the net external force is zero $(\vec{\mathrm{F}}_{\mathrm{net}} = 0)$.
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.

- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

a

Glossary

conservation of momentum principle
when the net external force is zero, the total momentum of the system is conserved or constant

isolated system
a system in which the net external force is zero

quark
fundamental constituent of matter and an elementary particle

Conceptual Questions

1: Professional Application

If you dive into water, you reach greater depths than if you do a belly flop. Explain this difference in depth using the concept of conservation of energy. Explain this difference in depth using what you have learned in this chapter.

2: Under what circumstances is momentum conserved?

3: Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?

4: Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

5: Professional Application

Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.

6: Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

7: Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

Problems & Exercises

1: Professional Application

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving

toward one another, the first having a mass of 150,000 kg and a velocity of 0.300 m/s, and the second having a mass of 110,000 kg and a velocity of -0.120 m/s. (The minus indicates direction of motion.) What is their final velocity?

2: Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

3: Professional Application

Consider the following question: *A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.* Would the answer to this question be different if the car with the 70-kg passenger had collided with a car that has a mass equal to and is traveling in the opposite direction and at the same speed? Explain your answer.

4: What is the velocity of a 900-kg car initially moving at 30.0 m/s, just after it hits a 150-kg deer initially running at 12.0 m/s in the same direction? Assume the deer remains on the car.

5: A 1.80-kg falcon catches a 0.650-kg dove from behind in midair. What is their velocity after impact if the falcon's velocity is initially 28.0 m/s and the dove's velocity is 7.00 m/s in the same direction?

Solutions

Problems & Exercises

1: 0.122 m/s

3: In a collision with an identical car, momentum is conserved. Afterwards $v_{\mathrm{f}} = 0$ for both cars. The change in momentum will be the same as in the crash with the tree. However, the force on the body is not determined since the time is not known. A padded stop will reduce injurious force on body.

5: 22.4 m/s in the same direction as the original motion

7.4 Elastic Collisions in One Dimension

Summary

- Describe an elastic collision of two objects in one dimension.
- Define internal kinetic energy.
- Derive an expression for conservation of internal kinetic energy in a one dimensional collision.
- Determine the final velocities in an elastic collision given masses and initial velocities.

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. In a one-dimensional problem we can omit the vector arrows as we are dealing solely with magnitudes. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. Figure 1 illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

ELASTIC COLLISION

An **elastic collision** is one that conserves internal kinetic energy.

INTERNAL KINETIC ENERGY

Internal kinetic energy is the sum of the kinetic energies of the objects in the system.

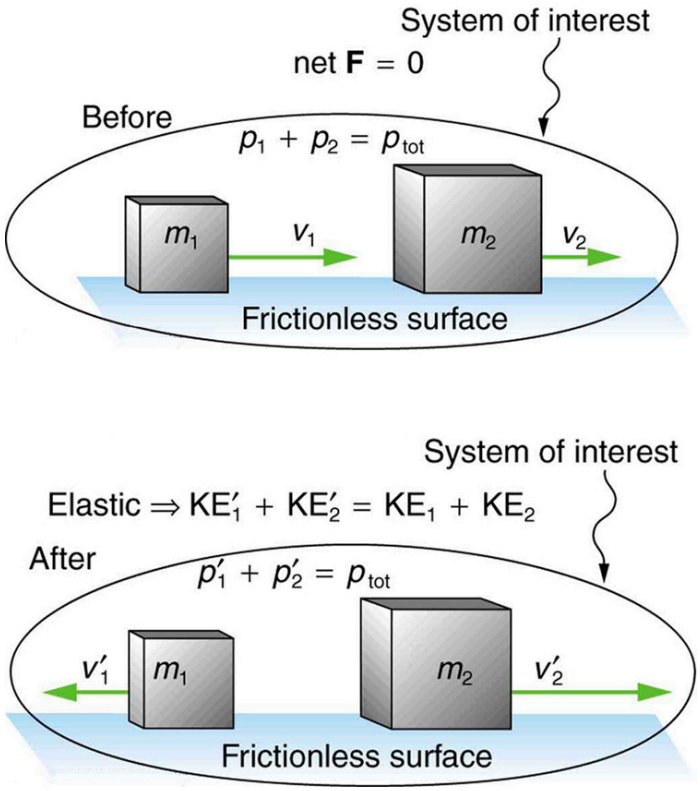


Figure 1. *An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.*

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1 + p_2 = p'_1 + p'_2 \ (F_{\text{net}} = 0)$$

or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \ (F_{\text{net}} = 0),$$

where the primes (‘) indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2 + \frac{1}{2} m_1 {v\prime_1}^2 + \frac{1}{2} m_2 {v\prime_2}^2 \ (\text{two-object elastic collision})$$

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Example 1: Calculating Velocities Following an Elastic Collision

Calculate the velocities of two objects following an elastic collision, given that

$$m_1 = 0.500 \text{ kg}\,, m_2 = 3.50 \text{ kg}, \; v_1 = 4.00 \text{ m/s}, \text{and } v_2 = 0.$$

Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in Figure 1 where both objects are initially moving. We are asked to find two unknowns (the final velocities v'_1 and v'_2). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus v_2 = 0. Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that v_2 = 0 and use conservation of momentum. Thus,

$$p_1 = p'_1 + p'_2$$

or

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2.$$

Using conservation of internal kinetic energy and that v_2 = 0,

$$\frac{1}{2} m_1 {v_1}^2 = \frac{1}{2} m_1 {v'_1}^2 + \frac{1}{2} m_2 {v'_2}^2.$$

Solving the first equation (momentum equation) for v'_2, we obtain

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v'_2, leaving only v'_1 as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

$$v'_1 = 4.00 \text{ m/s}$$

and

$$v'_1 = -3.00 \text{ m/s}.$$

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution v'_1 = -3.00 m/s) is negative, meaning that the first object bounces backward. When this negative value of v'_1 is used to find the velocity of the second object after the collision, we get

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}}[4.00 - (-3.00 \text{ m/s})]$$

or

$$v_2' = 1.00 \text{ m/s}.$$

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

MAKING CONNECTIONS: TAKE-HOME INVESTIGATION—ICE CUBES AND ELASTIC COLLISION

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using momentum.

PHET EXPLORATIONS: COLLISION LAB

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

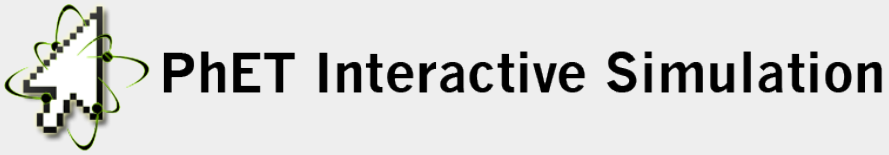


Figure 2. *Collision Lab*

Section Summary

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

Glossary

elastic collision
a collision that also conserves internal kinetic energy

internal kinetic energy
the sum of the kinetic energies of the objects in a system

Conceptual Questions

1: What is an elastic collision?

Problems & Exercises

1: Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

2: Professional Application

Two manned satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of 4.00×10^3 kg, and the second a mass of 7.50×10^3 kg. If the two satellites collide elastically rather than dock, what is their final relative velocity?

3: A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

Solutions

Problems & Exercises

2: 0.250 m/s

7.5 Inelastic Collisions in One Dimension

Summary

- Define inelastic collision.
- Explain perfectly inelastic collision.
- Apply an understanding of collisions to sports.
- Determine recoil velocity and loss in kinetic energy given mass and initial velocity.

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

INELASTIC COLLISION

An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

Figure 1 shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

PERFECTLY INELASTIC COLLISION

A collision in which the objects stick together is sometimes called “perfectly inelastic.”

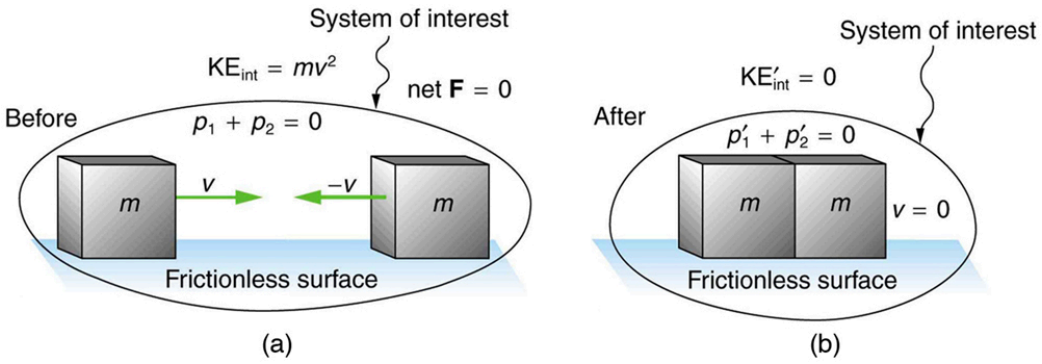


Figure 1. *An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.*

Example 1: Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie

(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision? Assume friction between the ice and the puck-goalie system is negligible. (See Figure 2)

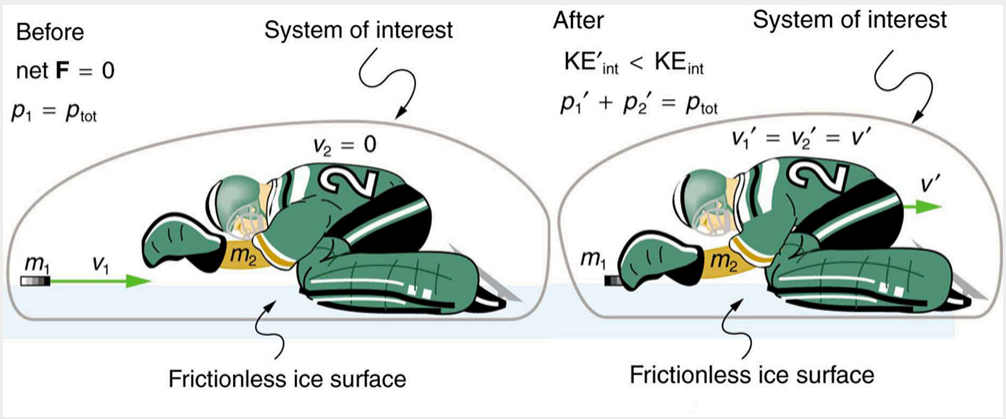


Figure 2. *An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.*

Strategy

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

Solution for (a)

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

$$p_1 + p_2 = p'_1 + p'_2$$

or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.$$

Because the goalie is initially at rest, we know $v_2 = 0$. Because the goalie catches the puck, the final velocities are equal, or $v'_1 = v'_2 = v'$. Thus, the conservation of momentum equation simplifies to

$$m_1 v_1 = (m_1 + m_2)v'.$$

Solving for v' yields

$$v' = \frac{m_1}{m_1 + m_2} v_1.$$

Entering known values in this equation, we get

$$v' = \left(\frac{0.150 \text{ kg}}{70.0 \text{ kg} + 0.150 \text{ kg}}\right)(35.0 \text{ m/s}) = 7.48 \times 10^{-2} \text{ m/s}.$$

Discussion for (a)

This recoil velocity is small and in the same direction as the puck's original velocity, as we might expect.

Solution for (b)

Before the collision, the internal kinetic energy KE_{int} of the system is that of the hockey puck, because the goalie is initially at rest. Therefore, KE_{int} is initially

$$\begin{aligned} KE_{\text{int}} &= \frac{1}{2}mv^2 = \frac{1}{2}(0.150 \text{ kg})(35.0 \text{ m/s})^2 \\ &= 91.9 \text{ J} \end{aligned}$$

After the collision, the internal kinetic energy is

$$\begin{aligned} KE\prime_{\text{int}} &= \frac{1}{2}(m + M)v^2 = \frac{1}{2}(70.15 \text{ kg})(7.48 \times 10^{-2} \text{ m/s})^2 \\ &= 0.196 \text{ J} \end{aligned}$$

The change in internal kinetic energy is thus

$$\begin{aligned} KE\prime_{\text{int}} - KE_{\text{int}} &= 0.196 \text{ J} - 91.9 \text{ J} \\ &= -91.7 \text{ J} \end{aligned}$$

where the minus sign indicates that the energy was lost.

Discussion for (b)

Nearly all of the initial internal kinetic energy is lost in this perfectly inelastic collision. KE_{int} is mostly converted to thermal energy and sound.

During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. Figure 3 shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. Example 2 deals with data from such a collision.

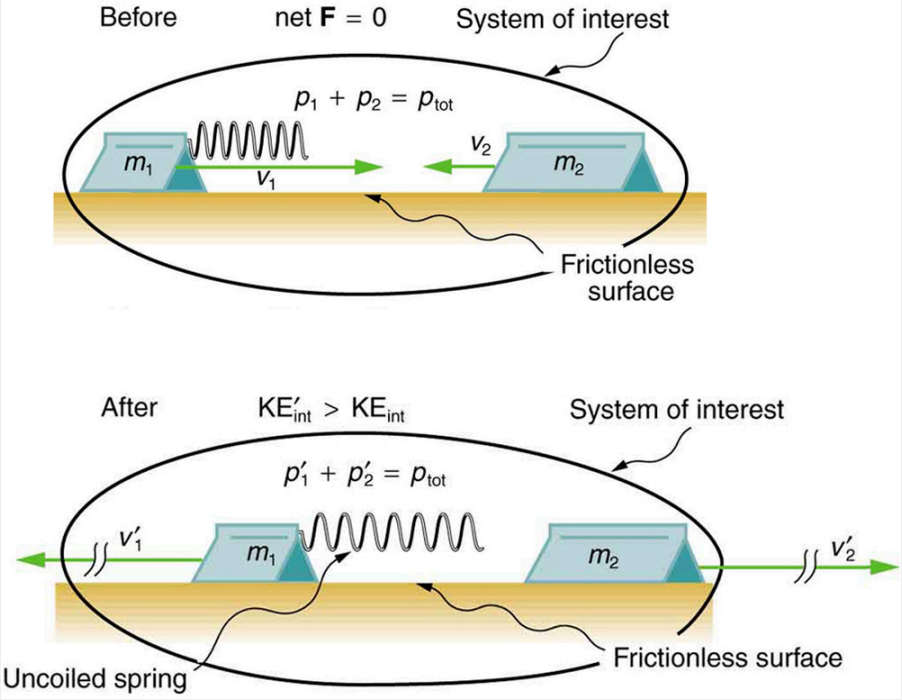


Figure 3. *An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in Example 2, the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.*

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the “sweet spot” on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

TAKE-HOME EXPERIMENT—BOUNCING OF TENNIS BALL

1. Find a racquet (a tennis, badminton, or other racquet will do). Place the racquet on the floor and stand on the handle. Drop a tennis ball on the strings from a measured height. Measure how high the ball bounces. Now ask a friend to hold the racquet firmly by the handle and drop a tennis ball from the same measured height above the racquet. Measure how high the ball bounces and observe what happens to your friend's hand during the collision. Explain your observations and measurements.
2. The coefficient of restitution (***c***) is a measure of the elasticity of a collision between a ball and an object, and is defined as the ratio of the speeds after and before the collision. A perfectly elastic collision has a ***c*** of 1. For a ball bouncing off the floor (or a racquet on the floor), ***c*** can be shown to be $c = (h/H)^{1/2}$ where h is the height to which the ball bounces and H is the height from which the ball is dropped. Determine c for the cases in Part 1 and for the case of a tennis ball bouncing off a concrete or wooden floor ($c = 0.85$ for new tennis balls used on a tennis court).

Example 2: Calculating Final Velocity and Energy Release: Two Carts Collide

In the collision pictured in Figure 3, two carts collide inelastically. Cart 1 (denoted m_1 carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00 m/s. Cart 2 (denoted m_2 in Figure 3) has a mass of 0.500 kg and an initial velocity of -0.500 m/s. After the collision, cart 1 is observed to recoil with a velocity of -4.00 m/s. (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

Strategy

We can use conservation of momentum to find the final velocity of cart 2, because $F_{\text{net}} = 0$ (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

Solution for (a)

As before, the equation for conservation of momentum in a two-object system is

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'.$$

The only unknown in this equation is v_2'. Solving for v_2' and substituting known values into the previous equation yields

$$\begin{aligned} v\prime_2 &= \frac{m_1 v_1 + m_2 v_2 - m_1 v\prime_1}{m_2} \\ &= \frac{(0.350\ \text{kg})(2.00\ \text{m/s}) + (0.500\ \text{kg})(-0.500\ \text{m/s})}{0.500\ \text{kg}} - \frac{(0.350\ \text{kg})(-4.00\ \text{m/s})}{0.500\ \text{kg}} \\ &= 3.70\ \text{m/s}. \end{aligned}$$

Solution for (b)

The internal kinetic energy before the collision is

$$\begin{aligned} KE_{\text{int}} &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\ &= \frac{1}{2}(0.350\ \text{kg})(2.00\ \text{m/s})^2 + \frac{1}{2}(0.500\ \text{kg})(-0.500\ \text{m/s})^2 \\ &= 0.763\ \text{J} \end{aligned}$$

After the collision, the internal kinetic energy is

$$\begin{aligned} KE\prime_{\text{int}} &= \frac{1}{2} m_1 v\prime_1^2 + \frac{1}{2} m_2 v\prime_2^2 \\ &= \frac{1}{2}(0.350\ \text{kg})(-4.00\ \text{m/s})^2 + \frac{1}{2}(0.500\ \text{kg})(3.70\ \text{m/s})^2 \\ &= 6.22\ \text{J}. \end{aligned}$$

The change in internal kinetic energy is thus

$$\begin{aligned} \text{KE}'_{\text{int}} - \text{KE}_{\text{int}} &= 6.22\ \text{J} - 0.763\ \text{J} \\ &= 5.46\ \text{J}. \end{aligned}$$

Discussion

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

Section Summary

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.
- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Glossary

inelastic collision
a collision in which internal kinetic energy is not conserved

perfectly inelastic collision
a collision in which the colliding objects stick together

Conceptual Questions

1: What is an inelastic collision? What is a perfectly inelastic collision?

2: Mixed-pair ice skaters performing in a show are standing motionless at arms length just before starting a routine. They reach out, clasp hands, and pull themselves together by only using their arms. Assuming there is no friction between the blades of their skates and the ice, what is their velocity after their bodies meet?

3: A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

Problems & Exercises

1: A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

2: During an ice show, a 60.0-kg skater leaps into the air and is caught by an initially stationary 75.0-kg skater. (a) What is their final velocity assuming negligible friction and that the 60.0-kg skater's original horizontal velocity is 4.00 m/s? (b) How much kinetic energy is lost?

3: Professional Application

Using mass and speed data from Chapter 7.1 Example 1 and assuming that the football player catches the ball with his feet off the ground with both of them moving horizontally, calculate: (a) the final velocity if the ball and player are going in the same direction and (b) the loss of kinetic energy in this case. (c) Repeat parts (a) and (b) for the situation in which the ball and the player are going in opposite directions. Might the loss of kinetic energy be related to how much it hurts to catch the pass?

4: A battleship that is 6.00×10^7 kg and is originally at rest fires a 1100-kg artillery shell horizontally with a velocity of 575 m/s. (a) If the shell is fired straight aft (toward the rear of the ship), there will be negligible friction opposing the ship's recoil. Calculate its recoil velocity. (b) Calculate the increase in internal kinetic energy (that is, for the ship and the shell). This energy is less than the energy released by the gun powder—significant heat transfer occurs.

5: Professional Application

Two manned satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of 4.00×10^3 kg, and the second a mass of 7.50×10^3 kg. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

6: Professional Application

A 30,000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110,000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?

7: Professional Application

Space probes may be separated from their launchers by exploding bolts. (They bolt away from one another.) Suppose a 4800-kg satellite uses this method to separate from the 1500-kg remains of its launcher, and that 5000 J of kinetic energy is supplied to the two parts. What are their subsequent velocities using the frame of reference in which they were at rest before separation?

8: A 0.0250-kg bullet is accelerated from rest to a speed of 550 m/s in a 3.00-kg rifle. The pain of the rifle's kick is much worse if you hold the gun loosely a few centimeters from your shoulder rather than holding it tightly against your shoulder. (a) Calculate the recoil velocity of the rifle if it is held loosely away from the shoulder. (b) How much kinetic energy does the rifle gain? (c) What is the recoil velocity if the rifle is held tightly against the shoulder, making the effective mass 28.0 kg? (d) How much kinetic energy is transferred to the rifle-shoulder combination? The pain is related to the amount of kinetic energy, which is significantly less in this latter situation. (e) Calculate the momentum of a 110-kg football player running at 8.00 m/s. Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s. Discuss its relationship to this problem.

9: Professional Application

One of the waste products of a nuclear reactor is plutonium-239 (^{239} Pu).] This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus (^{4}He + ^{235}U), the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is 8.40×10^{-13} J and is entirely converted to kinetic energy of the helium and uranium nuclei. The mass of the helium nucleus is 6.68×10^{-27} kg, while that of the uranium is 3.92×10^{-25} kg (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

10: Professional Application

The Moon's craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of 5.00×10^{12} kg (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is 7.36×10^{22} kg) ? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon. (c) In October 2009, NASA crashed a rocket into the Moon, and analyzed the plume produced by the impact. (Significant amounts of water were detected.) Answer part (a) and (b) for this real-life experiment. The mass of the rocket was 2000 kg and its speed upon impact was 9000 km/h. How does the plume produced alter these results?

11: Professional Application

Two football players collide head-on in midair while trying to catch a thrown football. The first player is 95.0 kg and has an initial velocity of 6.00 m/s, while the second player is 115 kg and has an initial velocity of –3.50 m/s. What is their velocity just after impact if they cling together?

12: What is the speed of a garbage truck that is 1.20×10^4 kg and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

13: During a circus act, an elderly performer thrills the crowd by catching a cannon ball shot at him. The cannon ball has a mass of 10.0 kg and the horizontal component of its velocity is 8.00 m/s when the 65.0-kg performer catches it. If the performer is on nearly frictionless roller skates, what is his recoil velocity?

14: (a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown's ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

Solutions

Problems & Exercises

1: (a) 86.4 N perpendicularly away from the bumper (b) 0.389 J (c) 64.0%

3: (a) 8.06 m/s (b) -56.0 J (c)(i) 7.88 m/s (ii) -223 J

5: (a) 0.163 m/s in the direction of motion of the more massive satellite (b) 81.6 J (c) 8.70×10^{-2} m/s in the direction of motion of the less massive satellite, 81.5 J. Because there are no external forces, the velocity of the center of mass of the two-satellite system is unchanged by the collision. The two velocities calculated above are the velocity of the center of mass in each of the two different individual reference frames. The loss in KE is the same in both reference frames because the KE lost to internal forces (heat, friction, etc.) is the same regardless of the coordinate system chosen.

7: 0.704 m/s and -2.25 m/s

8: (a) 4.58 m/s away from the bullet (b) 31.5 J (c) -0.491 m/s (d) 3.38 J

10: (a) 1.02×10^{-6} m/s (b) 5.63×10^{20} J (almost all KE lost) (c) Recoil speed is 6.79×10^{-17} m/s, energy lost is 6.25×10^{9} J. The plume will not affect the momentum result because the plume is still part of the Moon system. The plume may affect the kinetic energy result because a significant part of the initial kinetic energy may be transferred to the kinetic energy of the plume particles.

12: 24.8 m/s

14: (a) 4.00 kg (b) 210 J (c) The clown does work to throw the barbell, so the kinetic energy comes from the muscles of the clown. The muscles convert the chemical potential energy of ATP into kinetic energy.

7.6 Collisions of Point Masses in Two Dimensions

Summary

- Discuss two dimensional collisions as an extension of one dimensional analysis.
- Define point masses.
- Derive an expression for conservation of momentum along x-axis and y-axis.
- Describe elastic collisions of two objects with equal mass.
- Determine the magnitude and direction of the final velocity given initial velocity, and scattering angle.

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of **point masses**—that is, structureless particles that cannot rotate or spin.

We start by assuming that $\vec{\mathrm{F}}_{\mathrm{net}} = 0$, so that momentum $\vec{\mathrm{p}}$ is conserved. The simplest collision is one in which one of the particles is initially at rest. (See Figure 1.) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in Figure 1. Because momentum is conserved, the components of momentum along the *x*– and *y*-axes (p_{x} and p_{y}) will also be conserved, but with the chosen coordinate system, p_{y} is initially zero and p_{x} is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)

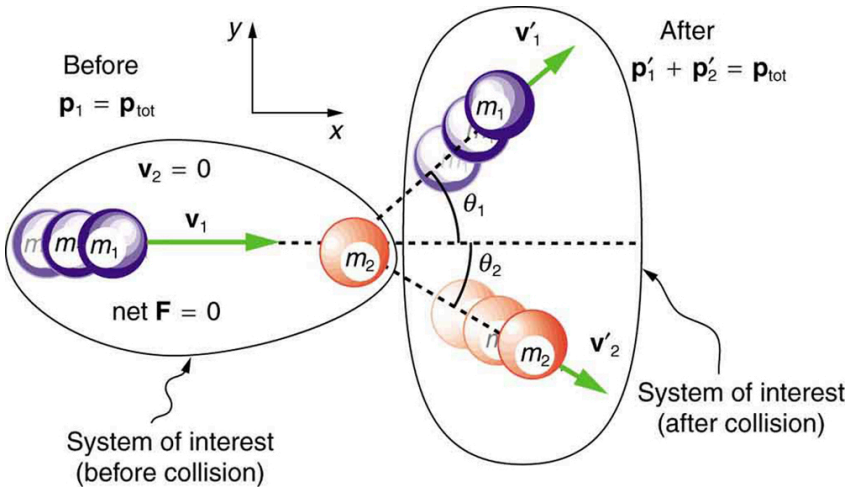


Figure 1. *A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the* ***x****-axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.*

Along the *x*-axis, the equation for conservation of momentum is

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}.$$

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}.$$

But because particle 2 is initially at rest, this equation becomes

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}.$$

The components of the velocities along the *x*-axis have the form $v \cos \theta$. Because particle 1 initially moves along the *x*-axis, we find $v_{1x} = v_1$.

Conservation of momentum along the *x*-axis gives the following equation:

$$m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2,$$

where θ_1 and θ_2 are as shown in Figure 1.

CONSERVATION OF MOMENTUM ALONG THE *x*-AXIS

$$m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2,$$

Along the *y*-axis, the equation for conservation of momentum is

$$p_{1y} + p_{2y} = p'_{1y} + p'_{2y}$$

or

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}.$$

But v_{1y} is zero, because particle 1 initially moves along the *x*-axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the *y*-axis becomes

$$0 = m_1 v'_{1y} + m_2 v'_{2y}.$$

The components of the velocities along the *y*-axis have the form $v \sin \theta$.

Thus, conservation of momentum along the *y*-axis gives the following equation:

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2.$$

CONSERVATION OF MOMENTUM ALONG THE *y*-AXIS

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$$

The equations of conservation of momentum along the *x*-axis and *y*-axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

Example 1: Determining the Final Velocity of an Unseen Object from the Scattering of Another Object

Suppose the following experiment is performed. A 0.250-kg object (m_1) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg (m_2). The 0.250-kg object emerges from the room at an angle of 45.0° with its incoming direction, as shown in Figure 2. The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity (v'_2 and θ_2) of the 0.400-kg object after the collision.

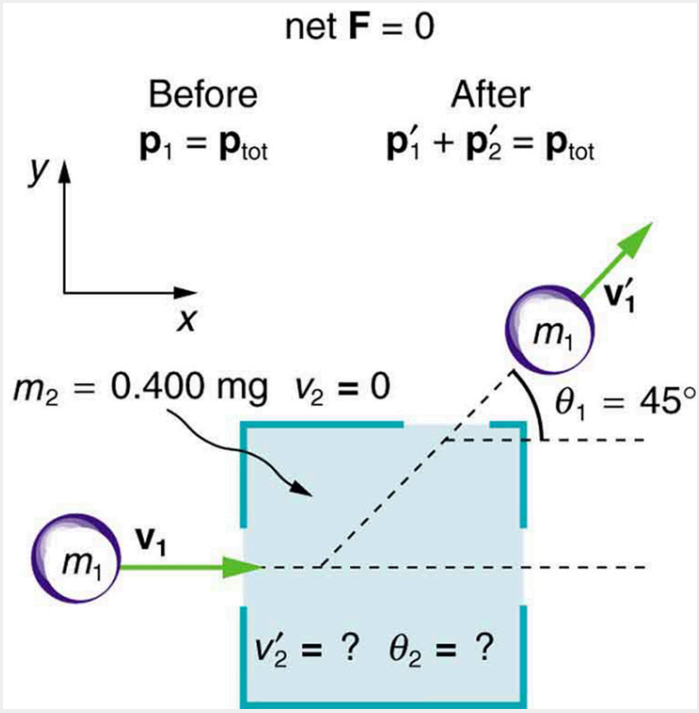


Figure 2. *A collision taking place in a dark room. The incoming object m_1 is scattered by an initially stationary object. Only the stationary object's mass m_2 is known. By measuring the angle and speed at which m_1 emerges from the room, it is possible to calculate the magnitude and direction of the initially stationary object's velocity after the collision.*

Strategy

Momentum is conserved because the surface is frictionless. The coordinate system shown in Figure 2 is one in which m_2 is originally at rest and the initial velocity is parallel to the x-axis, so that conservation of momentum along the x– and y-axes is applicable.

Everything is known in these equations except v'_2 and θ_2, which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the x– and y– directions.

Solution

Solving $m_1v_1 = m_1v'_1 \cos\theta_1 + m_2v'_2 \cos\theta_2$ for $v'_2 \cos\theta_2$ and $0 = m_1v'_1 \sin\theta_1 + m_2v'_2 \sin\theta_2$ for $v'_2 \sin\theta_2$ and taking the ratio yields an equation (in which θ_2 is the only unknown quantity. Applying the identity $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$, we obtain:

$$\tan\theta_2 = \frac{v'_1 \sin\theta_1}{v'_1 \cos\theta_1 - v_1}.$$

Entering known values into the previous equation gives

$$\tan\theta_2 = \frac{(1.50\ \text{m/s})(0.7071)}{(1.50\ \text{m/s})(0.7071) - 2.00\ \text{m/s}} = -1.129.$$

Thus,

$$\theta_2 = \tan^{-1} -1.129 = 311.5^0 \approx 312^0.$$

Angles are defined as positive in the counter clockwise direction, so this angle indicates that m_2 is scattered to the right in Figure 2, as expected (this angle is in the fourth quadrant). Either equation for the *x*– or *y*-axis can now be used to solve for v'_2, but the latter equation is easiest because it has fewer terms.

$$v'_2 = -\frac{m_1}{m_2} v'_1 \frac{\sin\theta_1}{\sin\theta_2}$$

Entering known values into this equation gives

$$v'_2 = -(\frac{0.250\ \text{kg}}{0.400\ \text{kg}})1.50\ \text{m/s} - (\frac{0.7071}{-0.7485}).$$

Thus,

$$v'_2 = 0.886\ \text{m/s}.$$

Discussion

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.

Optional: Elastic Collisions in 2D for Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool). (Refer to Figure 1 for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 (m_2) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

$$\frac{1}{2}mv_1^2 = \frac{1}{2}m{v'_1}^2 + \frac{1}{2}m{v'_2}^2.$$

Because the masses are equal, $m_1 = m_2 = m$. Algebraic manipulation (left to the reader) of conservation of momentum in the *x*– and *y*-directions can show that

$$\frac{1}{2}mv_1^2 = \frac{1}{2}m{v'_1}^2 + \frac{1}{2}m{v'_2}^2 + mv'_1v'_2(\cos\theta_1 - \theta_2).$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

$$mv'_1 v'_2 \cos(\theta_1 - \theta_2) = 0.$$

There are three ways that this term can be zero. They are

- $v'_1 = 0$ head-on collision; incoming ball stops
- $v'_2 = 0$ no collision; incoming ball continues unaffected
- $\cos(\theta_1 - \theta_2) = 0$ angle of separation $(\theta_1 - \theta_2)$ is 90° after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to 90° after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called *angular momentum*, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

Section Summary

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the *x*-axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the *x*-axis), stated by $m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2$ and along the direction perpendicular to the initial direction (the *y*-axis) stated by $0 = m_1 v'_{1y} + m_2 v'_{2y}$.
- The internal kinetic before and after the collision of two objects that have equal masses is
$$\frac{1}{2}mv_1^2 = \frac{1}{2}m{v'_1}^2 + \frac{1}{2}m{v'_2}^2 + mv'_1 v'_2 \cos(\theta_1 - \theta_2).$$
- Point masses are structureless particles that cannot spin.

Glossary

point masses
structureless particles with no rotation or spin

Problems & Exercises

1: Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of 30.0°, what is the velocity (magnitude and direction)

of the second puck? (You may use the result that $\theta_1 - \theta_2 = 90°$ for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

2: Confirm that the results of the example Example 1 do conserve momentum in both the *x*– and *y*-directions.

3: A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of 20.0° above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

4: Professional Application

A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 85.0° to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic? (c) Linear kinetic energy is greater after the collision. Discuss how spin on the ball might be converted to linear kinetic energy in the collision.

5: Professional Application

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei (^{4}He) from gold-197 nuclei (^{197}Au).The energy of the incoming helium nucleus was 8.00×10^{-13} J, and the masses of the helium and gold nuclei were 6.68×10^{-27} kg and 3.29×10^{-25} kg, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of 120° during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

6: Professional Application

Two cars collide at an icy intersection and stick together afterward. The first car has a mass of 1200 kg and is approaching at 8.00 m/s due south. The second car has a mass of 850 kg and is approaching at 17.0 m/s due west. (a) Calculate the final velocity (magnitude and direction) of the cars. (b) How much kinetic energy is lost in the collision? (This energy goes into deformation of the cars.) Note that because both cars have an initial velocity, you cannot use the equations for conservation of momentum along the *x*-axis and *y*-axis; instead, you must look for other simplifying aspects.

7: Starting with equations $m_1v_1 = m_1v'_1 \cos\theta_1 + m_2v'_2 \cos\theta_2$ and $0 = m_1v'_1 \sin\theta_1 + m_2v'_2 \sin\theta_2$ for conservation of momentum in the *x*– and *y*-directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + mv'_1v'_2\cos(\theta_1 - \theta_2)$$

as discussed in the text.

8: Integrated Concepts

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

Solutions

Problems & Exercises

1: (a) 3.00 m/s and 60 degrees below *x*-axis (b) Find speed of first puck after collision:

$$0 = mv_1' \sin 30^0 - mv_2' \sin 60^0 \Rightarrow v_1' = v_2' \frac{\sin 60^0}{\sin 30^0} = 5.196 \text{ m/s}$$

Verify that ratio of initial to final KE equals one:

$$\begin{aligned} \text{KE} &= \frac{1}{2}mv_1^2 = 18 \text{ mJ} \\ &= \frac{1}{2}mv\prime_1^2 + \frac{1}{2}mv\prime_2^2 = 18 \text{ mJ} \\ \frac{KE}{KE\prime} &= 1.00 \end{aligned}$$

3: (a) -2.26 m/s (b) 7.63 x 10^3 J (c) The ground will exert a normal force to oppose recoil of the cannon in the vertical direction. The momentum in the vertical direction is transferred to the earth. The energy is transferred into the ground, making a dent where the cannon is. After long barrages, cannon have erratic aim because the ground is full of divots.

5: (a) 5.36 x 10^5 m/s at -29.5^o (b) 7.52 x 10^{-13} J

6. 8.46 m/s at an angle of 33.6 degrees south of west.

total energy before = 161000 J total kinetic energy after = 73400 J, so 87900 J was lost in the collision.

7: We are given that $m_1 = m_2 \equiv m$. The given equations then become:

$$v_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2$$

and

$$0 = v_1' \sin\theta_1 + v_2' \sin\theta_2 .$$

Square each equation to get

\begin{align*} v_1^2&=v\prime_1^2 \cos^2 \theta_1 + v\prime_2^2 \cos^2 \theta_2 + 2v\prime_1v\prime_2 \cos \theta_1 \cos \theta_2 \\ 0&=v\prime_1^2 \sin^2 \theta_1 + v\prime_2^2 \sin^2 \theta_2 + 2v\prime_1v\prime_2 \sin \theta_1 \sin \theta_2 \end{align*}

Add these two equations and simplify:

$$\begin{aligned} v_1^2 &= v\prime_1^2 + v\prime_2^2 + 2v\prime_1 v\prime_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \\ &= v\prime_1^2 + v\prime_2^2 + 2v\prime_1 v\prime_2 (\frac{1}{2}\cos(\theta_1 - \theta_2) + \frac{1}{2}\cos(\theta_1 + \theta_2) + \frac{1}{2}\cos(\theta_1 - \theta_2) - \frac{1}{2}\cos(\theta_1 + \theta_2)) \\ &= v\prime_1^2 + v\prime_2^2 + 2v\prime_1 v\prime_2 \cos(\theta_1 - \theta_2) \end{aligned}$$

Multiply the entire equation by $\frac{1}{2}m$ to recover the kinetic energy:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}m{v_1'}^2 + \frac{1}{2}m{v_2'}^2 + mv_1'v_2'\cos(\theta_1 - \theta_2)$$

8: Player recoil velocity = 0.075 m/s Assuming constant velocity, the puck takes a time of = 15.0 m / 45.0 m/s = 0.333 seconds. The player travels a distance backward = 0.075 m/s x 0.333 s = 0.025 m

Chapter 8 Statics and Torque

Figure 1. *On a short time scale, rocks like these in Australia's Kings Canyon are static, or motionless relative to the Earth. (credit: freeaussiestock.com)*

What might desks, bridges, buildings, trees, and mountains have in common—at least in the eyes of a physicist? The answer is that they are ordinarily motionless relative to the Earth. Furthermore, their acceleration is zero because they remain motionless. That means they also have something in common with a car moving at a constant velocity, because anything with a constant velocity also has an acceleration of zero. Now, the important part—Newton's second law states that net $\vec{F} = m\vec{a}$, and so the net external force is zero for all stationary objects and for all objects moving at constant velocity. There are forces acting, but they are balanced. That is, they are in *equilibrium*.

STATICS

Statics is the study of forces in equilibrium, a large group of situations that makes up a special case of Newton's second law. We have already considered a few such situations; in this chapter, we cover the topic more thoroughly, including consideration of such possible effects as the rotation and deformation of an object by the forces acting on it.

How can we guarantee that a body is in equilibrium and what can we learn from systems that are in equilibrium? There are actually two conditions that must be satisfied to achieve equilibrium. These conditions are the topics of the first two sections of this chapter.

8.1 The First Condition for Equilibrium

Summary

- State the first condition of equilibrium.
- Explain static equilibrium.
- Explain dynamic equilibrium.

The first condition necessary to achieve equilibrium is the one already mentioned: the net external force on the system must be zero. Expressed as an equation, this is simply

$$\text{net } \vec{\text{F}} = 0$$

Note that if net *F* is zero, then the net external force in *any* direction is zero. For example, the net external forces along the typical *x*– and *y*-axes are zero. This is written as

$$\text{net } F_x = 0 \text{ and } F_y = 0$$

Figure 1 and Figure 2 illustrate situations where net *F* = 0 for both static equilibrium (motionless), and dynamic equilibrium (constant velocity).

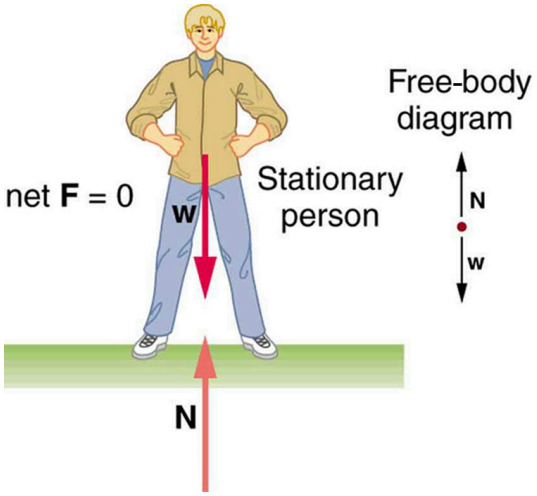


Figure 1. *This motionless person is in static equilibrium. The forces acting on him add up to zero. Both forces are vertical in this case.*

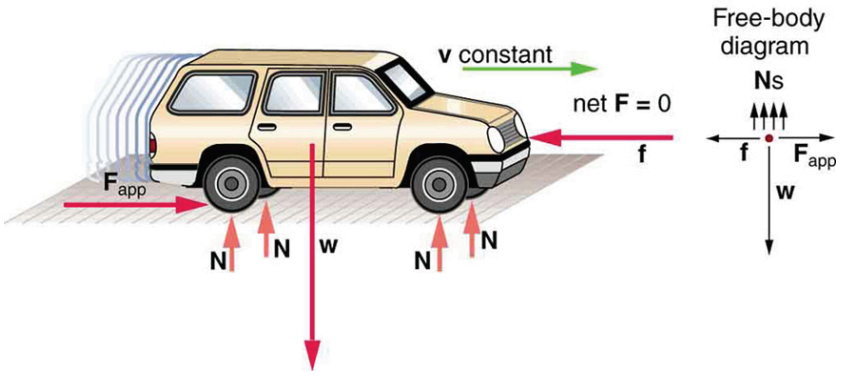


Figure 2. *This car is in dynamic equilibrium because it is moving at constant velocity. There are horizontal and vertical forces, but the net external force in any direction is zero. The applied force F_{app} between the tires and the road is balanced by air friction, and the weight of the car is supported by the normal forces, here shown to be equal for all four tires.*

However, it is not sufficient for the net external force of a system to be zero for a system to be in equilibrium. Consider the two situations illustrated in Figure 3 and Figure 4 where forces are applied to an ice hockey stick lying flat on ice. The net external force is zero in both situations shown in the figure; but in one case, equilibrium is achieved, whereas in the other, it is not. In Figure 3, the ice hockey stick remains motionless. But in Figure 4, with the same forces applied in different places, the stick experiences accelerated rotation. Therefore, we know that the point at which a force is applied is another factor in determining whether or not equilibrium is achieved. This will be explored further in the next section.

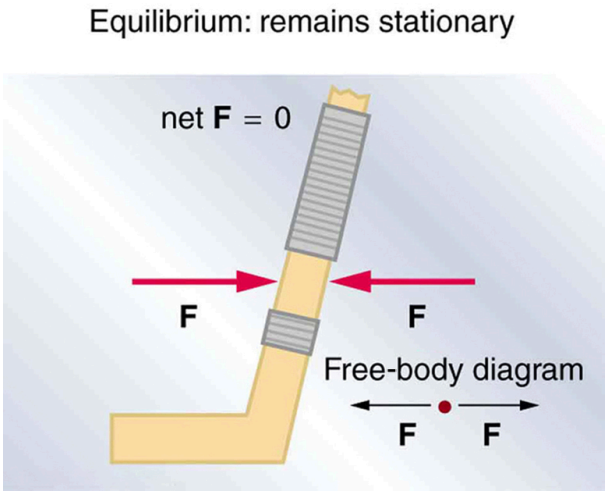


Figure 3. *An ice hockey stick lying flat on ice with two equal and opposite horizontal forces applied to it. Friction is negligible, and the gravitational force is balanced by the support of the ice (a normal force). Thus, net F = 0. Equilibrium is achieved, which is static equilibrium in this case.*

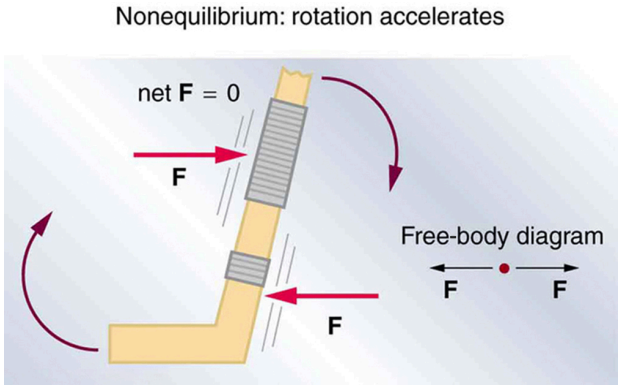


Figure 4. *The same forces are applied at other points and the stick rotates—in fact, it experiences an accelerated rotation. Here net F = 0 but the system is not at equilibrium. Hence, the net F = 0 is a necessary—but not sufficient—condition for achieving equilibrium.*

PHET EXPLORATIONS: TORQUE

Investigate how torque causes an object to rotate. Discover the relationships between angular acceleration, moment of inertia, angular momentum and torque.

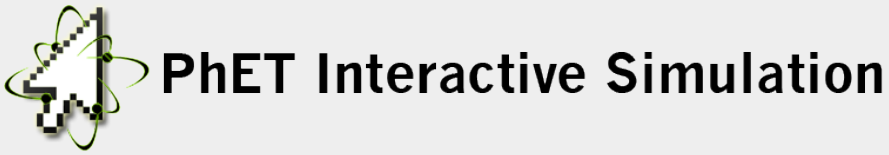


Figure 5. *Torque*

Section Summary

- Statics is the study of forces in equilibrium.
- Two conditions must be met to achieve equilibrium, which is defined to be motion without linear or rotational acceleration.
- The first condition necessary to achieve equilibrium is that the net external force on the system must be zero, so that net $F = 0$.

Glossary

static equilibrium
a state of equilibrium in which the net external force and torque acting on a system is zero

dynamic equilibrium
a state of equilibrium in which the net external force and torque on a system moving with constant velocity are zero

Conceptual Questions

1: What can you say about the velocity of a moving body that is in dynamic equilibrium? Draw a sketch of such a body using clearly labeled arrows to represent all external forces on the body.

2: Under what conditions can a rotating body be in equilibrium? Give an example.

8.2 The Second Condition for Equilibrium

Summary

- State the second condition that is necessary to achieve equilibrium.
- Explain torque and the factors on which it depends.
- Describe the role of torque in rotational mechanics

TORQUE

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity. A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it. To understand what factors affect rotation, let us think about what happens when you open an ordinary door by rotating it on its hinges.

Several familiar factors determine how effective you are in opening the door. See Figure 1. First of all, the larger the force, the more effective it is in opening the door—obviously, the harder you push, the more rapidly the door opens. Also, the point at which you push is crucial. If you apply your force too close to the hinges, the door will open slowly, if at all. Most people have been embarrassed by making this mistake and bumping up against a door when it did not open as quickly as expected. Finally, the direction in which you push is also important. The most effective direction is perpendicular to the door—we push in this direction almost instinctively.

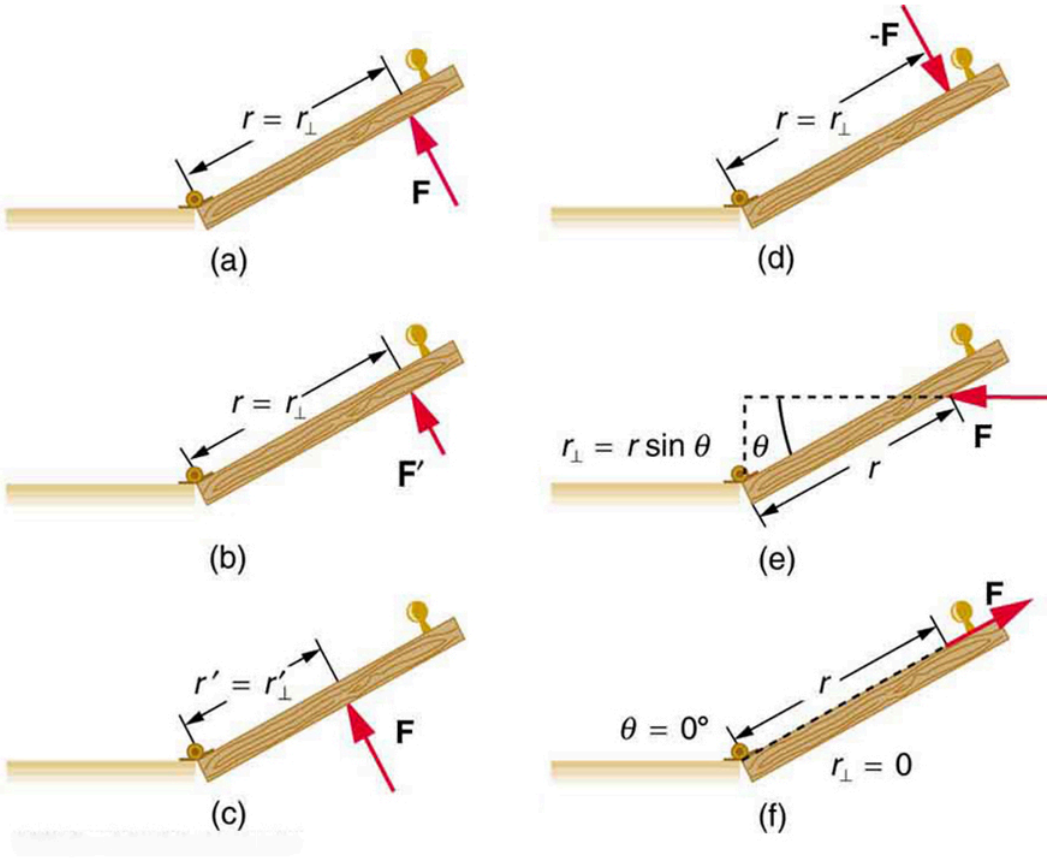


Figure 1. *Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) Counterclockwise torque is produced by this force, which means that the door will rotate in a counterclockwise due to F. Note that $r_\perp$ is the perpendicular distance of the pivot from the line of action of the force. (b) A smaller counterclockwise torque is produced by a smaller force F' acting at the same distance from the hinges (the pivot point). (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) The same force as in (a), but acting in the opposite direction, produces a clockwise torque. (e) A smaller counterclockwise torque is produced by the same magnitude force acting at the same point but in a different direction. Here, θ is less than 90º. (f) Torque is zero here since the force just pulls on the hinges, producing no rotation. In this case, θ = 0º.*

The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called torque. **Torque** is the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time). In equation form, the magnitude of torque is defined to be

$$\tau = rF \sin \theta$$

where τ (the Greek letter tau) is the symbol for torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between the force and the vector directed from the point of application to the pivot point, as seen in Figure 1 and Figure 2. An alternative expression for torque is given in terms of the **perpendicular lever arm** $r_\perp$ as shown in Figure 1 and Figure 2, which is defined as

$$r_\perp = r \sin \theta$$

so that

$$\tau = r_{\perp} F.$$

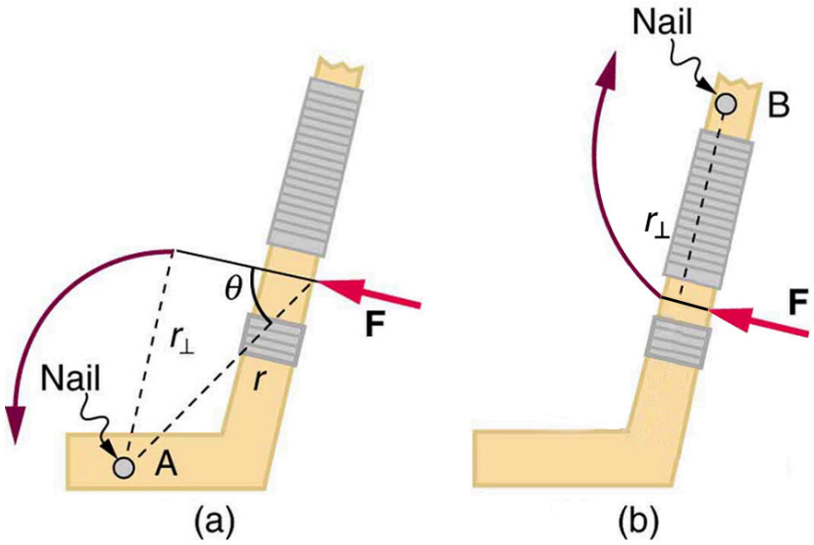


Figure 2. *A force applied to an object can produce a torque, which depends on the location of the pivot point. (a) The three factors r, F, and θ for pivot point A on a body are shown here—r is the distance from the chosen pivot point to the point where the force F is applied, and θ is the angle between F and the vector directed from the point of application to the pivot point. If the object can rotate around point A, it will rotate counterclockwise. This means that torque is counterclockwise relative to pivot A. (b) In this case, point B is the pivot point. The torque from the applied force will cause a clockwise rotation around point B, and so it is a clockwise torque relative to B.*

The perpendicular lever arm $r_\perp$ is the shortest distance from the pivot point to the line along which $\vec{\mathrm{F}}$ acts; it is shown as a dashed line in Figure 1 and Figure 2. This line is know was the **line of action** of the force. Note that the line segment that defines the distance $r_\perp$ is perpendicular to $\vec{\mathrm{F}}$, as its name implies. It is sometimes easier to find or visualize $r_\perp$ than to find both r and θ. In such cases, it may be more convenient to use $\tau = r_\perp F$ rather than $\tau = rF \sin\theta$ for torque, but both are equally valid.

The **SI unit of torque** is newtons times meters, usually written as N·m. For example, if you push perpendicular to the door with a force of 40 N at a distance of 0.800 m from the hinges, you exert a torque of 32 N·m (0.800 m × 40 N × sin 90°) relative to the hinges. If you reduce the force to 20 N, the torque is reduced to 16 N·m, and so on.

The torque is always calculated with reference to some chosen pivot point. For the same applied force, a different choice for the location of the pivot will give you a different value for the torque, since both r and θ depend on the location of the pivot. Any point in any object can be chosen to calculate the torque about that point. The object may not actually pivot about the chosen “pivot point.”

For rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point, as illustrated for points B and A, respectively, in Figure 2. If the object can rotate about point A, it will rotate counterclockwise, which means that the torque for the force is shown as counterclockwise relative to A. But if the object can rotate about point

B, it will rotate clockwise, which means the torque for the force shown is clockwise relative to B. Also, the magnitude of the torque is greater when the lever arm is longer.

Now, *the second condition necessary to achieve equilibrium* is that *the net external torque on a system must be zero*. An external torque is one that is created by an external force. You can choose the point around which the torque is calculated. The point can be the physical pivot point of a system or any other point in space—but it must be the same point for all torques. If the second condition (net external torque on a system is zero) is satisfied for one choice of pivot point, it will also hold true for any other choice of pivot point in or out of the system of interest. (This is true only in an inertial frame of reference.) The second condition necessary to achieve equilibrium is stated in equation form as

$$\text{net } \tau = 0$$

where net means total. Torques, which are in opposite directions are assigned opposite signs. A common convention is to call counterclockwise (ccw) torques positive and clockwise (cw) torques negative.

Center of Mass and Center of Gravity

In the previous chapter on Force and Newton's Laws, we made the assumption that all objects were point masses. Clearly, that assumption is no longer valid in problems involving torques and extended objects (i.e., objects that are not considered point masses). As we have seen above, applying a force typically results in a torque. However, if a force is applied to the **center of mass** of an object (or the line of action of the force runs through the center of mass), the force will only result in linear motion (i.e., the force will not result in any torque). For objects with a uniform mass density, the center of mass is the geometric center of the object. The center of mass is a hypothetical point where it can be assumed all the mass of the object is concentrated to help visualize the linear motion of the object.

A closely related concept is the **center of gravity** of an object, which is the point where it is assumed that the gravitational force acts on the object. *For objects where the acceleration due to gravity does not change over the size of the object, the center of gravity is located at the same point as the center of mass* (this will be true for all problems you encounter in this book). Depending on the location of the center of gravity in relation to the pivot point, it is possible for the gravitational force to exert a torque. For example, in Figure 3a the center of gravity of the pencil is directly above the pivot point, so it does not exert a force. On the other hand, in Figure 3b the gravitational force exerts a counter clockwise torque. However, in both cases the net force on the pencil is zero.

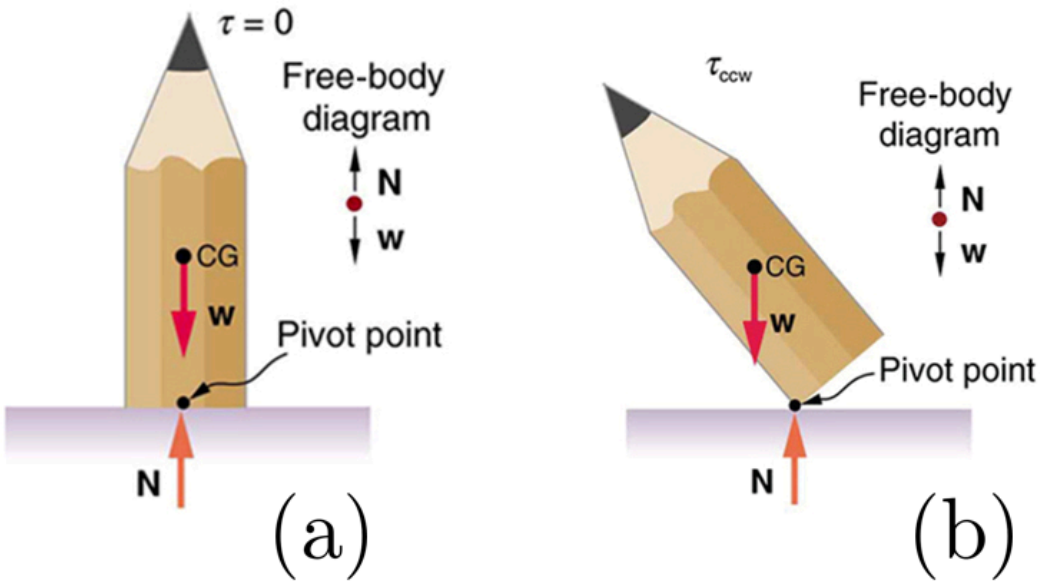


Figure 3. *In both cases, the net force on the pencil is zero. In Figure 3a the center of gravity of the pencil is directly above the pivot point, so it does not exert a torque. In this case, the pencil is in equilibrium. In Figure 3b the gravitational force exerts a counter clockwise torque so the pencil is not in equilibrium.*

Example 1: She Saw Torques On A Seesaw

When two children balance a seesaw as shown in Figure 4, they satisfy the two conditions for equilibrium. Most people have perfect intuition about seesaws, knowing that the lighter child must sit farther from the pivot and that a heavier child can keep a lighter one off the ground indefinitely. The two children shown in Figure 4 are balanced on a seesaw of negligible mass. (This assumption is made to keep the example simple—more involved examples will follow.) The first child has a mass of 26.0 kg and sits 1.60 m from the pivot.(a) If the second child has a mass of 32.0 kg, how far is she from the pivot? (b) What is F_p, the supporting force exerted by the pivot?

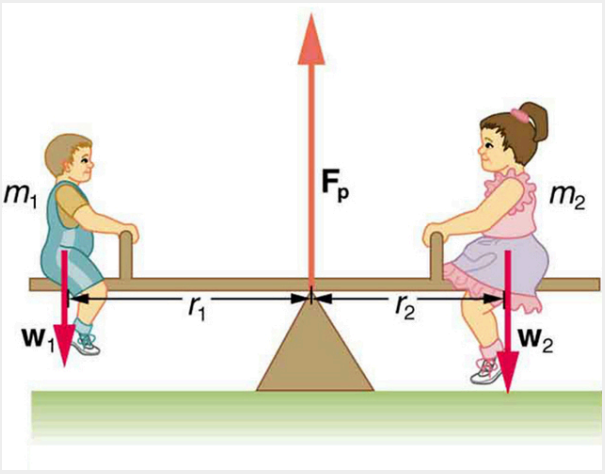


Figure 4. *Two children balancing a seesaw satisfy both conditions for equilibrium. The lighter child sits farther from the pivot to create a torque equal in magnitude to that of the heavier child.*

Strategy

Both conditions for equilibrium must be satisfied. In part (a), we are asked for a distance; thus, the second condition (regarding torques) must be used, since the first (regarding only forces) has no distances in it. To apply the second condition for equilibrium, we first identify the system of interest to be the seesaw plus the two children. We take the supporting pivot to be the point about which the torques are calculated. We then identify all external forces acting on the system.

Solution (a)

The three external forces acting on the system are the weights of the two children and the supporting force of the pivot. Let us examine the torque produced by each. Torque is defined to be

$$\tau = rF \sin \theta.$$

Here $\theta = 90°$, so that $\sin \theta = 1$ for all three forces. That means $r_\perp = r$ for all three. The torques exerted by the three forces are first,

$$\tau_1 = r_1 w_1$$

second,

$$\tau_2 = -r_2 w_2$$

and third,

$$\begin{aligned} \tau_\text{p} &= r_\text{p} F_\text{p} \\ &= 0 \cdot F_\text{p} \\ &= 0. \end{aligned}$$

Note that a minus sign has been inserted into the second equation because this torque is clockwise and is therefore negative by convention. Since F_p acts directly on the pivot point, the distance r_p is zero. A force acting on the pivot cannot cause a rotation, just as pushing directly on the hinges of a door will not cause it to rotate. Now, the second condition for equilibrium is that the sum of the torques on both children is zero. Therefore

$$\tau_2 = -\tau_1,$$

or

$$r_2 w_2 = r_1 w_1.$$

Weight is mass times the acceleration due to gravity. Entering *mg* for *w*, we get

$$r_2 m_2 g = r_1 m_1 g.$$

Solve this for the unknown r_2:

$$r_2 = r_1 \frac{m_1}{m_2}.$$

The quantities on the right side of the equation are known; thus, r_2 is

$$r_2 = (1.60 \text{ m}) \frac{26.0 \text{ kg}}{32.0 \text{ kg}} = 1.30 \text{ m}.$$

As expected, the heavier child must sit closer to the pivot (1.30 m versus 1.60 m) to balance the seesaw.

Solution (b)

This part asks for a force F_p. The easiest way to find it is to use the first condition for equilibrium, which is

$$\text{net } \vec{\text{F}} = 0.$$

The forces are all vertical, so that we are dealing with a one-dimensional problem along the vertical axis; hence, the condition can be written as

$$\text{net } F_y = 0$$

where we again call the vertical axis the *y*-axis. Choosing upward to be the positive direction, and using plus and minus signs to indicate the directions of the forces, we see that

$$F_\text{p} - w_1 - w_2 = 0.$$

This equation yields what might have been guessed at the beginning:

$$F_\text{p} = w_1 + w_2.$$

So, the pivot supplies a supporting force equal to the total weight of the system:

$$F_\text{p} = m_1 g + m_2 g.$$

Entering known values gives

$$\begin{aligned} F_\text{p} &= (26.0 \text{ kg})(9.80 \text{ m/s}^2) + (32.0 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 568 \text{ N}. \end{aligned}$$

Discussion

The two results make intuitive sense. The heavier child sits closer to the pivot. The pivot supports the weight of the two children. Part (b) can also be solved using the second condition for equilibrium, since both distances are known, but only if the pivot point is chosen to be somewhere other than the location of the seesaw's actual pivot!

Several aspects of the preceding example have broad implications. First, the choice of the pivot as the point around which torques are calculated simplified the problem. Since F_p is exerted on the pivot point,

its lever arm is zero. Hence, the torque exerted by the supporting force F_p is zero relative to that pivot point. The second condition for equilibrium holds for any choice of pivot point, and so we choose the pivot point to simplify the solution of the problem.

Second, the acceleration due to gravity canceled in this problem, and we were left with a ratio of masses. *This will not always be the case*. Always enter the correct forces—do not jump ahead to enter some ratio of masses.

Third, the weight of each child is distributed over an area of the seesaw, yet we treated the weights as if each force were exerted at a single point. This is not an approximation—the distances r_1 and r_2 are the distances to points directly below the **center of gravity** of each child. As we shall see in the next section, the mass and weight of a system can act as if they are located at a single point.

Finally, note that the concept of torque has an importance beyond static equilibrium. *Torque plays the same role in rotational motion that force plays in linear motion.* We will examine this in the next chapter.

TAKE-HOME EXPERIMENT

Take a piece of modeling clay and put it on a table, then mash a cylinder down into it so that a ruler can balance on the round side of the cylinder while everything remains still. Put a penny 8 cm away from the pivot. Where would you need to put two pennies to balance? Three pennies?

Section Summary

- The second condition assures those torques are also balanced. Torque is the rotational equivalent of a force in producing a rotation and is defined to be
$$\tau = rF \sin \theta$$
where τ is torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between F and the vector directed from the point where the force acts to the pivot point. The perpendicular lever arm $r_\perp$ is defined to be
$$r_\perp = r \sin \theta$$
so that
$$\tau = r_\perp F.$$
- The perpendicular lever arm $r_\perp$ is the shortest distance from the pivot point to the line along which F acts. The SI unit for torque is newton-meter (N·m). The second condition necessary to achieve equilibrium is that the net external torque on a system must be zero:
$$\text{net } \tau = 0$$
By convention, counterclockwise torques are positive, and clockwise torques are negative.

Glossary

torque
: turning or twisting effectiveness of a force

perpendicular lever arm
: the shortest distance from the pivot point to the line along which FF lies

SI units of torque
: newton times meters, usually written as N·m

center of mass
a hypothetical point where it can be assumed all the mass of the object is concentrated to help visualize the linear motion of the object.

center of gravity
: the point where the total weight of the body is assumed to be concentrated, which is the point where the force of gravity acts

Conceptual Questions

1: What three factors affect the torque created by a force relative to a specific pivot point?

2: A wrecking ball is being used to knock down a building. One tall unsupported concrete wall remains standing. If the wrecking ball hits the wall near the top, is the wall more likely to fall over by rotating at its base or by falling straight down? Explain your answer. How is it most likely to fall if it is struck with the same force at its base? Note that this depends on how firmly the wall is attached at its base.

3: Mechanics sometimes put a length of pipe over the handle of a wrench when trying to remove a very tight bolt. How does this help? (It is also hazardous since it can break the bolt.)

Problems & Exercises

1: (a) When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850m from the hinges. What torque are you exerting relative to the hinges? (b) Does it matter if you push at the same height as the hinges?

2: When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. (a) How much torque are you exerting in newton × meters (relative to the center of the bolt)? (b) Convert this torque to footpounds.

3: Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.

4: Use the second condition for equilibrium (net τ = 0) to calculate F_p in Example 1, employing any data given or solved for in part (a) of the example.

5: Repeat the seesaw problem in Example 1 with the center of mass of the seesaw 0.160 m to the left of the pivot (on the side of the lighter child) and assuming a mass of 12.0 kg for the seesaw. The other data given in the example remain unchanged. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium.

Solutions

Problems & Exercises

1: (a) 46.8 N·m (b) It does not matter at what height you push. The torque depends on only the magnitude of the force applied and the perpendicular distance of the force's application from the hinges. (Children don't have a tougher time opening a door because they push lower than adults, they have a tougher time because they don't push far enough from the hinges.)

3: 23.3 N

5: Given:

$$\begin{aligned} m_1 &= 26.0\ \text{kg},\ m_2 = 32.0\ \text{kg},\ m_s = 12.0\ \text{kg}, \\ r_1 &= 1.60\ \text{m},\ r_s = 0.160\ \text{m},\ \text{find (a)}r_2,\ \text{(b)}F_\text{p} \end{aligned}$$

a) Since children are balancing:

$$\begin{aligned} \text{net}\ \tau_\text{cw} &= -\text{net}\tau_\text{ccw} \\ \Rightarrow w_1 r_1 + m_\text{s} g r_\text{s} &= w_2 r_2 \end{aligned}$$

So, solving for r_2 gives:

\begin{align*}r_2 &= \dfrac{w_1r_1+m_{\text{s}}
gr_{\text{s}}}{w_2}=\dfrac{m_1gr_1+m_{\text{s}}gr_{\text{s}}}{m_2g}=\dfrac{m_1r_1+m_{\tex
t{s}} r_{\text{s}}}{m_2} \\&=
\dfrac{(26.0~\text{kg})(1.60~\text{m})+(12.0~\text{kg})(0.160~\text{m})}{32.0~\text{kg}}\\&=1.3
6~\text{m}\end{align*}

b) Since the children are not moving:

$$\begin{aligned} \text{net F} = 0 &= F_\text{p} - w_1 - w_2 - w_s \\ \Rightarrow F_\text{p} &= w_1 + w_2 + w_\text{s} \end{aligned}$$

So that

$$\begin{aligned} F_\text{p} &= (26.0\ \text{kg} + 32.0\ \text{kg} + 12.0\ \text{kg})(9.80\ \text{m/s}^2) \\ &= 686\ \text{N} \end{aligned}$$

8.3 Applications of Statics and Problem-Solving Strategies

Summary

- Discuss the applications of Statics in real life.
- State and discuss various problem-solving strategies in Statics.

Statics can be applied to a variety of situations, ranging from raising a drawbridge to bad posture and back strain. We begin with a discussion of problem-solving strategies specifically used for statics. Since statics is a special case of Newton's laws, both the general problem-solving strategies and the special strategies for Newton's laws, discussed in Chapter 4.7 Problem-Solving Strategies, still apply.

PROBLEM-SOLVING STRATEGY: STATIC EQUILIBRIUM SITUATIONS

1. The first step is to determine whether or not the system is in **static equilibrium**. This condition is always the case when the *acceleration of the system is zero and accelerated rotation does not occur*.
2. It is particularly important to *draw a free body diagram for the system of interest*. Carefully label all forces, and note their relative magnitudes, directions, and points of application whenever these are known.
3. Solve the problem by applying either or both of the conditions for equilibrium (represented by the equations net $F = 0$ and net $\tau = 0$, depending on the list of known and unknown factors. If the second condition is involved, *choose the pivot point to simplify the solution*. Any pivot point can be chosen, but the most useful ones cause torques by unknown forces to be zero. (Torque is zero if the force is applied at the pivot (then $r = 0$), or along a line through the pivot point (then $\theta = 0$). Always choose a convenient coordinate system for projecting forces.
4. *Check the solution to see if it is reasonable* by examining the magnitude, direction, and units of the answer. The importance of this last step never diminishes, although in unfamiliar applications, it is usually more difficult to judge reasonableness. These judgments become progressively easier with experience.

Now let us apply this problem-solving strategy for the pole vaulter shown in the three figures below. The pole is uniform and has a mass of 5.00 kg. In Figure 1, the pole's cg lies halfway between the vaulter's hands. It seems reasonable that the force exerted by each hand is equal to half the weight of the pole, or 24.5 N. This obviously satisfies the first condition for equilibrium (net $F = 0$). The second condition (net

$\tau = 0$) is also satisfied, as we can see by choosing the cg to be the pivot point. The weight exerts no torque about a pivot point located at the cg, since it is applied at that point and its lever arm is zero. The equal forces exerted by the hands are equidistant from the chosen pivot, and so they exert equal and opposite torques. Similar arguments hold for other systems where supporting forces are exerted symmetrically about the cg. For example, the four legs of a uniform table each support one-fourth of its weight.

In Figure 1, a pole vaulter holding a pole with its cg halfway between his hands is shown. Each hand exerts a force equal to half the weight of the pole, $F_R = F_L = w/2$. (b) The pole vaulter moves the pole to his left, and the forces that the hands exert are no longer equal. See Figure 1. If the pole is held with its cg to the left of the person, then he must push down with his right hand and up with his left. The forces he exerts are larger here because they are in opposite directions and the cg is at a long distance from either hand.

Similar observations can be made using a meter stick held at different locations along its length.

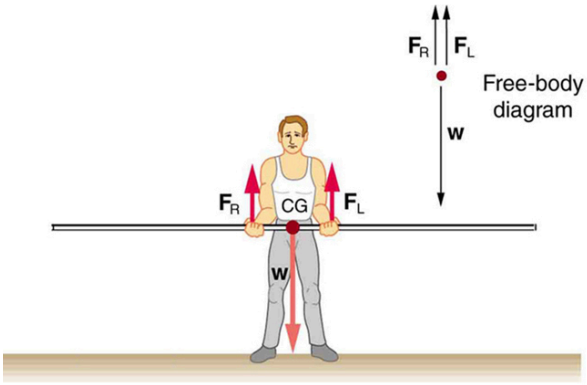


Figure 1. *A pole vaulter holds a pole horizontally with both hands.*

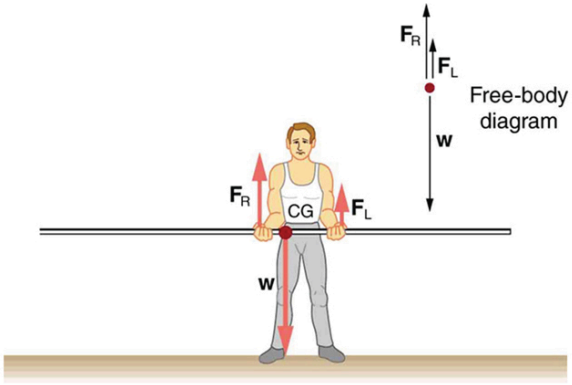


Figure 2. *A pole vaulter is holding a pole horizontally with both hands. The center of gravity is near his right hand.*

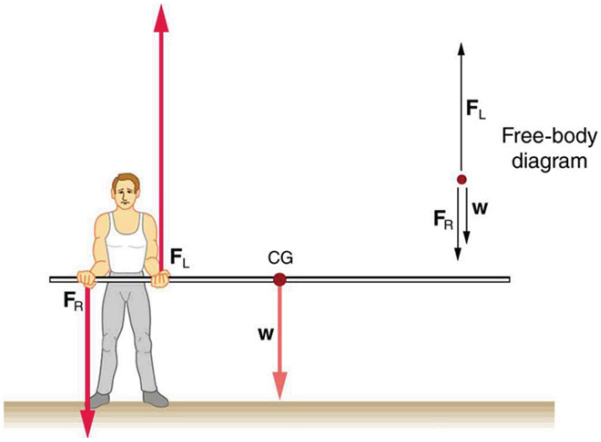


Figure 3. *A pole vaulter is holding a pole horizontally with both hands. The center of gravity is to the left side of the vaulter.*

If the pole vaulter holds the pole as shown in Figure 2, the situation is not as simple. The total force he exerts is still equal to the weight of the pole, but it is not evenly divided between his hands. (If $F_L = F_R$, then the torques about the cg would not be equal since the lever arms are different.) Logically, the right hand should support more weight, since it is closer to the cg. In fact, if the right hand is moved directly under the cg, it will support all the weight. This situation is exactly analogous to two people carrying a load; the one closer to the cg carries more of its weight. Finding the forces F_L and F_R is straightforward, as the next example shows.

If the pole vaulter holds the pole from near the end of the pole (Figure 3), the direction of the force applied by the right hand of the vaulter reverses its direction.

Example 1: What Force Is Needed to Support a Weight Held Near Its CG?

For the situation shown in Figure 2, calculate: (a) F_R, the force exerted by the right hand, and (b) F_L, the force exerted by the left hand. The hands are 0.900 m apart, and the cg of the pole is 0.600 m from the left hand.

Strategy

Figure 2 includes a free body diagram for the pole, the system of interest. There is not enough information to use the first condition for equilibrium (net $F = 0$), since two of the three forces are unknown and the hand forces cannot be assumed to be equal in this case. There is enough information to use the second condition for equilibrium (net $\tau = 0$) if the pivot point is chosen to be at either hand, thereby making the torque from that hand zero. We choose to locate the pivot at the left hand in this part of the problem, to eliminate the torque from the left hand.

Solution for (a)

There are now only two nonzero torques, those from the gravitational force (τ_w) and from the push or pull of the right hand (τ_R). Stating the second condition in terms of clockwise and counterclockwise torques,

$$\text{net}\tau_{\text{cw}} = -\text{net } \tau_{\text{ccw}}.$$

or the algebraic sum of the torques is zero.

Here this is

$$\tau_{\text{R}} = -\tau_{\text{w}}$$

since the weight of the pole creates a counterclockwise torque and the right hand counters with a clockwise torque. Using the definition of torque, $\tau = rF \sin\theta$, noting that $\theta = 90°$, and substituting known values, we obtain

$$(0.900 \text{ m})(F_{\text{R}}) = (0.600 \text{ m})(mg)$$

Thus,

$$\begin{aligned} F_{\text{R}} &= (0.667)(5.00 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 32.7 \text{ N}. \end{aligned}$$

Solution for (b)

The first condition for equilibrium is based on the free body diagram in the figure. This implies that by Newton's second law:

$$F_{\text{L}} + F_{\text{R}} - mg = 0$$

From this we can conclude:

$$F_{\text{L}} + F_{\text{R}} = w = mg$$

Solving for F_{L}, we obtain

$$\begin{aligned} F_{\text{L}} &= mg - F_{\text{R}} \\ &= mg - 32.7 \text{ N} \\ &= (5.00 \text{ kg})(9.80 \text{ m/s}^2) - 32.7 \text{ N} \\ &= 16.3 \text{ N}. \end{aligned}$$

Discussion

$F_{\text{L}}$is seen to be exactly half of F_{R}, as we might have guessed, since F_{L} is applied twice as far from the cg as F_{R}.

If the pole vaulter holds the pole as he might at the start of a run, shown in Figure 3, the forces change again. Both are considerably greater, and one force reverses direction.

TAKE-HOME EXPERIMENT

This is an experiment to perform while standing in a bus or a train. Stand facing sideways. How do you move your body to readjust the distribution of your mass as the bus accelerates and decelerates? Now stand facing forward. How do you move your body to readjust the distribution of your mass as

the bus accelerates and decelerates? Why is it easier and safer to stand facing sideways rather than forward? Note: For your safety (and those around you), make sure you are holding onto something while you carry out this activity!

PHET EXPLORATIONS: BALANCING ACT

Play with objects on a teeter totter to learn about balance. Test what you've learned by trying the Balance Challenge game.

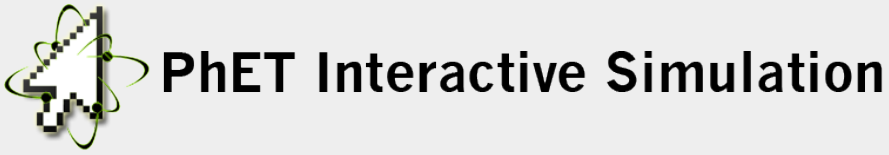


Figure 4. *Balancing Act*

Summary

- Statics can be applied to a variety of situations, ranging from raising a drawbridge to bad posture and back strain. We have discussed the problem-solving strategies specifically useful for statics. Statics is a special case of Newton's laws, both the general problem-solving strategies and the special strategies for Newton's laws, discussed in Chapter 4.7 Problem-Solving Strategies, still apply.

Glossary

static equilibrium
equilibrium in which the acceleration of the system is zero and accelerated rotation does not occur

Conceptual Questions

1: When visiting some countries, you may see a person balancing a load on the head. Explain why the center of mass of the load needs to be directly above the person's neck vertebrae.

Problems & Exercises

1: Suppose a horse leans against a wall as in Figure 10. Calculate the force exerted on the wall assuming that force is horizontal while using the data in the schematic representation of the situation. Note that the force exerted on the wall is equal in magnitude and opposite in direction to the force exerted on the horse, keeping it in equilibrium. The total mass of the horse and rider is 500 kg. Take the data to be accurate to three digits.

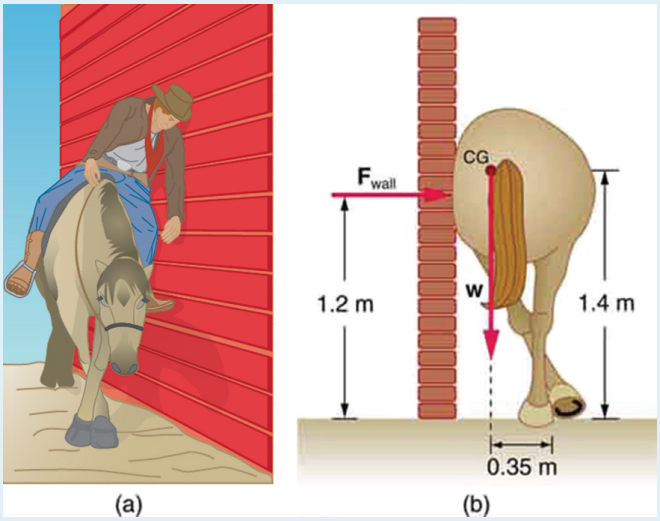


Figure 10.

2: Two children of mass 20.0 kg and 30.0 kg sit balanced on a seesaw with the pivot point located at the center of the seesaw. If the children are separated by a distance of 3.00 m, at what distance from the pivot point is the small child sitting in order to maintain the balance?

3: (a) Calculate the magnitude and direction of the force on each foot of the horse in Figure 10 (two are on the ground), assuming the center of mass of the horse is midway between the feet. The total mass of the horse and rider is 500kg. (b) What is the minimum coefficient of friction between the hooves and ground? Note that the force exerted by the wall is horizontal.

4: A person carries a plank of wood 2.00 m long with one hand pushing down on it at one end with a force F_1 and the other hand holding it up at .500 m from the end of the plank with force F_2. If the plank has a mass of 20.0 kg and its center of gravity is at the middle of the plank, what are the magnitudes of the forces F_1 and F_2?

5: A 17.0-m-high and 11.0-m-long wall under construction and its bracing are shown in Figure 11. The wall is in stable equilibrium without the bracing but can pivot at its base. Calculate the force exerted by each of the 10 braces if a strong wind exerts a horizontal force of 650 N on each square meter of the wall. Assume that the net force from the wind acts at a height halfway up the wall and that all braces exert equal forces parallel to their lengths. Neglect the thickness of the wall.

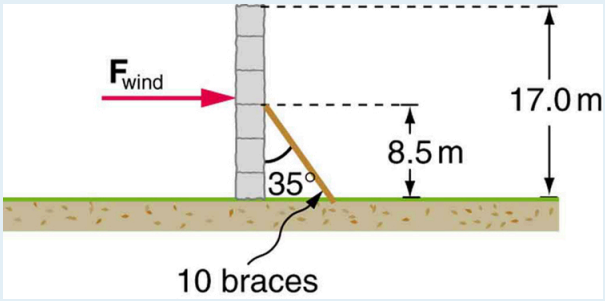


Figure 11.

6: (a) What force must be exerted by the wind to support a 2.50-kg chicken in the position shown in Figure 12? (b) What is the ratio of this force to the chicken's weight? (c) Does this support the contention that the chicken has a relatively stable construction?

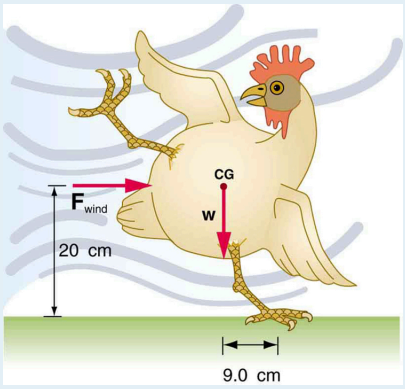


Figure 12.

7: Suppose the weight of the drawbridge in Figure 13 is supported entirely by its hinges and the opposite shore, so that its cables are slack. (a) What fraction of the weight is supported by the opposite shore if the point of support is directly beneath the cable attachments? (b) What is the direction and magnitude of the force the hinges exert on the bridge under these circumstances? The mass of the bridge is 2500 kg.

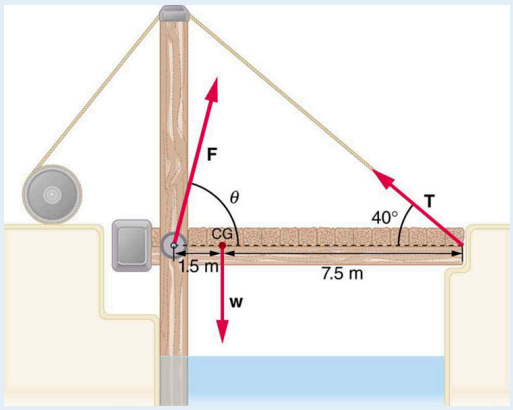


Figure 13. *A small drawbridge, showing the forces on the hinges (F), its weight (w), and the tension in its wires (T).*

8: Suppose a 900-kg car is on the bridge in Figure 13 with its center of mass halfway between the hinges and

the cable attachments. (The bridge is supported by the cables and hinges only.) (a) Find the force in the cables. (b) Find the direction and magnitude of the force exerted by the hinges on the bridge.

9: A sandwich board advertising sign is constructed as shown in Figure 14. The sign's mass is 8.00 kg. (a) Calculate the tension in the chain assuming no friction between the legs and the sidewalk. (b) What force is exerted by each side on the hinge?

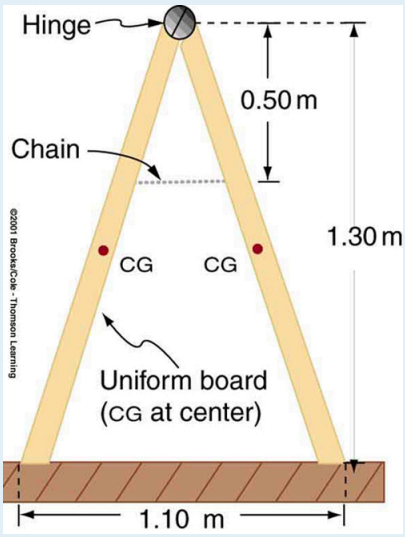


Figure 14. *A sandwich board advertising sign demonstrates tension.*

10: (a) What minimum coefficient of friction is needed between the legs and the ground to keep the sign in Figure 14 in the position shown if the chain breaks? (b) What force is exerted by each side on the hinge?

11: A gymnast is attempting to perform splits. From the information given in Figure 15, calculate the magnitude and direction of the force exerted on each foot by the floor.

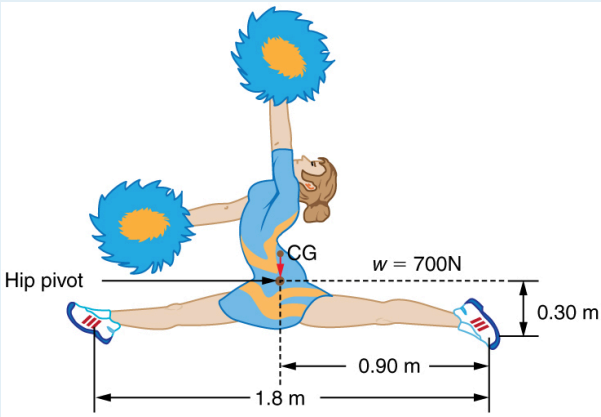


Figure 15. *A gymnast performs full split. The center of gravity and the various distances from it are shown.*

12: To get up on the roof, a person (mass 70.0 kg) places a 6.00-m aluminum ladder (mass 10.0 kg) against the house on a concrete pad with the base of the ladder 2.00 m from the house. The ladder rests against a

plastic rain gutter, which we can assume to be frictionless. The center of mass of the ladder is 2 m from the bottom. The person is standing 3 m from the bottom. What are the magnitudes of the forces on the ladder at the top and bottom?

13: In Figure 3, the cg of the pole held by the pole vaulter is 2.00 m from the left hand, and the hands are 0.700 m apart. Calculate the force exerted by (a) his right hand and (b) his left hand. (c) If each hand supports half the weight of the pole in Figure 1, show that the second condition for equilibrium (net $\tau = 0$) is satisfied for a pivot other than the one located at the center of gravity of the pole. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium described above.

Problems & Exercises

1: To get up on the roof, a person (mass 70.0 kg) places a 6.00-m aluminum ladder (mass 10.0 kg) against the house on a concrete pad with the base of the ladder 2.00 m from the house. The ladder rests against a plastic rain gutter, which we can assume to be frictionless. The center of mass of the ladder is 2 m from the bottom. The person is standing 3 m from the bottom. What are the magnitudes of the forces on the ladder at the top and bottom?

2: In Figure 3, the cg of the pole held by the pole vaulter is 2.00 m from the left hand, and the hands are 0.700 m apart. Calculate the force exerted by (a) his right hand and (b) his left hand. (c) If each hand supports half the weight of the pole in Figure 1, show that the second condition for equilibrium (net $\tau = 0$) is satisfied for a pivot other than the one located at the center of gravity of the pole. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium described above.

Solutions

Problems & Exercises

1: $F_{wall} = 1.43 \times 10^3$ N

3: (a) 2.55×10^3 N, 16.3^o to the left of vertical (i.e, toward the wall) (b) 0.292

5: $F_B = 2.12 \times 10^4$ N

7: (a) 0.167, or about one-sixth of the weight is supported by the opposite shore. (b) $F = 2.0 \times 10^4$ N straight up.

9: (a) 21.6 N (b) 21.6 N

11: 350 N directly upwards

8.4 Forces and Torques in Muscles and Joints

Summary

- Explain the forces exerted by muscles.
- State how a bad posture causes back strain.
- Discuss the benefits of skeletal muscles attached close to joints.
- Discuss various complexities in the real system of muscles, bones, and joints.

Muscles, bones, and joints are some of the most interesting applications of statics. There are some surprises. Muscles, for example, exert far greater forces than we might think. Figure 1 shows a forearm holding a book and a schematic diagram of an analogous lever system. The schematic is a good approximation for the forearm, which looks more complicated than it is, and we can get some insight into the way typical muscle systems function by analyzing it.

Muscles can only contract, so they occur in pairs. In the arm, the biceps muscle is a flexor—that is, it closes the limb. The triceps muscle is an extensor that opens the limb. This configuration is typical of skeletal muscles, bones, and joints in humans and other vertebrates. Most skeletal muscles exert much larger forces within the body than the limbs apply to the outside world. The reason is clear once we realize that most muscles are attached to bones via tendons close to joints, causing these systems to have mechanical advantages much less than one. Viewing them as simple machines, the input force is much greater than the output force, as seen in Figure 1.

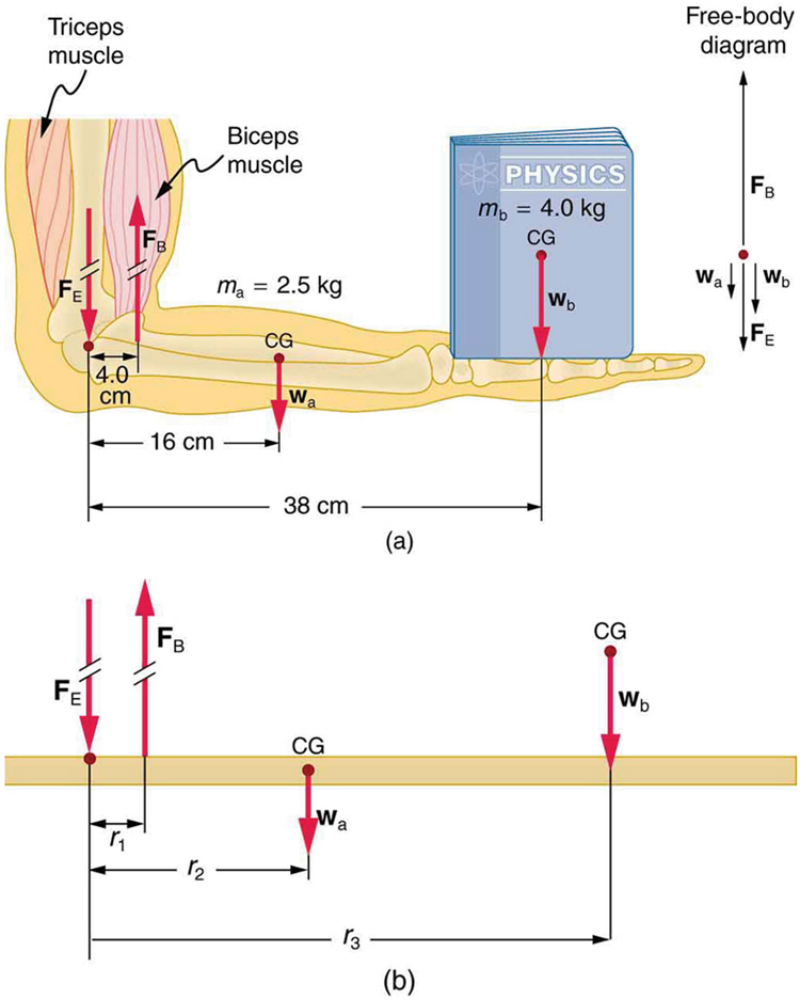


Figure 1. *(a) The figure shows the forearm of a person holding a book. The biceps exert a force F_B to support the weight of the forearm and the book. The triceps are assumed to be relaxed. (b) Here, you can view an approximately equivalent mechanical system with the pivot at the elbow joint as seen in Example 1.*

Example 1: Muscles Exert Bigger Forces Than You Might Think

Calculate the force the biceps muscle must exert to hold the forearm and its load as shown in Figure 1, and compare this force with the weight of the forearm plus its load. You may take the data in the figure to be accurate to three significant figures.

Strategy

There are four forces acting on the forearm and its load (the system of interest). The magnitude of the force of the biceps is F_B; that of the elbow joint is F_E; that of the weights of the forearm is w_a, and its load is w_b.Two of these are unknown (F_B and F_E), so that the first condition for equilibrium cannot by

itself yield F_B. But if we use the second condition and choose the pivot to be at the elbow, then the torque due to F_E is zero, and the only unknown becomes F_B.

Solution

The torques created by the weights are clockwise relative to the pivot, while the torque created by the biceps is counterclockwise; thus, the second condition for equilibrium (net τ = 0) becomes

$$r_2 w_a + r_3 w_b = r_1 F_B .$$

Note that sin θ = 1 for all forces, since θ = 90° for all forces. This equation can easily be solved for F_B in terms of known quantities, yielding

$$F_B = \frac{r_2 w_a + r_3 w_b}{r_1}.$$

Entering the known values gives

$$F_B = \frac{(0.160\ \text{m})(2.50\ \text{kg})(9.80\ \text{m/s}^2) + (0.380\ \text{m})(4.00\ \text{kg})(9.80\ \text{m/s}^2)}{0.0400\ \text{m}}$$

which yields

$$F_B = 470\ \text{N}.$$

Now, the combined weight of the arm and its load is (6.50 kg)(9.80 m/s^2) = 63.7 N, so that the ratio of the force exerted by the biceps to the total weight is

$$\frac{F_B}{w_a + w_b} = \frac{470}{63.7} = 7.38.$$

Discussion

This means that the biceps muscle is exerting a force 7.38 times the weight supported.

In the above example of the biceps muscle, the angle between the forearm and upper arm is 90°. If this angle changes, the force exerted by the biceps muscle also changes. In addition, the length of the biceps muscle changes. The force the biceps muscle can exert depends upon its length; it is smaller when it is shorter than when it is stretched.

Very large forces are also created in the joints. In the previous example, the downward force F_E exerted by the humerus at the elbow joint equals 407 N, or 6.38 times the total weight supported. (The calculation of F_E is straightforward and is left as an end-of-chapter problem.) Because muscles can contract, but not expand beyond their resting length, joints and muscles often exert forces that act in opposite directions and thus subtract. (In the above example, the upward force of the muscle minus the downward force of the joint equals the weight supported—that is, 470 N – 407 N = 63 N, approximately equal to the weight supported.) Forces in muscles and joints are largest when their load is a long distance from the joint, as the book is in the previous example.

In racquet sports such as tennis the constant extension of the arm during game play creates large forces in this way. The mass times the lever arm of a tennis racquet is an important factor, and many players use the heaviest racquet they can handle. It is no wonder that joint deterioration and damage to the tendons in the elbow, such as “tennis elbow,” can result from repetitive motion, undue torques, and possibly

poor racquet selection in such sports. Various tried techniques for holding and using a racquet or bat or stick not only increases sporting prowess but can minimize fatigue and long-term damage to the body. For example, tennis balls correctly hit at the “sweet spot” on the racquet will result in little vibration or impact force being felt in the racquet and the body—less torque as explained in Chapter 9.8 Collisions of Extended Bodies in Two Dimensions. Twisting the hand to provide top spin on the ball or using an extended rigid elbow in a backhand stroke can also aggravate the tendons in the elbow.

Training coaches and physical therapists use the knowledge of relationships between forces and torques in the treatment of muscles and joints. In physical therapy, an exercise routine can apply a particular force and torque which can, over a period of time, revive muscles and joints. Some exercises are designed to be carried out under water, because this requires greater forces to be exerted, further strengthening muscles. However, connecting tissues in the limbs, such as tendons and cartilage as well as joints are sometimes damaged by the large forces they carry. Often, this is due to accidents, but heavily muscled athletes, such as weightlifters, can tear muscles and connecting tissue through effort alone.

The back is considerably more complicated than the arm or leg, with various muscles and joints between vertebrae, all having mechanical advantages less than 1. Back muscles must, therefore, exert very large forces, which are borne by the spinal column. Discs crushed by mere exertion are very common. The jaw is somewhat exceptional—the masseter muscles that close the jaw have a mechanical advantage greater than 1 for the back teeth, allowing us to exert very large forces with them. A cause of stress headaches is persistent clenching of teeth where the sustained large force translates into fatigue in muscles around the skull.

Figure 2 shows how bad posture causes back strain. In part (a), we see a person with good posture. Note that her upper body’s cg is directly above the pivot point in the hips, which in turn is directly above the base of support at her feet. Because of this, her upper body’s weight exerts no torque about the hips. The only force needed is a vertical force at the hips equal to the weight supported. No muscle action is required, since the bones are rigid and transmit this force from the floor. This is a position of unstable equilibrium, but only small forces are needed to bring the upper body back to vertical if it is slightly displaced. Bad posture is shown in part (b); we see that the upper body’s cg is in front of the pivot in the hips. This creates a clockwise torque around the hips that is counteracted by muscles in the lower back. These muscles must exert large forces, since they have typically small mechanical advantages. (In other words, the perpendicular lever arm for the muscles is much smaller than for the cg.) Poor posture can also cause muscle strain for people sitting at their desks using computers. Special chairs are available that allow the body’s CG to be more easily situated above the seat, to reduce back pain. Prolonged muscle action produces muscle strain. Note that the cg of the entire body is still directly above the base of support in part (b) of Figure 2. This is compulsory; otherwise the person would not be in equilibrium. We lean forward for the same reason when carrying a load on our backs, to the side when carrying a load in one arm, and backward when carrying a load in front of us, as seen in Figure 3.

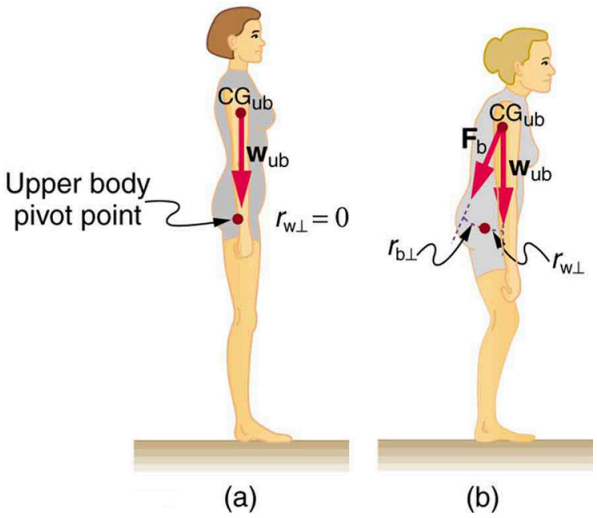


Figure 2. *(a) Good posture places the upper body's cg over the pivots in the hips, eliminating the need for muscle action to balance the body. (b) Poor posture requires exertion by the back muscles to counteract the clockwise torque produced around the pivot by the upper body's weight. The back muscles have a small effective perpendicular lever arm, $r_{b\perp}$, and must therefore exert a large force F_b. Note that the legs lean backward to keep the cg of the entire body above the base of support in the feet.*

You have probably been warned against lifting objects with your back. This action, even more than bad posture, can cause muscle strain and damage discs and vertebrae, since abnormally large forces are created in the back muscles and spine.

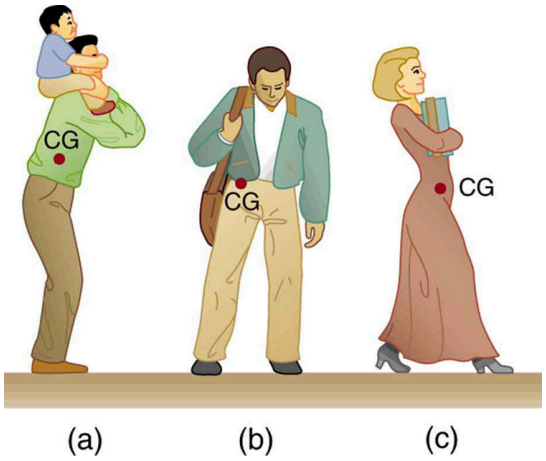


Figure 3. *People adjust their stance to maintain balance. (a) A father carrying his son piggyback leans forward to position their overall cg above the base of support at his feet. (b) A student carrying a shoulder bag leans to the side to keep the overall cg over his feet. (c) Another student carrying a load of books in her arms leans backward for the same reason.*

Example 2: Do Not Lift with Your Back

Consider the person lifting a heavy box with his back, shown in Figure 4. (a) Calculate the magnitude of the force F_{B}– in the back muscles that is needed to support the upper body plus the box and compare this with his weight. The mass of the upper body is 55.0 kg and the mass of the box is 30.0 kg. (b) Calculate the magnitude and direction of the force F_{V}– exerted by the vertebrae on the spine at the indicated pivot point. Again, data in the figure may be taken to be accurate to three significant figures.

Strategy

By now, we sense that the second condition for equilibrium is a good place to start, and inspection of the known values confirms that it can be used to solve for F_{B}– if the pivot is chosen to be at the hips. The torques created by w_{ub} and w_{box}– are clockwise, while that created by $\vec{\mathrm{F}}_{\mathrm{B}}$– is counterclockwise.

Solution for (a)

Using the perpendicular lever arms given in the figure, the second condition for equilibrium (net τ = 0) becomes

$$(0.350\ \mathrm{m})(55.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2}) + (0.500\ \mathrm{m})(30.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2}) = (0.0800\ \mathrm{m})F_{\mathrm{B}}.$$

Solving for F_{B} yields

$$F_{\mathrm{B}} = 4.20 \times 10^3\ \mathrm{N}.$$

The ratio of the force the back muscles exert to the weight of the upper body plus its load is

$$\frac{F_{\mathrm{B}}}{w_{\mathrm{ub}} + w_{\mathrm{box}}} = \frac{4200\ \mathrm{N}}{833\ \mathrm{N}} = 5.04.$$

This force is considerably larger than it would be if the load were not present.

Solution for (b)

More important in terms of its damage potential is the force on the vertebrae F_{V}. The first condition for equilibrium (net F = 0) can be used to find its magnitude and direction. Using y for vertical and x for horizontal, the condition for the net external forces along those axes to be zero

$$\text{net } F_y = 0 \text{ and net } F_x = 0.$$

Starting with the vertical (y) components, this yields

$$F_{\mathrm{Vy}} - w_{\mathrm{ub}} - w_{\mathrm{box}} - F_{\mathrm{B}} \sin 29.0^0 = 0.$$

Thus,

$$\begin{aligned} F_{\mathrm{Vy}} &= w_{\mathrm{ub}} + w_{\mathrm{box}} + F_{\mathrm{B}} \sin 29.0^0 \\ &= 833\ \mathrm{N} + (4200\ \mathrm{N}) \sin 29.0^0 \end{aligned}$$

yielding

$$F_{\mathrm{Vy}} = 2.87 \times 10^3\ \mathrm{N}.$$

Similarly, for the horizontal (x) components,

$$F_{\mathrm{Vx}} - F_{\mathrm{B}} \cos 29.0^0 = 0$$

yielding

$$F_{\text{Vx}} = 3.67 \times 10^3 \text{ N}.$$

The magnitude of $\vec{\text{F}}_{\text{V}}$ is given by the Pythagorean theorem:

$$F_{\text{V}} = \sqrt{F_{\text{Vx}}^2 + F_{\text{Vy}}^2} = 4.66 \times 10^3 \text{ N}.$$

The direction of $\vec{\text{F}}_{\text{V}}$ is

$$\theta = \tan^{-1}\left(\frac{F_{\text{Vy}}}{F_{\text{Vx}}}\right) = 38.0^0.$$

Note that the ratio of F_{V} to the weight supported is

$$\frac{F_{\text{V}}}{w_{\text{ub}} + w_{\text{box}}} = \frac{4660 \text{ N}}{833 \text{ N}} = 5.59.$$

Discussion

This force is about 5.6 times greater than it would be if the person were standing erect. The trouble with the back is not so much that the forces are large—because similar forces are created in our hips, knees, and ankles—but that our spines are relatively weak. Proper lifting, performed with the back erect and using the legs to raise the body and load, creates much smaller forces in the back—in this case, about 5.6 times smaller.

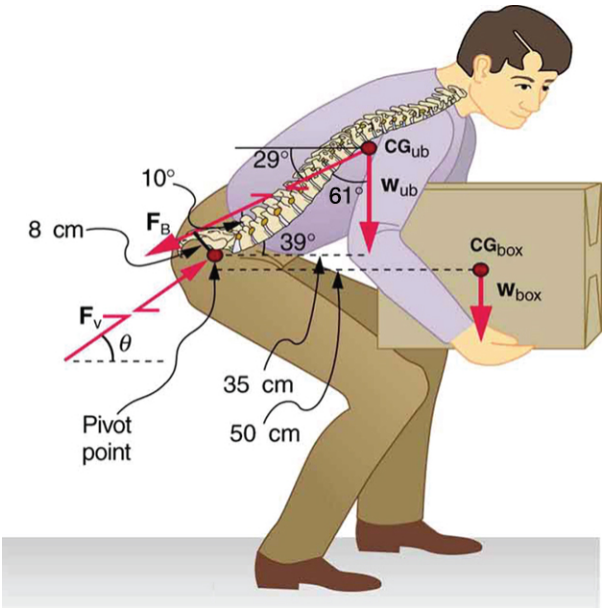


Figure 4. *This figure shows that large forces are exerted by the back muscles and experienced in the vertebrae when a person lifts with their back, since these muscles have small effective perpendicular lever arms. The data shown here are analyzed in the preceding example, Example 2.*

What are the benefits of having most skeletal muscles attached so close to joints? One advantage is speed because small muscle contractions can produce large movements of limbs in a short period of time. Other advantages are flexibility and agility, made possible by the large numbers of joints and the ranges over which they function. For example, it is difficult to imagine a system with biceps muscles attached at the wrist that would be capable of the broad range of movement we vertebrates possess.

There are some interesting complexities in real systems of muscles, bones, and joints. For instance, the pivot point in many joints changes location as the joint is flexed, so that the perpendicular lever arms and the mechanical advantage of the system change, too. Thus the force the biceps muscle must exert to hold up a book varies as the forearm is flexed. Similar mechanisms operate in the legs, which explain, for example, why there is less leg strain when a bicycle seat is set at the proper height. The methods employed in this section give a reasonable description of real systems provided enough is known about the dimensions of the system. There are many other interesting examples of force and torque in the body—a few of these are the subject of end-of-chapter problems.

Section Summary

- Statics plays an important part in understanding everyday strains in our muscles and bones.
- Many lever systems in the body have a mechanical advantage of significantly less than one, as many of our muscles are attached close to joints.
- Someone with good posture stands or sits in such as way that their center of gravity lies directly above the pivot point in their hips, thereby avoiding back strain and damage to disks.

Conceptual Questions

1: Why are the forces exerted on the outside world by the limbs of our bodies usually much smaller than the forces exerted by muscles inside the body?

2: Explain why the forces in our joints are several times larger than the forces we exert on the outside world with our limbs. Can these forces be even greater than muscle forces?

3: Certain types of dinosaurs were bipedal (walked on two legs). What is a good reason that these creatures invariably had long tails if they had long necks?

4: Swimmers and athletes during competition need to go through certain postures at the beginning of the race. Consider the balance of the person and why start-offs are so important for races.

5: If the maximum force the biceps muscle can exert is 1000 N, can we pick up an object that weighs 1000 N? Explain your answer.

6: Suppose the biceps muscle was attached through tendons to the upper arm close to the elbow and the forearm near the wrist. What would be the advantages and disadvantages of this type of construction for the motion of the arm?

7: Explain one of the reasons why pregnant women often suffer from back strain late in their pregnancy.

Problems & Exercises

1: Verify that the force in the elbow joint in Example 1 is 407 N, as stated in the text.

2: Two muscles in the back of the leg pull on the Achilles tendon as shown in Figure 5. What total force do they exert?

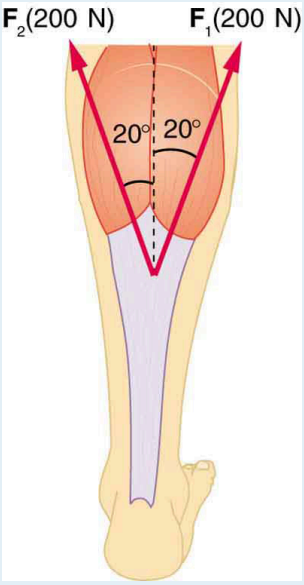


Figure 5. *The Achilles tendon of the posterior leg serves to attach plantaris, gastrocnemius, and soleus muscles to calcaneus bone.*

3: The upper leg muscle (quadriceps) exerts a force of 1250 N, which is carried by a tendon over the kneecap (the patella) at the angles shown in Figure 6. Find the direction and magnitude of the force exerted by the kneecap on the upper leg bone (the femur).

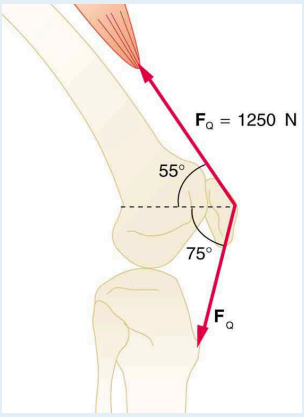


Figure 6. *The knee joint works like a hinge to bend and straighten the lower leg. It permits a person to sit, stand, and pivot.*

4: A device for exercising the upper leg muscle is shown in Figure 7, together with a schematic representation of an equivalent lever system. Calculate the force exerted by the upper leg muscle to lift the mass at a constant speed. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium in Chapter 8.3 Applications of Statistics, Including Problem-Solving Strategies.

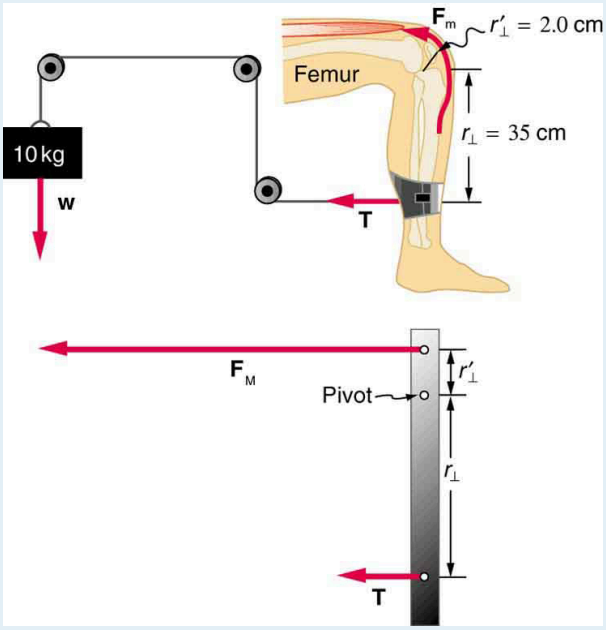


Figure 7. *A mass is connected by pulleys and wires to the ankle in this exercise device.*

5: A person working at a drafting board may hold her head as shown in Figure 8, requiring muscle action to support the head. The three major acting forces are shown. Calculate the direction and magnitude of the force supplied by the upper vertebrae F_V to hold the head stationary, assuming that this force acts along a line through the center of mass as do the weight and muscle force.

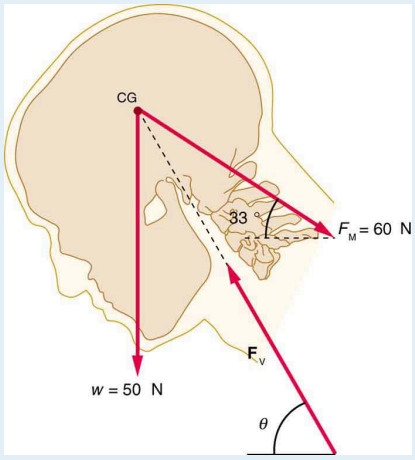


Figure 8.

6: We analyzed the biceps muscle example with the angle between forearm and upper arm set at 90°. Using the same numbers as in Example 1, find the force exerted by the biceps muscle when the angle is 120° and the forearm is in a downward position.

7: Even when the head is held erect, as in Figure 9, its center of mass is not directly over the principal point of support (the atlanto-occipital joint). The muscles at the back of the neck should therefore exert a force to keep the head erect. That is why your head falls forward when you fall asleep in the class. (a) Calculate the force exerted by these muscles using the information in the figure. (b) What is the force exerted by the pivot on the head?

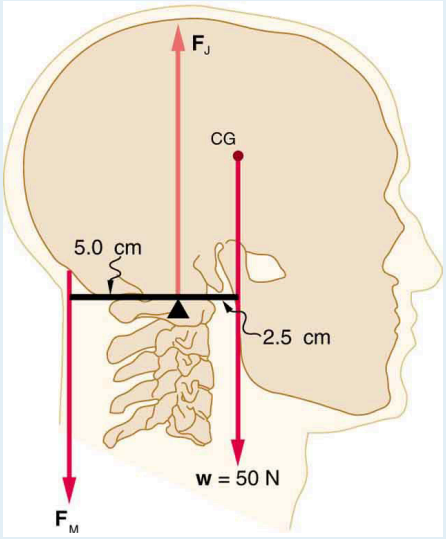


Figure 9. *The center of mass of the head lies in front of its major point of support, requiring muscle action to hold the head erect. A simplified lever system is shown.*

8: A 75-kg man stands on his toes by exerting an upward force through the Achilles tendon, as in Figure 10. (a) What is the force in the Achilles tendon if he stands on one foot? (b) Calculate the force at the pivot of the simplified lever system shown—that force is representative of forces in the ankle joint.

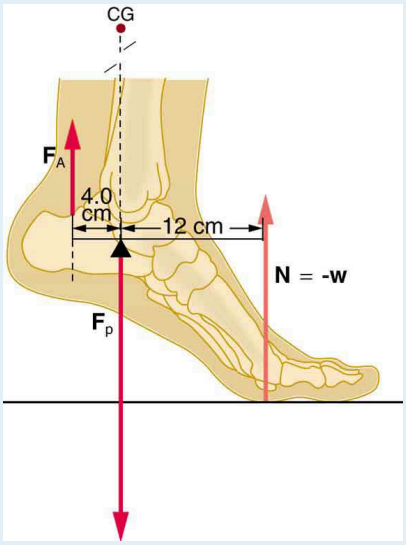


Figure 10. *The muscles in the back of the leg pull the Achilles tendon when one stands on one's toes. A simplified lever system is shown.*

9: A father lifts his child as shown in Figure 11. What force should the upper leg muscle exert to lift the child at a constant speed?

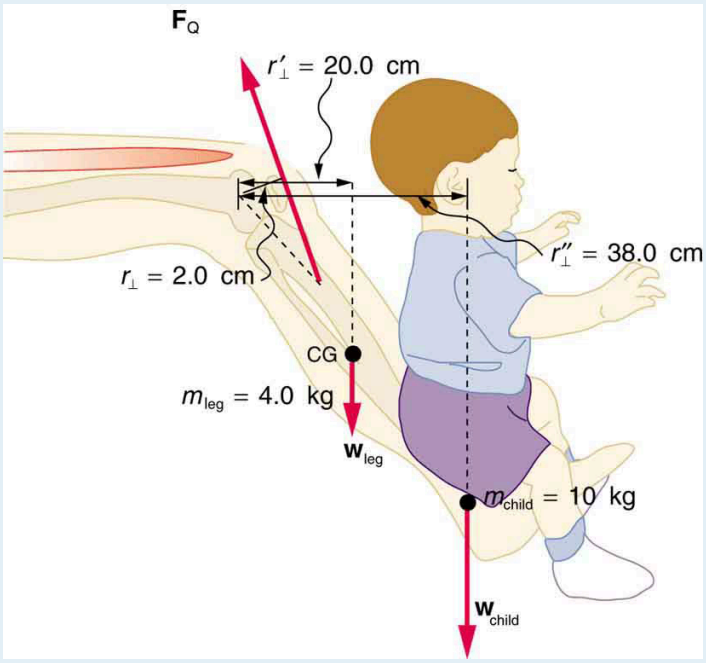


Figure 11. *A child being lifted by a father's lower leg.*

10: Unlike most of the other muscles in our bodies, the masseter muscle in the jaw, as illustrated in Figure 12, is attached relatively far from the joint, enabling large forces to be exerted by the back teeth. (a) Using the information in the figure, calculate the force exerted by the lower teeth on the bullet. (b) Calculate the force on the joint.

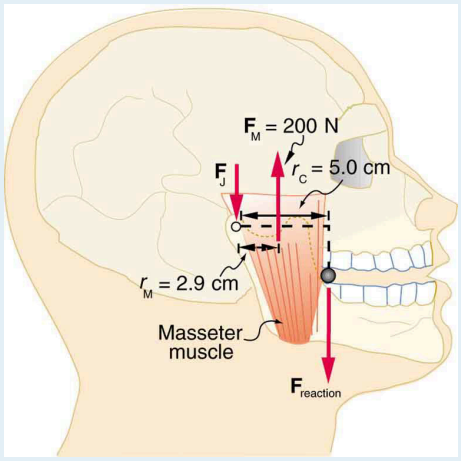


Figure 12. *A person clenching a bullet between his teeth.*

11: Integrated Concepts

Suppose we replace the 4.0-kg book in Exercise 6 of the biceps muscle with an elastic exercise rope that obeys Hooke's Law. Assume its force constant k=600 N/m. (a) How much is the rope stretched (past equilibrium) to provide the same force F_B as in this example? Assume the rope is held in the hand at the same

location as the book. (b) What force is on the biceps muscle if the exercise rope is pulled straight up so that the forearm makes an angle of 25° with the horizontal? Assume the biceps muscle is still perpendicular to the forearm.

12: (a) What force should the woman in Figure 13 exert on the floor with each hand to do a push-up? Assume that she moves up at a constant speed. (b) The triceps muscle at the back of her upper arm has an effective lever arm of 1.75 cm, and she exerts force on the floor at a horizontal distance of 20.0 cm from the elbow joint. Calculate the magnitude of the force in each triceps muscle, and compare it to her weight. (c) How much work does she do if her center of mass rises 0.240 m? (d) What is her useful power output if she does 25 pushups in one minute?

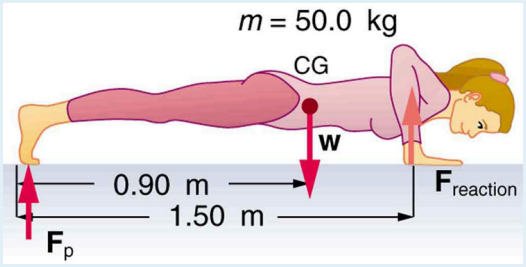


Figure 13. *A woman doing pushups.*

13: You have just planted a sturdy 2-m-tall palm tree in your front lawn for your mother's birthday. Your brother kicks a 500 g ball, which hits the top of the tree at a speed of 5 m/s and stays in contact with it for 10 ms. The ball falls to the ground near the base of the tree and the recoil of the tree is minimal. (a) What is the force on the tree? (b) The length of the sturdy section of the root is only 20 cm. Furthermore, the soil around the roots is loose and we can assume that an effective force is applied at the tip of the 20 cm length. What is the effective force exerted by the end of the tip of the root to keep the tree from toppling? Assume the tree will be uprooted rather than bend. (c) What could you have done to ensure that the tree does not uproot easily?

14: Unreasonable Results

Suppose two children are using a uniform seesaw that is 3.00 m long and has its center of mass over the pivot. The first child has a mass of 30.0 kg and sits 1.40 m from the pivot. (a) Calculate where the second 18.0 kg child must sit to balance the seesaw. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

15: Construct Your Own Problem

Consider a method for measuring the mass of a person's arm in anatomical studies. The subject lies on her back, extends her relaxed arm to the side and two scales are placed below the arm. One is placed under the elbow and the other under the back of her hand. Construct a problem in which you calculate the mass of the arm and find its center of mass based on the scale readings and the distances of the scales from the shoulder joint. You must include a free body diagram of the arm to direct the analysis. Consider changing the position of the scale under the hand to provide more information, if needed. You may wish to consult references to obtain reasonable mass values.

Solutions

Problems & Exercises

1: $r_1 = 4.00 \text{ cm}; w_\text{a} = 2.50 \text{ kg}; r_2 = 16.0 \text{ cm}; w_\text{b} = 4.00 \text{ kg}; r_3 = 38.0 \text{ cm}$

$$F_\text{B} = 470 \text{ N}$$

$$F_\text{E} = w_\text{a}(\frac{r_2}{r_1} - 1) + w_\text{b}(\frac{r_3}{r_1} - 1)$$

$$= (2.50 \text{ kg})(9.80 \text{ m/s}^2)(\frac{16.0 \text{ cm}}{4.0 \text{ cm}} - 1) + (4.00 \text{ kg})(9.80 \text{ m/s}^2)(\frac{38.0 \text{ cm}}{4.00 \text{ cm}} - 1)$$

$$= 407 \text{ N}$$

3: $1.1 \times 10^3 \text{ N}, \theta = 190^0 \text{ ccw from positive } x \text{ axis}$

5: $F_\text{V} = 97 \text{ N}, \theta = 59^0$

7: (a) 25 N downward (b) 75 N upward

8: (a) $F_\text{A} = 2.21 \times 10^3 \text{ N}$ upward (b) $F_\text{B} = 2.94 \times 10^3 \text{ N}$ downward

10: (a) $F_{\text{teeth on bullet}} = 1.2 \times 10^2 \text{ N}$ upward (b) $F_\text{J} = 84 \text{ N}$ downward

12: (a) 147 N downward (b) 1689 N, which is 3.4 times her weight (c) 118 J (d) 49.0 W

14: (a)$\bar{x}_2 = 2.33 \text{ m}$ (b) The seesaw is 3.0 m long, and hence, there is only 1.50 m of board on the other side of the pivot. The second child is off the board. (c) The position of the first child must be shortened, i.e. brought closer to the pivot.

Chapter 9 Rotational Motion and Angular Momentum

Figure 1. *The mention of a tornado conjures up images of raw destructive power. Tornadoes blow houses away as if they were made of paper and have been known to pierce tree trunks with pieces of straw. They descend from clouds in funnel-like shapes that spin violently, particularly at the bottom where they are most narrow, producing winds as high as 500 km/h. (credit: Daphne Zaras, U.S. National Oceanic and Atmospheric Administration).*

Why do tornadoes spin at all? And why do tornados spin so rapidly? The answer is that air masses that produce tornadoes are themselves rotating, and when the radii of the air masses decrease, their rate of rotation increases. An ice skater increases her spin in an exactly analogous manner as seen in Figure 2. The skater starts her rotation with outstretched limbs and increases her spin by pulling them in toward her body. The same physics describes the exhilarating spin of a skater and the wrenching force of a tornado.

Clearly, force, energy, and power are associated with rotational motion. These and other aspects of rotational motion are covered in this chapter. We shall see that all important aspects of rotational motion either have already been defined for linear motion or have exact analogs in linear motion. First, we look at angular acceleration—the rotational analog of linear acceleration.

Figure 2. *This figure skater increases her rate of spin by pulling her arms and her extended leg closer to her axis of rotation. (credit: Luu, Wikimedia Commons)*

9.1 Rotation Angle and Angular Velocity

Summary

- Define arc length, rotation angle, radius of curvature and angular velocity.
- Calculate the angular velocity of a car wheel spin.

In 1D Kinematics, we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration. 2D kinematics dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does moves in a curve (a special case of this type of motion is uniform circular motion). We begin the study of rotational motion with rotation kinematics, and defining two angular quantities needed to describe this new type of motion.

Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in Figure 1 rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each pit used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle** $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r}.$$

Figure 1. *All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle Δθ in a time Δt.*

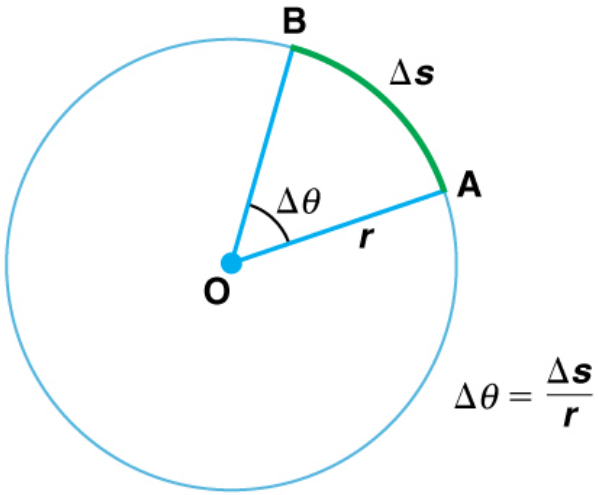


Figure 2. *The radius of a circle is rotated through an angle Δθ. The arc length Δs is described on the circumference.*

The **arc length** Δ*s* is the distance traveled along a circular path as shown in Figure 2 Note that *r* is the **radius of curvature** of the circular path.

We know that for one complete revolution, the arc length is the circumference of a circle of radius *r*. The circumference of a circle is 2π*r*. Thus for one complete revolution the rotation angle is

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi.$$

This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians** (rad), defined so that

$$2\pi \text{ rad} = 1 \text{ revolution}.$$

A comparison of some useful angles expressed in both degrees and radians is shown in Table 1.

Degree Measures	Radian Measure
30^0	$\frac{\pi}{6}$
60^0	$\frac{\pi}{3}$
90^0	$\frac{\pi}{2}$
120^0	$\frac{2\pi}{3}$
135^0	$\frac{3\pi}{4}$
180^0	π

Table 1. Comparison of Angular Units.

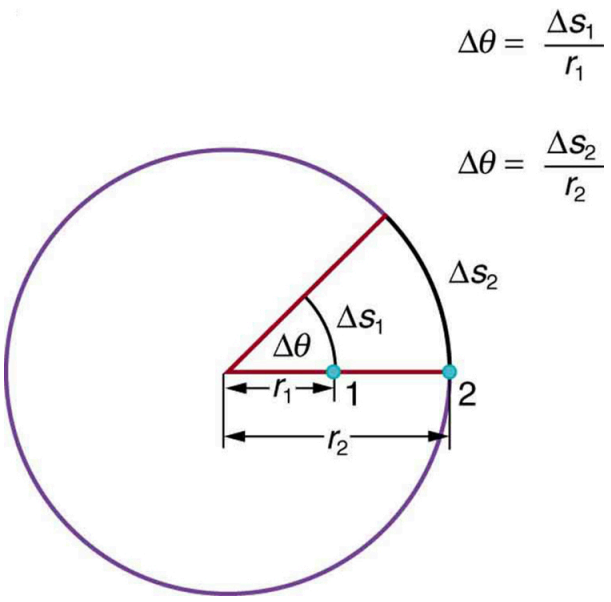


Figure 3. *Points 1 and 2 rotate through the same angle (Δθ), but point 2 moves through a greater arc length (Δs) because it is at a greater distance from the center of rotation (r).*

If Δθ = 2π rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are 360° in a circle or one revolution, the relationship between radians and degrees is thus

$$2\pi \text{ rad} = 360^0$$

so that

$$1 \text{ rad} = \frac{360^0}{2\pi} \approx 57.3^0.$$

Angular Velocity

How fast is an object rotating? We define **angular velocity ω** as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where an angular rotation $\Delta\theta$ takes place in a time Δt. The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s). Angular velocity is often expressed in units of rev/min ("rpm" or "revolutions per minute"). You can convert from rev/min to rad/s using the fact that that $2\pi \text{ rad} = 1 \text{ rev}$ and 1 min = 60 sec.

Angular velocity ω is analogous to linear velocity *v*. To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length Δs in a time Δt, and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}.$$

From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for *v* gives

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega.$$

We write this relationship in two different ways and gain two different insights:

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

The first relationship in $v = r\omega$ states that the linear velocity *v* is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest *r*), as you might expect. We can also call this linear speed *v* of a point on the rim the *tangential speed*. The second relationship in $v = r\omega$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed *v* of the car. See Figure 4. So the faster the car moves, the faster the tire spins—large *v* means a large ω, because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (*v*) for the car.

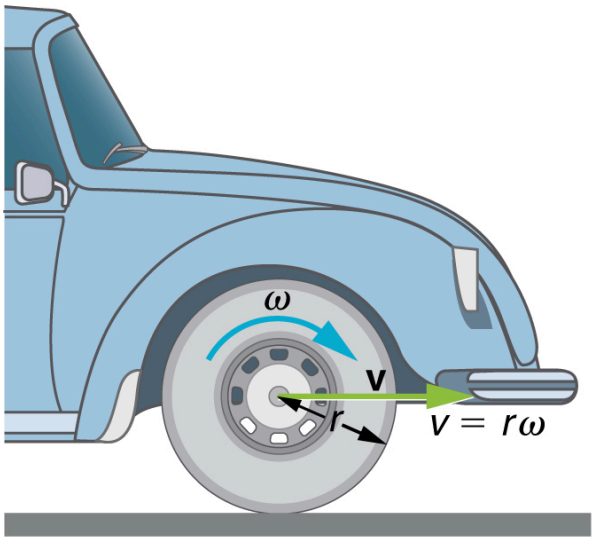


Figure 4. *A car moving at a velocity v to the right has a tire rotating with an angular velocity ω. The speed of the tread of the tire relative to the axle is v, the same as if the car were jacked up. Thus the car moves forward at linear velocity v=rω, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.*

Example 1: How Fast Does a Car Tire Spin?

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at 15.0 m/s (about 54 km/h). See Figure 4.

Strategy

Because the linear speed of the tire rim is the same as the speed of the car, we have v = 15.0 m/s. The radius of the tire is given to be r = 0.300 m. Knowing v and r, we can use the second relationship in $v = r\omega$, $\omega = \frac{v}{r}$ to calculate the angular velocity.

Solution

To calculate the angular velocity, we will use the following relationship:

$$\omega = \frac{v}{r}.$$

Substituting the knowns,

$$\omega = \frac{15.0 \text{ m/s}}{0.300 \text{ m}} = 50.0 \text{ rad/s}.$$

Discussion

When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units

of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity

$$\omega = (15.0 \text{ m/s})/(1.20 \text{ m}) = 12.5 \text{ rad/s}.$$

Both ω and v have directions (hence they are angular and linear *velocities*, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in Figure 5.

Connection to Centripetal Acceleration

We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration** ($\boldsymbol{a_c}$); centripetal means “toward the center” or “center seeking”. In Section 5.1, we saw that the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r},$$

which is the acceleration of an object in a circle of radius r at a speed v. It is also useful to express a_c in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find $a_c = (r\omega)^2/r = r\omega^2$. We can express the magnitude of centripetal acceleration using either of two equations:

$$a_c = \frac{v^2}{r};\ a_c = r\omega^2.$$

Example 2: How Big Is The Centripetal Acceleration in an Ultracentrifuge?

Calculate the centripetal acceleration of a point 7.50 cm from the axis of an **ultracentrifuge** spinning at 7.5×10^4 rev/min. Determine the ratio of this acceleration to that due to gravity. Note that this example is similar to Example 2 in the Section 5.2 on centripetal acceleration.

Strategy

The term rev/min stands for revolutions per minute. By converting this to radians per second, we obtain the angular velocity ω. Because r is given, we can use the second expression in the equation $a_c = \frac{v^2}{r};\ a_c = r\omega^2$ to calculate the centripetal acceleration.

Solution

To convert 7.5×10^4 rev/min to radians per second, we use the facts that one revolution is 2π rad and one minute is 60.0 s. Thus,

$$\omega = 7.50 \times 10^4 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60.0 \text{ s}} = 7854 \text{ rad/s}.$$

Now the centripetal acceleration is given by the second expression in $a_\text{c} = \frac{v^2}{r};\ a_\text{c} = r\omega^2$ as

$$a_\text{c} = r\omega^2.$$

Converting 7.50 cm to meters and substituting known values gives

$$a_\text{c} = (0.0750 \text{ m})(7854 \text{ rad/s})^2 = 4.63 \times 10^6 \text{ m/s}^2.$$

Note that the unitless radians are discarded in order to get the correct units for centripetal acceleration. Taking the ratio of a_c to g yields

$$\frac{a_\text{c}}{g} = \frac{4.63 \times 10^6}{9.80} = 4.72 \times 10^5.$$

Discussion

This last result means that the centripetal acceleration is 472,000 times as strong as g. It is no wonder that such high ω centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

TAKE-HOME EXPERIMENT

Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.

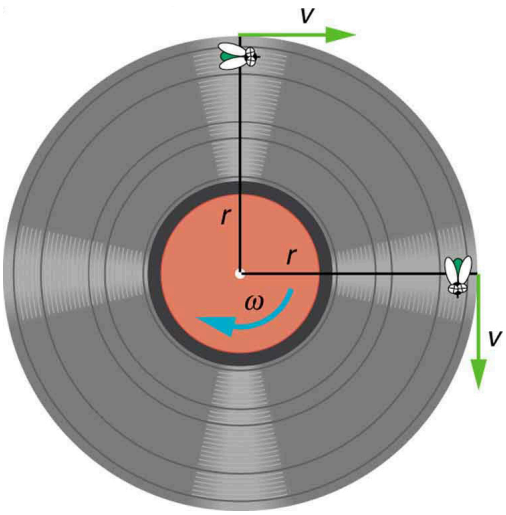


Figure 5. *As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is always tangent to the circle. The direction of the angular velocity is clockwise in this case.*

PHET EXPLORATIONS: LADYBUG REVOLUTION

Figure 6. *Ladybug Revolution*

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

Section Summary

- Uniform circular motion is motion in a circle at constant speed. The rotation angle $\Delta\theta$ is defined as the ratio of the arc length to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r},$$

  where arc length Δs is distance traveled along a circular path and r is the radius of curvature of the circular path. The quantity $\Delta\theta$ is measured in units of radians (rad), for which

$$2\pi \text{ rad} = 360^0 = 1 \text{ revolution}.$$

- The conversion between radians and degrees is 1 rad = 57.3°.
- Angular velocity ω is the rate of change of an angle,

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where a rotation $\Delta\theta$ takes place in a time Δt. The units of angular velocity are radians per second (rad/s). Linear velocity v and angular velocity ω are related by

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

Glossary

arc length
Δs, the distance traveled by an object along a circular path

pit
a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD

rotation angle
the ratio of the arc length to the radius of curvature on a circular path:

$$\Delta\theta = \frac{\Delta s}{r}$$

radius of curvature
radius of a circular path

radians
a unit of angle measurement

angular velocity
ω, the rate of change of the angle with which an object moves on a circular path

Conceptual Questions

1: There is an analogy between rotational and linear physical quantities. What rotational quantities are analogous to distance and velocity?

Problems & Exercises

1: Semi-trailer trucks have an odometer on one hub of a trailer wheel. The hub is weighted so that it does not rotate, but it contains gears to count the number of wheel revolutions—it then calculates the distance traveled. If the wheel has a 1.15 m diameter and goes through 200,000 rotations, how many kilometers should the odometer read?

2: Microwave ovens rotate at a rate of about 6 rev/min. What is this in revolutions per second? What is the angular velocity in radians per second?

3: An automobile with 0.260 m radius tires travels 80,000 km before wearing them out. How many revolutions do the tires make, neglecting any backing up and any change in radius due to wear?

4: (a) What is the period of rotation of Earth in seconds? (b) What is the angular velocity of Earth? (c) Given that Earth has a radius of 6.4×10^6 m at its equator, what is the linear velocity at Earth's surface?

5: A baseball pitcher brings his arm forward during a pitch, rotating the forearm about the elbow. If the velocity of the ball in the pitcher's hand is 35.0 m/s and the ball is 0.300 m from the elbow joint, what is the angular velocity of the forearm?

6: In lacrosse, a ball is thrown from a net on the end of a stick by rotating the stick and forearm about the elbow. If the angular velocity of the ball about the elbow joint is 30.0 rad/s and the ball is 1.30 m from the elbow joint, what is the velocity of the ball?

7: A truck with 0.420-m-radius tires travels at 32.0 m/s. What is the angular velocity of the rotating tires in radians per second? What is this in rev/min?

8: A rotating space station is said to create "artificial gravity"—a loosely-defined term used for an acceleration that would be crudely similar to gravity. The outer wall of the rotating space station would become a floor for the astronauts, and centripetal acceleration supplied by the floor would allow astronauts to exercise and maintain muscle and bone strength more naturally than in non-rotating space environments. If the space station is 200 m in diameter, what angular velocity would produce an "artificial gravity" of 9.80 m/s^2 at the rim?

9: An ordinary workshop grindstone has a radius of 7.50 cm and rotates at 6500 rev/min.

(a) Calculate the magnitude of the centripetal acceleration at its edge in meters per second squared and convert it to multiples of *g*.

(b) What is the linear speed of a point on its edge?

10: Helicopter blades withstand tremendous stresses. In addition to supporting the weight of a helicopter, they are spun at rapid rates and experience large centripetal accelerations, especially at the tip.

(a) Calculate the magnitude of the centripetal acceleration at the tip of a 4.00 m long helicopter blade that rotates at 300 rev/min.

(b) Compare the linear speed of the tip with the speed of sound (taken to be 340 m/s).

11: Olympic ice skaters are able to spin at about 5 rev/s.

(a) What is their angular velocity in radians per second?

(b) What is the centripetal acceleration of the skater's nose if it is 0.120 m from the axis of rotation?

(c) An exceptional skater named Dick Button was able to spin much faster in the 1950s than anyone since—at about 9 rev/s. What was the centripetal acceleration of the tip of his nose, assuming it is at 0.120 m radius?

(d) Comment on the magnitudes of the accelerations found. It is reputed that Button ruptured small blood vessels during his spins.

12: Verify that the linear speed of an ultracentrifuge is about 0.50 km/s, and Earth in its orbit is about 30 km/s by calculating:

(a) The linear speed of a point on an ultracentrifuge 0.100 m from its center, rotating at 50,000 rev/min.

(b) The linear speed of Earth in its orbit about the Sun (use data from the text on the radius of Earth's orbit and approximate it as being circular).

13: At takeoff, a commercial jet has a 60.0 m/s speed. Its tires have a diameter of 0.850 m.

(a) At how many rev/min are the tires rotating?

(b) What is the centripetal acceleration at the edge of the tire?

(c) With what force must a determined 1.00×10^{-15} kg bacterium cling to the rim?

(d) Take the ratio of this force to the bacterium's weight.

14: Integrated Concepts

When kicking a football, the kicker rotates his leg about the hip joint.

(a) If the velocity of the tip of the kicker's shoe is 35.0 m/s and the hip joint is 1.05 m from the tip of the shoe, what is the shoe tip's angular velocity?

(b) The shoe is in contact with the initially stationary 0.500 kg football for 20.0 ms. What average force is exerted on the football to give it a velocity of 20.0 m/s?

(c) Find the maximum range of the football, neglecting air resistance.

15: Construct Your Own Problem

Consider an amusement park ride in which participants are rotated about a vertical axis in a cylinder with vertical walls. Once the angular velocity reaches its full value, the floor drops away and friction between the walls and the riders prevents them from sliding down. Construct a problem in which you calculate the necessary angular velocity that assures the riders will not slide down the wall. Include a free body diagram of a single rider. Among the variables to consider are the radius of the cylinder and the coefficients of friction between the riders' clothing and the wall.

Solutions

Problems & Exercises

1: 723 km

3: 5×10^7 rotations

5: 117 rad/s

7: 76.2 rad/s 728 rpm

9: (a) 3.47×10^4 m/s^2, 3.55×10^3 g (b) 51.1 m/s

11: (a) 31.4 rad/s (b) 118 m/s (c) 384 m/s (d) The centripetal acceleration felt by Olympic skaters is 12 times larger than the acceleration due to gravity. That's quite a lot of acceleration in itself. The centripetal acceleration felt by Button's nose was 39.2 times larger than the acceleration due to gravity. It is no wonder that he ruptured small blood vessels in his spins.

13: (a) 1.35×10^3 rpm (b) 8.47×10^3 m/s^2 (c) 8.47×10^{-12} N (d) 865

14: (a) 33.3 rad/s (b) 500 N (c) 40.8 m

9.2 Angular Acceleration

Summary

- Describe uniform circular motion.
- Explain non-uniform circular motion.
- Calculate angular acceleration of an object.
- Observe the link between linear and angular acceleration.

Chapter 5 Uniform Circular Motion and Gravitation discussed only uniform circular motion, which is motion in a circle at constant speed and, hence, constant angular velocity. Recall that angular velocity ω was defined as the time rate of change of angle θ:

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where θ is the angle of rotation as seen in Figure 1. The relationship between angular velocity ω and linear velocity v was also defined in Chapter 9.1 Rotation Angle and Angular Velocity as

$$v = r\omega$$

or

$$\omega = \frac{v}{r},$$

where r is the radius of curvature, also seen in Figure 1. According to the sign convention, the counter clockwise direction is considered as positive direction and clockwise direction as negative

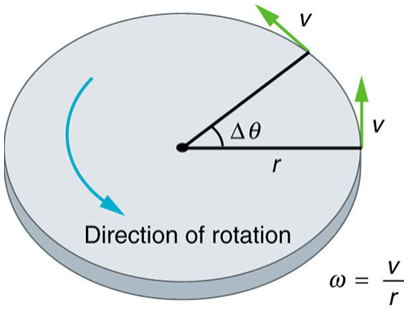


Figure 1. *This figure shows uniform circular motion and some of its defined quantities.*

Angular velocity is not constant when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer's hard disk slows to a halt when switched off. In all these cases, there

is an **angular acceleration**, in which ω changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration α is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

$$\alpha = \frac{\Delta\omega}{\Delta t},$$

where Δω is the change in angular velocity and Δ*t* is the change in time. The units of angular acceleration are (rad/s)/s, or rad/s^2. If ω increases, then α is positive. If ω decreases, then α is negative.

Example 1: Calculating the Angular Acceleration and Deceleration of a Bike Wheel

Suppose a teenager puts her bicycle on its back and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the angular acceleration in rad/s^2. (b) If she now slams on the brakes, causing an angular acceleration of -87.3 rad/s^2, how long does it take the wheel to stop?

Strategy for (a)

The angular acceleration can be found directly from its definition in $\alpha = \frac{\Delta\omega}{\Delta t}$ because the final angular velocity and time are given. We see that Δω is 250 rpm and Δ*t* is 5.00 s.

Solution for (a)

Entering known information into the definition of angular acceleration, we get

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{250 \text{ rpm}}{5.00 \text{ s.}}\end{aligned}$$

Because Δω is in revolutions per minute (rpm) and we want the standard units of rad/s^2 for angular acceleration, we need to convert Δω from rpm to rad/s:

$$\begin{aligned}\Delta\omega &= 250\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \\ &= 26.2 \text{ rads.}\end{aligned}$$

Entering this quantity into the expression for α, we get

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{26.2 \text{ rad/s}}{5.00 \text{ s}} \\ &= 5.24 \text{ rad/s.}^2\end{aligned}$$

Strategy for (b)

In this part, we know the angular acceleration and the initial angular velocity. We can find the stoppage time by using the definition of angular acceleration and solving for Δt, yielding

$$\Delta t = \frac{\Delta\omega}{\alpha}.$$

Solution for (b)

Here the angular velocity decreases from 26.2 rad/s (250 rpm) to zero, so that Δω is -26.2 rad/s, and α is given to be -87.3 rad/s^2. Thus,

$$\begin{aligned}\Delta t &= \frac{-26.2\ \text{rad/s}}{-87.3\ \text{rad/s}^2} \\ &= 0.300\ \text{s}.\end{aligned}$$

Discussion

Note that the angular acceleration as the girl spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero. In both cases, the relationships are analogous to what happens with linear motion. For example, there is a large deceleration when you crash into a brick wall—the velocity change is large in a short time interval.

If the bicycle in the preceding example had been on its wheels instead of upside-down, it would first have accelerated along the ground and then come to a stop. This connection between circular motion and linear motion needs to be explored. For example, it would be useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is *tangent* to the circle at the point of interest, as seen in Figure 2. Thus, linear acceleration is called tangential acceleration a_t.

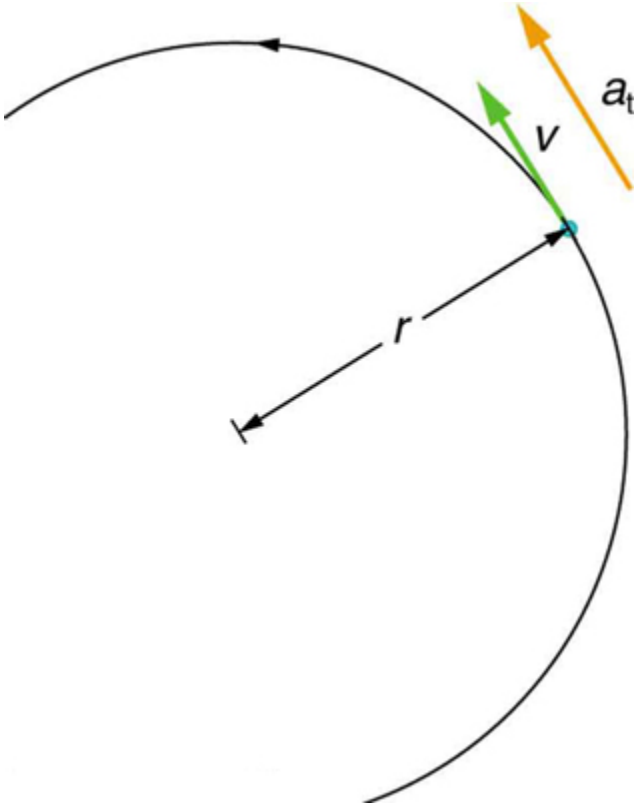


Figure 2. *In circular motion, linear acceleration a, occurs as the magnitude of the velocity changes: a is tangent to the motion. In the context of circular motion, linear acceleration is also called tangential acceleration* a_t.

Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. We know from uniform circular motion that in circular motion centripetal acceleration, a_c, refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration, as seen in Figure 3. Thus, a_t and a_c are perpendicular and independent of one another. Tangential acceleration a_t is directly related to the angular acceleration α and is linked to an increase or decrease in the velocity, but not its direction.

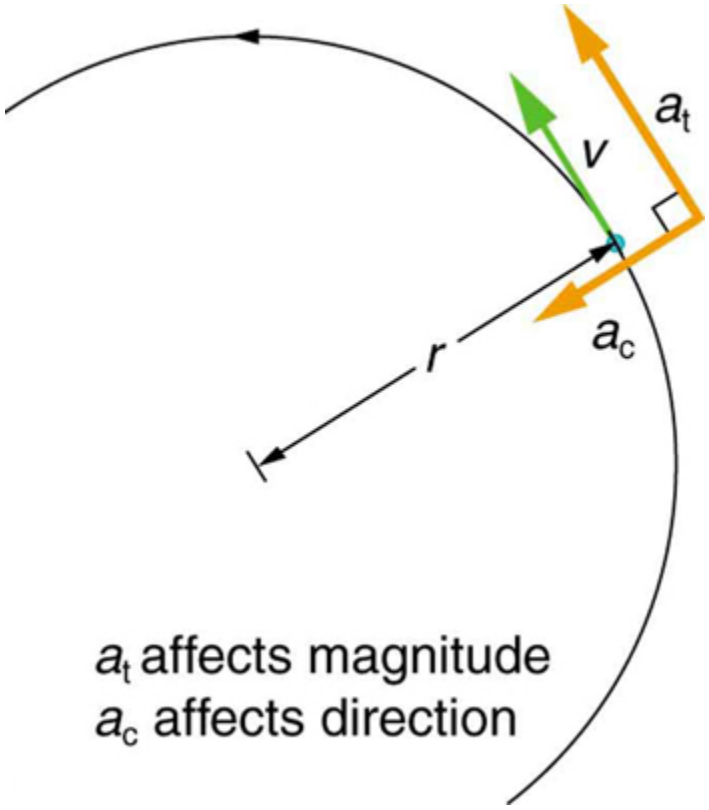


Figure 3. *Centripetal acceleration a_c occurs as the direction of velocity changes; it is perpendicular to the circular motion. Centripetal and tangential acceleration are thus perpendicular to each other.*

Now we can find the exact relationship between linear acceleration a_t and angular acceleration α. Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined (as it was in Chapter 2 One-Dimensional Kinematics) to be

$$a_t = \frac{\Delta v}{\Delta t}.$$

For circular motion, note that $v = r\omega$, so that

$$a_t = \frac{\Delta(r\omega)}{\Delta t}.$$

The radius r is constant for circular motion, and so $\Delta(r\omega) = r(\Delta\omega)$. Thus,

$$a_t = r\frac{\Delta\omega}{\Delta t}.$$

By definition, $\alpha = \frac{\Delta\omega}{\Delta t}$. Thus,

$$a_t = r\alpha,$$

or

$$\alpha = \frac{a_{\text{t}}}{r}.$$

These equations mean that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration is, and vice versa. For example, the greater the angular acceleration of a car's drive wheels, the greater the acceleration of the car. The radius also matters. For example, the smaller a wheel, the smaller its linear acceleration for a given angular acceleration α.

Example 2: Calculating the Angular Acceleration of a Motorcycle Wheel

A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels? (See Figure 4.)

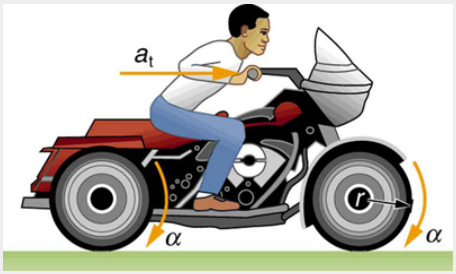


Figure 4. *The linear acceleration of a motorcycle is accompanied by an angular acceleration of its wheels.*

Strategy

We are given information about the linear velocities of the motorcycle. Thus, we can find its linear acceleration a_{t}. Then, the expression $\alpha = \frac{a_{\text{t}}}{r}$ can be used to find the angular acceleration.

Solution

The linear acceleration is

$$\begin{aligned} a_{\text{t}} &= \frac{\Delta v}{\Delta t} \\ &= \frac{30.0\ \text{m/s}}{4.20\ \text{s}} \\ &= 7.14\ \text{m/s}^2. \end{aligned}$$

We also know the radius of the wheels. Entering the values for a_{t} and r into $\alpha = \frac{a_{\text{t}}}{r}$, we get

\begin{align*} \alpha &= \dfrac{a_{\text{t}}}{r} \\ &= \dfrac{7.14~\text{m/s}}{20.320~\text{m}} \\ &= 22.3~\text{rad/s}^2. \end{align*}

Discussion

Units of radians are dimensionless and appear in any relationship between angular and linear quantities.

So far, we have defined three rotational quantities— θ, ω, and α. These quantities are analogous to the translational quantities *x*, *v*, and *a*. Table 1 displays rotational quantities, the analogous translational quantities, and the relationships between them.

Rotational	Translational	Relationship
θ	x	$\theta = \frac{x}{r}$
ω	v	$\omega = \frac{v}{r}$
α	a	$\alpha = \frac{a_{\text{t}}}{r}$

Table 1. Rotational and Translational Quantities.

MAKING CONNECTIONS: TAKE-HOME EXPERIMENT

Sit down with your feet on the ground on a chair that rotates. Lift one of your legs such that it is unbent (straightened out). Using the other leg, begin to rotate yourself by pushing on the ground. Stop using your leg to push the ground but allow the chair to rotate. From the origin where you began, sketch the angle, angular velocity, and angular acceleration of your leg as a function of time in the form of three separate graphs. Estimate the magnitudes of these quantities.

Check Your Understanding

1: Angular acceleration is a vector, having both magnitude and direction. How do we denote its magnitude and direction? Illustrate with an example.

PHET EXPLORATIONS: LADYBUG REVOLUTION

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

PhET Interactive Simulation

Figure 5. Ladybug Revolution

Section Summary

- Uniform circular motion is the motion with a constant angular velocity $\omega = \frac{\Delta\theta}{\Delta t}$.
- In non-uniform circular motion, the velocity changes with time and the rate of change of angular velocity (i.e. angular acceleration) is $\alpha = \frac{\Delta\omega}{\Delta t}$.
- Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction, given as $a_{\text{t}} = \frac{\Delta v}{\Delta t}$.
- For circular motion, note that $v = r\omega$, so that
$$a_{\text{t}} = \frac{\Delta(r\omega)}{\Delta t}.$$
- The radius r is constant for circular motion, and so $\Delta(r\omega) = r\Delta\omega$. Thus,
$$a_{\text{t}} = r\frac{\Delta\omega}{\Delta t}.$$
- By definition, $\Delta\omega / \Delta t = \alpha$. Thus,
$$a_{\text{t}} = r\alpha$$
or
$$\alpha\frac{a_{\text{t}}}{r}.$$

Glossary

angular acceleration
the rate of change of angular velocity with time

change in angular velocity
the difference between final and initial values of angular velocity

tangential acceleration
the acceleration in a direction tangent to the circle at the point of interest in circular motion

Conceptual Questions

1: Analogies exist between rotational and translational physical quantities. Identify the rotational term analogous to each of the following: acceleration, force, mass, work, translational kinetic energy, linear momentum, impulse.

2: Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.

3: In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.

4: Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) The plate starts to spin? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

Problems & Exercises

1: At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity in revolutions per second?

2: Integrated Concepts

An ultracentrifuge accelerates from rest to 100,000 rpm in 2.00 min. (a) What is its angular acceleration in rad/s^2? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the radial acceleration in m/s^2 and multiples of g of this point at full rpm?

3: Integrated Concepts

You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?

4: Unreasonable Results

You are told that a basketball player spins the ball with an angular acceleration of 100 rad/s^2. (a) What is the

ball's final angular velocity if the ball starts from rest and the acceleration lasts 2.00 s? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

Solutions

Check Your Understanding

1: The magnitude of angular acceleration is α and its most common units are rad/s^2. The direction of angular acceleration along a fixed axis is denoted by a + or a – sign, just as the direction of linear acceleration in one dimension is denoted by a + or a – sign. For example, consider a gymnast doing a forward flip. Her angular momentum would be parallel to the mat and to her left. The magnitude of her angular acceleration would be proportional to her angular velocity (spin rate) and her moment of inertia about her spin axis.

Problems & Exercises

1: $\omega = 0.737$ rev/s

3: (a) -0.26 rad/s^2 (b) 27 rev

9.3 Kinematics of Rotational Motion

Summary

- Observe the kinematics of rotational motion.
- Derive rotational kinematic equations.
- Evaluate problem solving strategies for rotational kinematics.

Just by using our intuition, we can begin to see how rotational quantities like θ, ω, and α are related to one another. For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotates through many revolutions. In more technical terms, if the wheel's angular acceleration α is large for a long period of time *t*, then the final angular velocity ω and angle of rotation θ are large. The wheel's rotational motion is exactly analogous to the fact that the motorcycle's large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematics is the description of motion. The **kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating ω, α, and *t*. To determine this equation, we recall a familiar kinematic equation for translational, or straight-line, motion:

$$v = v_0 + at \text{ (constant a)}$$

Note that in rotational motion $a = a_t$, and we shall use the symbol *a* for tangential or linear acceleration from now on. As in linear kinematics, we assume *a* is constant, which means that angular acceleration α is also a constant, because α. Now, let us substitute $v = r\omega$ and $a = r\alpha$ into the linear equation above:

$$r\omega = r\omega_0 + r\alpha t.$$

The radius *r* cancels in the equation, yielding

$$\omega = \omega_0 + at \text{ (constant a)},$$

where $\boldsymbol{\omega_0}$ is the initial angular velocity. This last equation is a *kinematic relationship* among ω, α, and *t* —that is, it describes their relationship without reference to forces or masses that may affect rotation. It is also precisely analogous in form to its translational counterpart.

MAKING CONNECTIONS

Kinematics for rotational motion is completely analogous to translational kinematics. Kinematics is concerned with the description of motion without regard to force or mass. We will find that translational kinematic quantities, such as displacement, velocity, and acceleration have direct analogs in rotational motion.

Starting with the four kinematic equations we developed in Chapter 2 One-Dimensional Kinematics, we can derive the following four rotational kinematic equations (presented together with their translational counterparts):

Rotational	**Translational**	
$\theta = \bar{\omega}t$	$x = \bar{v}t$	
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	(constant α, a)
$\omega = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$	(constant α, a)
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$	(constant α, a)

Table 2. Rotational Kinematic Equations.

In these equations, the subscript 0 denotes initial values (θ_0, x_0, and t_0 are initial values), and the average angular velocity $\bar{\omega}$ and average velocity $\bar{v}$ are defined as follows:

$$\bar{\omega} = \frac{\omega_0 + \omega}{2} \text{ and } \bar{v} = \frac{v_0 + v}{2}.$$

The equations given above in Table 2 can be used to solve any rotational or translational kinematics problem in which a and α are constant.

PROBLEM-SOLVING STRATEGY FOR ROTATIONAL KINEMATICS

1. *Examine the situation to determine that rotational kinematics (rotational motion) is involved.* Rotation must be involved, but without the need to consider forces or masses that affect the motion.
2. *Identify exactly what needs to be determined in the problem (identify the unknowns).* A

sketch of the situation is useful.

3. *Make a list of what is given or can be inferred from the problem as stated (identify the knowns).*
4. *Solve the appropriate equation or equations for the quantity to be determined (the unknown).* It can be useful to think in terms of a translational analog because by now you are familiar with such motion.
5. *Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units*. Be sure to use units of radians for angles.
6. *Check your answer to see if it is reasonable: Does your answer make sense*?

Example 1: Calculating the Acceleration of a Fishing Reel

A deep-sea fisherman hooks a big fish that swims away from the boat pulling the fishing line from his fishing reel. The whole system is initially at rest and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of 110 rad/s^2 for 2.00 s as seen in Figure 1.

(a) What is the final angular velocity of the reel?

(b) At what speed is fishing line leaving the reel after 2.00 s elapses?

(c) How many revolutions does the reel make?

(d) How many meters of fishing line come off the reel in this time?

Strategy

In each part of this example, the strategy is the same as it was for solving problems in linear kinematics. In particular, known values are identified and a relationship is then sought that can be used to solve for the unknown.

Solution for (a)

Here α and *t* are given and ω needs to be determined. The most straightforward equation to use is $\omega = \omega_0 + \alpha t$ because the unknown is already on one side and all other terms are known. That equation states that

$$\omega = \omega_0 + \alpha t.$$

We are also given that $\omega_0 = 0$ (it starts from rest), so that

$$\omega = 0 + (110 \text{ rad/s}^2)(2.00 \text{ s}) = 220 \text{ rad/s}.$$

Solution for (b)

Now that ω is known, the speed *v* can most easily be found using the relationship

$$v = r\omega,$$

where the radius *r* of the reel is given to be 4.50 cm; thus,

$$v = (0.0450\ \text{m})(220\ \text{rad/s}) = 9.90\ \text{m/s}.$$

Note again that radians must always be used in any calculation relating linear and angular quantities. Also, because radians are dimensionless, we have m × rad = m.

Solution for (c)

Here, we are asked to find the number of revolutions. Because 1 rev = 2π rad, we can find the number of revolutions by finding θ in radians. We are given α and *t*, and we know ω_0 is zero, so that θ can be obtained using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$.

$$\begin{aligned}\theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= 0 + (0.500)(110\ \text{rad/s}^2)(2.00\ \text{s})^2 = 220\ \text{rad}.\end{aligned}$$

Converting radians to revolutions gives

$$\theta = (220\ \text{rad})\frac{1\ \text{rev}}{2\pi\ \text{rad}} = 35.0\ \text{rev}.$$

Solution for (d)

The number of meters of fishing line is *x*, which can be obtained through its relationship with θ:

$$x = r\theta = (0.0450\ \text{m})(220\ \text{rad}) = 9.90\ \text{m}.$$

Discussion

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. We also see in this example how linear and rotational quantities are connected. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.) The amount of fishing line played out is 9.90 m, about right for when the big fish bites.

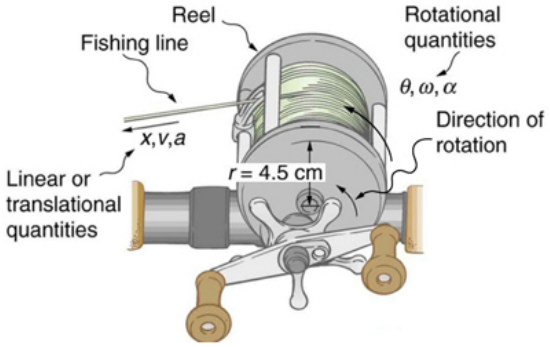


Figure 1. *Fishing line coming off a rotating reel moves linearly. Example 1 and Example 2 consider relationships between rotational and linear quantities associated with a fishing reel.*

Example 2: Calculating the Duration When the Fishing Reel Slows Down and Stops

Now let us consider what happens if the fisherman applies a brake to the spinning reel, achieving an angular acceleration of -300 rad/s^2. How long does it take the reel to come to a stop?

Strategy

We are asked to find the time *t* for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is ω_0 = 220 rad/s and the final angular velocity ω is zero. The angular acceleration is given to be $\alpha = -300 \text{ rad/s}^2$. Examining the available equations, we see all quantities but *t* are known in $\omega = \omega_0 + \alpha t$, making it easiest to use this equation.

Solution

The equation states

$$\omega = \omega_0 + \alpha t.$$

We solve the equation algebraically for *t*, and then substitute the known values as usual, yielding

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 220 \text{ rad/s}}{-300 \text{ rad/s}^2} = 0.733 \text{ s}.$$

Discussion

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish will be slower, requiring a smaller acceleration.

Example 3: Calculating the Slow Acceleration of Trains and Their Wheels

Large freight trains accelerate very slowly. Suppose one such train accelerates from rest, giving its 0.350-m-radius wheels an angular acceleration of 0.250 rad/s^2. After the wheels have made 200 revolutions (assume no slippage): (a) How far has the train moved down the track? (b) What are the final angular velocity of the wheels and the linear velocity of the train?

Strategy

In part (a), we are asked to find *x*, and in (b) we are asked to find ω and *v*. We are given the number of revolutions θ, the radius of the wheels *r*, and the angular acceleration α.

Solution for (a)

The distance *x* is very easily found from the relationship between distance and rotation angle:

$$\theta = \frac{x}{r}.$$

Solving this equation for *x* yields

$$x = r\theta.$$

Before using this equation, we must convert the number of revolutions into radians, because we are dealing with a relationship between linear and rotational quantities:

$$\theta = (200 \text{ rev})\frac{2\pi \text{ rad}}{1 \text{ rev}} = 1257 \text{ rad}.$$

Now we can substitute the known values into $x = r\theta$ to find the distance the train moved down the track:

$$x = r\theta = (0.350 \text{ m})(1257 \text{ rad}) = 440 \text{ m}.$$

Solution for (b)

We cannot use any equation that incorporates *t* to find ω, because the equation would have at least two unknown values. The equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ will work, because we know the values for all variables except ω:

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Taking the square root of this equation and entering the known values gives

$$\begin{aligned}\omega &= [0 + 2(0.250 \text{ rad/s}^2)(1257 \text{ rad})]^{1/2} \\ &= 25.1 \text{ rad/s}.\end{aligned}$$

We can find the linear velocity of the train, *v*, through its relationship to ω:

$$v = r\omega = (0.350 \text{ m})(25.1 \text{ rad/s}) = 8.77 \text{ m/s}.$$

Discussion

The distance traveled is fairly large and the final velocity is fairly slow (just under 32 km/h).

There is translational motion even for something spinning in place, as the following example illustrates. Figure 2 shows a fly on the edge of a rotating microwave oven plate. The example below calculates the total distance it travels.

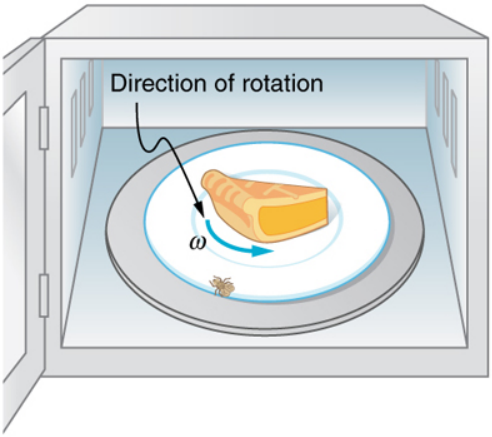


Figure 2. *The image shows a microwave plate. The fly makes revolutions while the food is heated (along with the fly).*

Example 4: Calculating the Distance Traveled by a Fly on the Edge of a Microwave Oven Plate

A person decides to use a microwave oven to reheat some lunch. In the process, a fly accidentally flies into the microwave and lands on the outer edge of the rotating plate and remains there. If the plate has a radius of 0.15 m and rotates at 6.0 rpm, calculate the total distance traveled by the fly during a 2.0-min cooking period. (Ignore the start-up and slow-down times.)

Strategy

First, find the total number of revolutions θ, and then the linear distance x traveled. $\theta = \bar{\omega}t$ can be used to find θ because $\bar{\omega}$ is given to be 6.0 rpm.

Solution

Entering known values into $\theta = \bar{\omega}t$ gives

$$\theta = \bar{\omega}t = (6.0 \text{ rpm})(2.0 \text{ min}) = 12 \text{ rev}.$$

As always, it is necessary to convert revolutions to radians before calculating a linear quantity like x from an angular quantity like θ:

$$\theta = (12 \text{ rev})\frac{2\pi \text{ rad}}{1 \text{ rev}} = 75.4 \text{ rad}.$$

Now, using the relationship between x and θ, we can determine the distance traveled:

$$x = r\omega = (0.15 \text{ m})(75.4 \text{ rad}) = 11 \text{ m}.$$

Discussion

Quite a trip (if it survives)! Note that this distance is the total distance traveled by the fly. Displacement is actually zero for complete revolutions because they bring the fly back to its original position. The distinction between total distance traveled and displacement was first noted in Chapter 2 One-Dimensional Kinematics.

Check Your Understanding

1: Rotational kinematics has many useful relationships, often expressed in equation form. Are these relationships laws of physics or are they simply descriptive? (Hint: the same question applies to linear kinematics.)

Section Summary

- Kinematics is the description of motion.
- The kinematics of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time.
- Starting with the four kinematic equations we developed in the Chapter 2 One-Dimensional Kinematics, we can derive the four rotational kinematic equations (presented together with their translational counterparts) seen in Table 2.
- In these equations, the subscript 0 denotes initial values (x_0 and t_0 are initial values), and the average angular velocity $\bar{\omega}$ and average velocity $\bar{v}$ are defined as follows:

$$\bar{\omega} = \frac{\omega_0 + \omega}{2} \text{ and } \bar{v} = \frac{v_0 + v}{2}.$$

Glossary

kinematics of rotational motion
describes the relationships among rotation angle, angular velocity, angular acceleration, and time

Problems & Exercises

1: With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s.

(a) What is its angular acceleration in rad/s^2?

(b) How many revolutions does it go through in the process?

2: Suppose a piece of dust finds itself on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)

3: A gyroscope slows from an initial rate of 32.0 rad/s at a rate of 0.700 rad/s^2.

(a) How long does it take to come to rest?

(b) How many revolutions does it make before stopping?

4: During a very quick stop, a car decelerates at 7.00 m/s^2.

(a) What is the angular acceleration of its 0.280-m-radius tires, assuming they do not slip on the pavement?

(b) How many revolutions do the tires make before coming to rest, given their initial angular velocity is 95.0 rad/s?

(c) How long does the car take to stop completely?

(d) What distance does the car travel in this time?

(e) What was the car's initial velocity?

(f) Do the values obtained seem reasonable, considering that this stop happens very quickly?

Figure 3. *Yo-yos are amusing toys that display significant physics and are engineered to enhance performance based on physical laws. (credit: Beyond Neon, Flickr)*

5: Everyday application: Suppose a yo-yo has a center shaft that has a 0.250 cm radius and that its string is being pulled.

(a) If the string is stationary and the yo-yo accelerates away from it at a rate of 1.50 m/s^2, what is the angular acceleration of the yo-yo?

(b) What is the angular velocity after 0.750 s if it starts from rest?

(c) The outside radius of the yo-yo is 3.50 cm. What is the tangential acceleration of a point on its edge?

Solutions

Check Your Understanding

1: Rotational kinematics (just like linear kinematics) is descriptive and does not represent laws of nature. With kinematics, we can describe many things to great precision but kinematics does not consider causes. For example, a large angular acceleration describes a very rapid change in angular velocity without any consideration of its cause.

Problems & Exercises

1: (a) $80\ \mathrm{rad/s^2}$ (b) 1.0 rev

3: (a) 45.7 s (b) 116 rev

5: (a) $600\ \mathrm{rad/s^2}$ (b) 450 rad/s (c) 21.0 m/s

9.4 Dynamics of Rotational Motion: Rotational Inertia

Summary

- Understand the relationship between force, mass and acceleration.
- Study the turning effect of force.
- Study the analogy between force and torque, mass and moment of inertia, and linear acceleration and angular acceleration.

If you have ever spun a bike wheel or pushed a merry-go-round, you know that force is needed to change angular velocity as seen in Figure 1. In fact, your intuition is reliable in predicting many of the factors that are involved. For example, we know that a door opens slowly if we push too close to its hinges. Furthermore, we know that the more massive the door, the more slowly it opens. The first example implies that the farther the force is applied from the pivot, the greater the angular acceleration; another implication is that angular acceleration is inversely proportional to mass. These relationships should seem very similar to the familiar relationships among force, mass, and acceleration embodied in Newton's second law of motion. There are, in fact, precise rotational analogs to both force and mass.

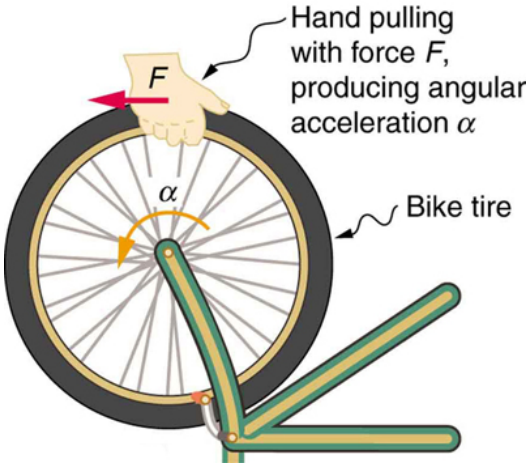


Figure 1. *Force is required to spin the bike wheel. The greater the force, the greater the angular acceleration produced. The more massive the wheel, the smaller the angular acceleration. If you push on a spoke closer to the axle, the angular acceleration will be smaller.*

To develop the precise relationship among force, mass, radius, and angular acceleration, consider what happens if we exert a force F on a point mass m that is at a distance r from a pivot point, as shown in

Figure 2. Because the force is perpendicular to *r*, an acceleration $a = \frac{F}{m}$ is obtained in the direction of *F*. We can rearrange this equation such that *F* = *ma* and then look for ways to relate this expression to expressions for rotational quantities. We note that *a* = *r*ω, and we substitute this expression into *F* = *ma*, yielding

$$F = mr\alpha.$$

Recall that **torque** is the turning effectiveness of a force. In this case, because $\vec{\mathrm{F}}$ is perpendicular to *r*, torque is simply τ = *Fr*. So, if we multiply both sides of the equation above by *r*, we get torque on the left-hand side. That is,

$$rF = mr^2\alpha$$

or

$$\tau = mr^2\alpha.$$

This last equation is the rotational analog of Newton's second law (*F* = *ma*), where torque is analogous to force, angular acceleration is analogous to translational acceleration, and mr^2 is analogous to mass (or inertia). The quantity mr^2 is called the **rotational inertia** or **moment of inertia** of a point mass *m* a distance *r* from the center of rotation.

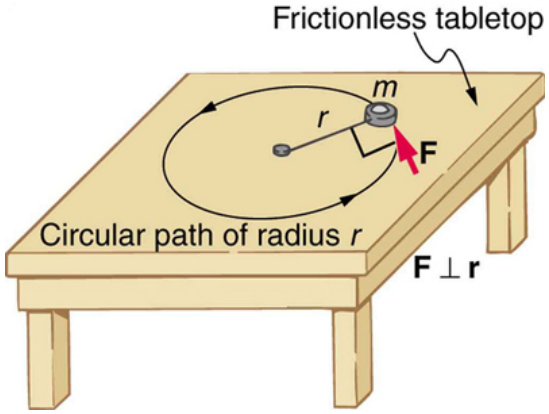


Figure 2. *An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force F is applied to the object perpendicular to the radius r, causing it to accelerate about the pivot point. The force is kept perpendicular to r.*

MAKING CONNECTIONS: ROTATIONAL MOTION DYNAMICS

Dynamics for rotational motion is completely analogous to linear or translational dynamics. Dynamics

is concerned with force and mass and their effects on motion. For rotational motion, we will find direct analogs to force and mass that behave just as we would expect from our earlier experiences.

Rotational Inertia and Moment of Inertia

Before we can consider the rotation of anything other than a point mass like the one in Figure 2, we must extend the idea of rotational inertia to all types of objects. To expand our concept of rotational inertia, we define the **moment of inertia** I of an object to be the sum of mr^2 for all the point masses of which it is composed. That is, $I = \sum mr^2$. Here I is analogous to m in translational motion. Because of the distance r, the moment of inertia for any object depends on the chosen axis. Actually, calculating I is beyond the scope of this text except for one simple case—that of a hoop, which has all its mass at the same distance from its axis. A hoop's moment of inertia around its axis is therefore MR^2, where M is its total mass and R its radius. (We use M and R for an entire object to distinguish them from m and r for point masses.) In all other cases, we must consult Figure 3 (note that the table is piece of artwork that has shapes as well as formulae) for formulas for I that have been derived from integration over the continuous body. Note that I has units of mass multiplied by distance squared ($\text{kg}\cdot\text{m}^2$), as we might expect from its definition.

The general relationship among torque, moment of inertia, and angular acceleration is

$$\text{net } \tau = I\alpha$$

or

$$\alpha = \frac{\text{net } \tau}{I},$$

where net τ is the total torque from all forces relative to a chosen axis. For simplicity, we will only consider torques exerted by forces in the plane of the rotation. Such torques are either positive or negative and add like ordinary numbers. The relationship in $\tau = I\alpha,\ \alpha = \frac{\text{net } \tau}{I}$ is the rotational analog to Newton's second law and is very generally applicable. This equation is actually valid for *any* torque, applied to *any* object, relative to *any* axis.

As we might expect, the larger the torque is, the larger the angular acceleration is. For example, the harder a child pushes on a merry-go-round, the faster it accelerates. Furthermore, the more massive a merry-go-round, the slower it accelerates for the same torque. The basic relationship between moment of inertia and angular acceleration is that the larger the moment of inertia, the smaller is the angular acceleration. But there is an additional twist. The moment of inertia depends not only on the mass of an object, but also on its *distribution* of mass relative to the axis around which it rotates. For example, it will be much easier to accelerate a merry-go-round full of children if they stand close to its axis than if they all stand at the outer edge. The mass is the same in both cases; but the moment of inertia is much larger when the children are at the edge.

TAKE-HOME EXPERIMENT

Cut out a circle that has about a 10 cm radius from stiff cardboard. Near the edge of the circle, write numbers 1 to 12 like hours on a clock face. Position the circle so that it can rotate freely about a horizontal axis through its center, like a wheel. (You could loosely nail the circle to a wall.) Hold the circle stationary and with the number 12 positioned at the top, attach a lump of blue putty (sticky material used for fixing posters to walls) at the number 3. How large does the lump need to be to just rotate the circle? Describe how you can change the moment of inertia of the circle. How does this change affect the amount of blue putty needed at the number 3 to just rotate the circle? Change the circle's moment of inertia and then try rotating the circle by using different amounts of blue putty. Repeat this process several times.

PROBLEM-SOLVING STRATEGY FOR ROTATIONAL DYNAMICS

1. *Examine the situation to determine that torque and mass are involved in the rotation.* Draw a careful sketch of the situation.
2. *Determine the system of interest.*
3. *Draw a free body diagram*. That is, draw and label all external forces acting on the system of interest.
4. *Apply* $\text{net } \tau = I\alpha,\ \alpha = \dfrac{\text{net } \tau}{I}$, *the rotational equivalent of Newton's second law, to solve the problem*. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
5. *As always, check the solution to see if it is reasonable.*

MAKING CONNECTIONS

In statics, the net torque is zero, and there is no angular acceleration. In rotational motion, net torque is the cause of angular acceleration, exactly as in Newton's second law of motion for rotation.

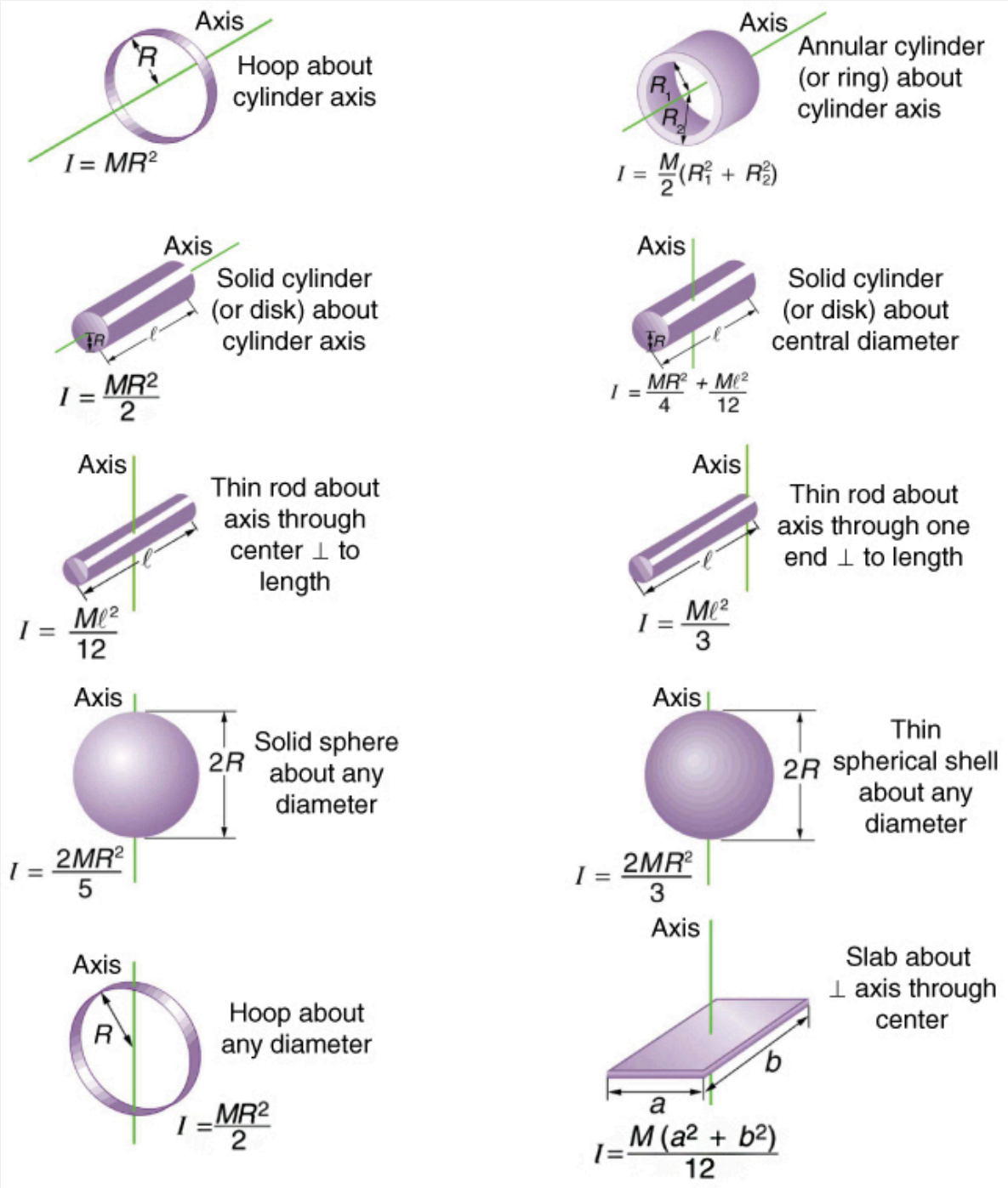


Figure 3. *Some rotational inertias.*

Example 1: Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in Figure 4. He exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible retarding friction.

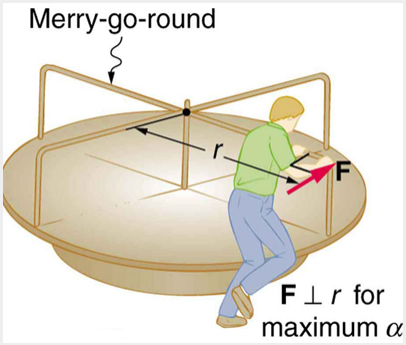


Figure 4. *A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.*

Strategy

Angular acceleration is given directly by the expression Formula does not parse

$$\alpha = \frac{\tau}{I}.$$

To solve for α, we must first calculate the torque τ (which is the same in both cases) and moment of inertia *I* (which is greater in the second case). To find the torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

$$\tau = rF\sin\theta = (1.50\text{ m})(250\text{ N}) = 375\text{ N}\cdot\text{m}.$$

Solution for (a)

The moment of inertia of a solid disk about this axis is given in Figure 3 to be

$$\frac{1}{2}MR^2,$$

where *M* = 50.0 kg and *R* = 1.50 m, so that

$$I = (0.500)(50.0\text{ kg})(1.50\text{ m})^2 = 56.25\text{ kg}\cdot\text{m}^2.$$

Now, after we substitute the known values, we find the angular acceleration to be

$$\alpha = \frac{\tau}{I} = \frac{375\text{ N}\cdot\text{m}}{56.25\text{ kg}\cdot\text{m}^2} = 6.67\frac{\text{rad}}{\text{s}^2}.$$

Solution for (b)

We expect the angular acceleration for the system to be less in this part, because the moment of inertia

is greater when the child is on the merry-go-round. To find the total moment of inertia I, we first find the child's moment of inertia I_c by considering the child to be equivalent to a point mass at a distance of 1.25 m from the axis. Then,

$$I_c = MR^2 = (18.0\ \text{kg})(1.25\ \text{m})^2 = 28.13\ \text{kg}\cdot\text{m}^2.$$

The total moment of inertia is the sum of moments of inertia of the merry-go-round and the child (about the same axis). To justify this sum to yourself, examine the definition of I:

$$I = 28.13\ \text{kg}\cdot\text{m}^2 + 56.25\ \text{kg}\cdot\text{m}^2 = 84.38\ \text{kg}\cdot\text{m}^2.$$

Substituting known values into the equation for α gives

$$\alpha = \frac{\tau}{I} = \frac{375\ \text{N}\cdot\text{m}}{84.38\ \text{kg}\cdot\text{m}^2} = 4.44\frac{rad}{s^2}.$$

Discussion

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case. Summer Olympics, here he comes! Confirmation of these numbers is left as an exercise for the reader.

Check Your Understanding

1: Torque is the analog of force and moment of inertia is the analog of mass. Force and mass are physical quantities that depend on only one factor. For example, mass is related solely to the numbers of atoms of various types in an object. Are torque and moment of inertia similarly simple?

Section Summary

- The farther the force is applied from the pivot, the greater is the angular acceleration; angular acceleration is inversely proportional to mass.
- If we exert a force F on a point mass m that is at a distance r from a pivot point and because the force is perpendicular to r, an acceleration $a = F/m$ is obtained in the direction of F. We can rearrange this equation such that

  $$F = ma,$$

  and then look for ways to relate this expression to expressions for rotational quantities. We note that $a = r\alpha$, and we substitute this expression into $F = ma$, yielding

  $$F = mr\alpha$$

- Torque is the turning effectiveness of a force. In this case, because F is perpendicular to r, torque is simply $\tau = rF$. If we multiply both sides of the equation above by r, we get torque on the left-hand side. That is,

$$rF = mr^2\alpha$$

or

$$\tau = mr^2\alpha.$$

- The moment of inertia I of an object is the sum of MR^2 for all the point masses of which it is composed. That is,

$$I = \sum mr^2.$$

- The general relationship among torque, moment of inertia, and angular acceleration is

$$\tau = I\alpha$$

or

$$\alpha = \frac{\text{net } \tau}{I}$$

Glossary

torque
the turning effectiveness of a force

rotational inertia
resistance to change of rotation. The more rotational inertia an object has, the harder it is to rotate

- moment of inertia
- mass times the square of perpendicular distance from the rotation axis; for a point mass, it is $I = mr^2$ and, because any object can be built up from a collection of point masses, this relationship is the basis for all other moments of inertia

Conceptual Questions

1: The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is $ML^2/3$. Why is this moment of inertia greater than it would be if you spun a point mass M at the location of the center of mass of the rod (at $L/2$)? (That would be $ML^2/4$.)

2: Why is the moment of inertia of a hoop that has a mass M and a radius R greater than the moment of inertia of a disk that has the same mass and radius? Why is the moment of inertia of a spherical shell that has a mass M and a radius R greater than that of a solid sphere that has the same mass and radius?

3: Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.

4: While reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the

tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle's frame?

Figure 5. *The image shows a side view of a racing bicycle. Can you see evidence in the design of the wheels on this racing bicycle that their moment of inertia has been purposely reduced? (credit: Jesús Rodriguez)*

5: A ball slides up a frictionless ramp. It is then rolled without slipping and with the same initial velocity up another frictionless ramp (with the same slope angle). In which case does it reach a greater height, and why?

Problems & Exercises

1: This problem considers additional aspects of Example 1. (a) How long does it take the father to give the merry-go-round an angular velocity of 1.50 rad/s? (b) How many revolutions must he go through to generate this velocity? (c) If he exerts a slowing force of 300 N at a radius of 1.35 m, how long would it take him to stop them?

2: Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximately a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.

3: The triceps muscle in the back of the upper arm extends the forearm. This muscle in a professional boxer exerts a force of 2.00×10^3 N with an effective perpendicular lever arm of 3.00 cm, producing an angular acceleration of the forearm of 120 rad/s^2. What is the moment of inertia of the boxer's forearm?

4: A soccer player extends her lower leg in a kicking motion by exerting a force with the muscle above the knee in the front of her leg. She produces an angular acceleration of 30.00 rad/s^2 and her lower leg has a moment of inertia of 0.750 kg·m^2. What is the force exerted by the muscle if its effective perpendicular lever arm is 1.90 cm?

5: Suppose you exert a force of 180 N tangential to a 0.280-m-radius 75.0-kg grindstone (a solid disk).

(a)What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c)

What is the angular acceleration if there is an opposing frictional force of 20.0 N exerted 1.50 cm from the axis?

6: Consider the 12.0 kg motorcycle wheel shown in Figure 6. Assume it to be approximately an annular ring with an inner radius of 0.280 m and an outer radius of 0.330 m. The motorcycle is on its center stand, so that the wheel can spin freely. (a) If the drive chain exerts a force of 2200 N at a radius of 5.00 cm, what is the angular acceleration of the wheel? (b) What is the tangential acceleration of a point on the outer edge of the tire? (c) How long, starting from rest, does it take to reach an angular velocity of 80.0 rad/s?

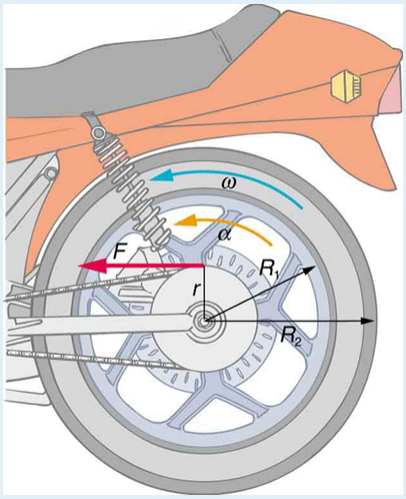


Figure 6. *A motorcycle wheel has a moment of inertia approximately that of an annular ring.*

7: Zorch, an archenemy of Superman, decides to slow Earth's rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Superman is not immediately concerned, because he knows Zorch can only exert a force of 4.00×10^7 N (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Superman time to devote to other villains.) Explicitly show how you follow the steps found in Problem-Solving Strategy for Rotational Dynamics.

8: An automobile engine can produce 200 N · m of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0 kg disk that has a 0.180 m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m. The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m. The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius.

9: Starting with the formula for the moment of inertia of a rod rotated around an axis through one end perpendicular to its length $\left(I = M\ell^2/3\right)$, prove that the moment of inertia of a rod rotated about an axis through its center perpendicular to its length is $I = M\ell^2/12$. You will find the graphics in Figure 3 useful in visualizing these rotations.

10: Unreasonable Results

A gymnast doing a forward flip lands on the mat and exerts a 500-N · m torque to slow and then reverse her angular velocity. Her initial angular velocity is 10.0 rad/s, and her moment of inertia is 0.050 kg·m^2. (a) What

time is required for her to exactly reverse her spin? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

11: Unreasonable Results

An advertisement claims that an 800-kg car is aided by its 20.0-kg flywheel, which can accelerate the car from rest to a speed of 30.0 m/s. The flywheel is a disk with a 0.150-m radius. (a) Calculate the angular velocity the flywheel must have if 95.0% of its rotational energy is used to get the car up to speed. (b) What is unreasonable about the result? (c) Which premise is unreasonable or which premises are inconsistent?

Solutions

Check Your Understanding

1: No. Torque depends on three factors: force magnitude, force direction, and point of application. Moment of inertia depends on both mass and its distribution relative to the axis of rotation. So, while the analogies are precise, these rotational quantities depend on more factors.

Problems & Exercises

1: (a) 0.338 s (b) 0.0403 rev (c) 0.313 s

3: $0.50\ \text{kg}\cdot\text{m}^2$

5: (a) $50.4\ \text{N}\cdot\text{m}$ (b) $17.1\ \text{rad/s}^2$ (c) $17.0\ \text{rad/s}^2$

7: $3.96 \times 10^{18}\ \text{s}$ or $1.26 \times 10^{11}\ \text{y}$

9:

$$I_{\text{end}} = I_{\text{center}} + m(\frac{l}{2})^2$$

$$\text{Thus, } I_{\text{center}} = I_{\text{end}} - \frac{1}{4}ml^2 = \frac{1}{3}ml^2 - \frac{1}{4}ml^2 = \frac{1}{12}ml^2$$

10: (a) 2.0 s (b) The time interval is too short. (c) The moment of inertia is much too small, by one to two orders of magnitude. A torque of $500\ \text{N}\cdot\text{m}$ is reasonable.

11: (a) 17,500 rpm (b) This angular velocity is very high for a disk of this size and mass. The radial acceleration at the edge of the disk is > 50,000 gs. (c) Flywheel mass and radius should both be much greater, allowing for a lower spin rate (angular velocity).

9.5 Rotational Kinetic Energy: Work and Energy Revisited

Summary

- Derive the equation for rotational work.
- Calculate rotational kinetic energy.
- Demonstrate the Law of Conservation of Energy.

In this module, we will learn about work and energy associated with rotational motion. Figure 1 shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable **rotational kinetic energy**.

Figure 1. *The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell)*

Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in Chapter 6.1 for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in Figure 2) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled:

$$\text{net } W = (\text{net } F)\Delta s.$$

To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by *r*, and gather terms:

$$\text{net } W = (r \text{ net } F)\frac{\Delta s}{r}.$$

We recognize that *r* net *F* = net τ and Δs/*r* = *θ*, so that

$$\text{net } W = (\text{net } \tau)\theta.$$

This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation net W = (net τ)*θ* is valid in general, even though it was derived for a special case.

To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that $\text{net } \tau = I\alpha$, so that

$$\text{net } W = I\alpha\theta.$$

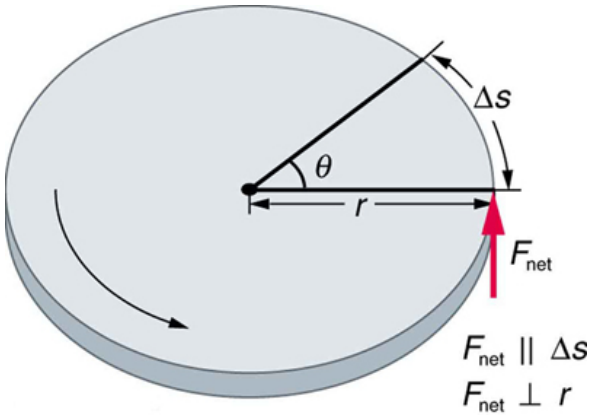


Figure 2. *The net force on this disk is kept perpendicular to its radius as the force causes the disk to rotate. The net work done is thus (net F)Δs. The net work goes into rotational kinetic energy.*

MAKING CONNECTIONS

Work and energy in rotational motion are completely analogous to work and energy in translational motion, first presented in Chapter 6 Work and Energy.

Now, we solve one of the rotational kinematics equations for $\alpha\theta$. We start with the equation

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Next, we solve for $\alpha\theta$

$$\alpha\theta = \frac{\omega^2 - \omega_0^2}{2}.$$

Substituting this into the equation for net *W* and gathering terms yields

$$\text{net } W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2.$$

This equation is the **work-energy theorem** for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term $(\frac{1}{2})I\omega^2$ to be **rotational kinetic energy** KE_{rot} for an object with a moment of inertia *I* and an angular velocity ω:

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with *I* being analogous to *m* and ω to *v*. Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in Figure 3.

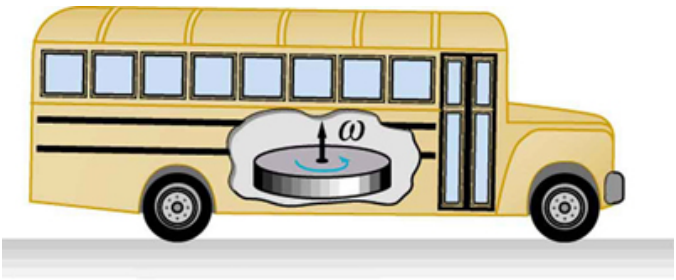


Figure 3. *Experimental vehicles, such as this bus, have been constructed in which rotational kinetic energy is stored in a large flywheel. When the bus goes down a hill, its transmission converts its gravitational potential energy into* KE_{rot}*. It can also convert translational kinetic energy, when the bus stops, into* KE_{rot}*. The flywheel's energy can then be used to accelerate, to go up another hill, or to keep the bus from going against friction.*

Example 1: Calculating the Work and Energy for Spinning a Grindstone

Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in Figure 4. In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of 1.00 rad(57.3°)? The force is kept perpendicular to the grindstone's 0.320-m radius at the point of application, and the effects of friction are negligible. (b)

What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.)

Strategy

To find the work, we can use the equation net W = (net τ)θ. We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in $\mathrm{KE_{rot}} = \frac{1}{2} I\omega^2$.

Solution for (a)

The net work is expressed in the equation

$$\text{net } W = (\text{net } \tau)\theta,$$

where net τ is the applied force multiplied by the radius (rF) because there is no retarding friction, and the force is perpendicular to r. The angle θ is given. Substituting the given values in the equation above yields

$$\begin{aligned} \text{net } W &= rF\theta = (0.320 \text{ m})(200 \text{ N})(1.00 \text{ rad}) \\ &= 64.0 \text{ N}\cdot\text{m}. \end{aligned}$$

Noting that 1 N·m = 1 J,

$$\text{net } W = 64.0 \text{ J}.$$

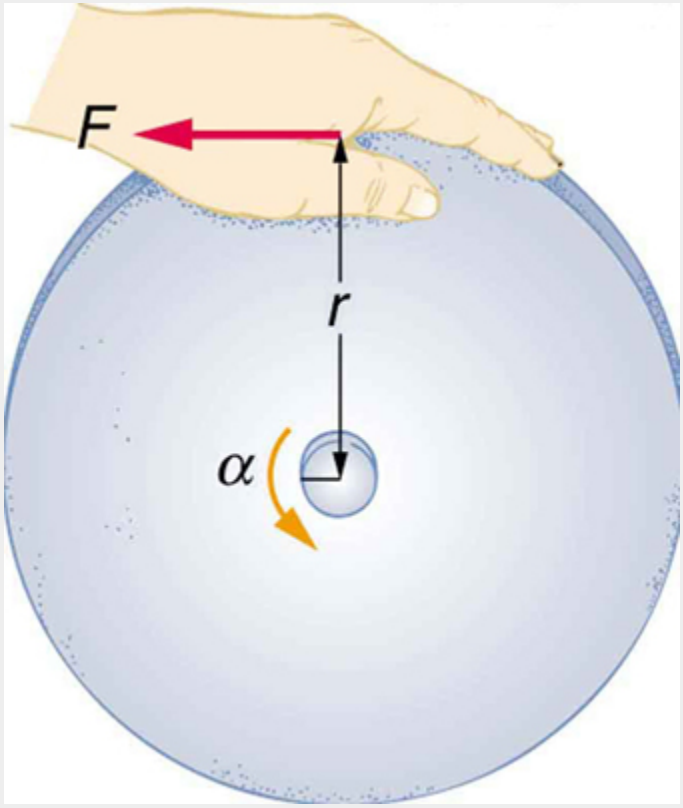


Figure 4. *A large grindstone is given a spin by a person grasping its outer edge.*

Solution for (b)

To find ω from the given information requires more than one step. We start with the kinematic relationship in the equation

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Note that $\omega_0 = 0$ because we start from rest. Taking the square root of the resulting equation gives

$$\omega = (2\alpha\theta)^{1/2}.$$

Now we need to find α. One possibility is

$$\alpha = \frac{\text{net } \tau}{I},$$

where the torque is

$$\text{net } \tau = rF = (0.320 \text{ m})(200 \text{ N}) = 64.0 \text{ N} \cdot \text{m}.$$

The formula for the moment of inertia for a disk is found in Making Connections:

$$I = \frac{1}{2}MR^2 = 0.5(85.0 \text{ kg})(0.320 \text{ m})^2 = 4.352 \text{ kg} \cdot \text{m}^2.$$

Substituting the values of torque and moment of inertia into the expression for α, we obtain

$$\alpha = \frac{64.0 \text{ N}\cdot\text{m}}{4.352 \text{ kg}\cdot\text{m}^2} = 14.7\frac{rad}{s^2}.$$

Now, substitute this value and the given value for θ into the above expression for ω:

$$\omega = (2\alpha\theta)^{1/2} = [2(14.7\frac{rad}{s^2})(1.00 \text{ rad})]^{1/2} = 5.42\frac{rad}{s}.$$

Solution for (c)

The final rotational kinetic energy is

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

Both I and ω were found above. Thus,

$$\text{KE}_{\text{rot}} = (0.5)(4.352 \text{ kg}\cdot\text{m}^2)(5.42 \text{ rad/s})^2 = 64.0 \text{ J}.$$

Discussion

The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples.

Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter's altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground.

PROBLEM-SOLVING STRATEGY FOR ROTATIONAL ENERGY

1. *Determine that energy or work is involved in the rotation*.
2. *Determine the system of interest*. A sketch usually helps.
3. *Analyze the situation to determine the types of work and energy involved*.
4. *For closed systems, mechanical energy is conserved*. That is, $\text{KE}_\text{i} + \text{PE}_\text{i} = \text{KE}_\text{f} + \text{PE}_\text{f}$. Note

that KE_i and KE_f may each include translational and rotational contributions.

5. *For open systems*, mechanical energy may not be conserved, and other forms of energy (referred to previously as OE), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary.
6. *Eliminate terms wherever possible to simplify the algebra.*
7. *Check the answer to see if it is reasonable.*

Example 2: Calculating Helicopter Energies

A typical small rescue helicopter, similar to the one in Figure 5, has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it?

Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy.

Solution for (a)

The rotational kinetic energy is

$$\text{KE}_{\text{rot}} = \frac{1}{2} I\omega^2.$$

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find KE_{rot}. The angular velocity ω is

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade will be that of a thin rod rotated about its end, found in Making Connections. The total *I* is four times this moment of inertia, because there are four blades. Thus,

$$I = 4\frac{M\ell^2}{3} = 4 \times \frac{(50.0 \text{ kg})(4.00 \text{ m})^2}{3} = 1067 \text{ kg} \cdot \text{m}^2.$$

Entering ω and *I* into the expression for rotational kinetic energy gives

$$\begin{aligned} \mathrm{KE_{rot}} &= 0.5(1067\ \mathrm{kg \cdot m^2})(31.4\ \mathrm{rad/s})^2 \\ &= 5.26 \times 10^5\ \mathrm{J} \end{aligned}$$

Solution for (b)

Entering the given values of mass and velocity for the translational kinetic energy, we obtain

$$\mathrm{KE_{trans}} = \frac{1}{2}mv^2 = 0.5(1000\ \mathrm{kg})(20.0\ \mathrm{m/s})^2 = 2.00 \times 10^5\ \mathrm{J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

$$\frac{2.00 \times 10^5\ \mathrm{J}}{5.26 \times 10^5\ \mathrm{J}} = 0.380.$$

Solution for (c)

At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies:

$$\mathrm{KE_{rot}} = \mathrm{PE_{grav}}$$

or

$$\frac{1}{2}I\omega^2 = mgh.$$

We now solve for *h* and substitute known values into the resulting equation

$$h = \frac{\frac{1}{2}I\omega^2}{mg} = \frac{5.26 \times 10^5\ \mathrm{J}}{(1000\ \mathrm{kg})(9.80\ \mathrm{m/s})^2} = 53.7\ \mathrm{m}.$$

Discussion

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades.

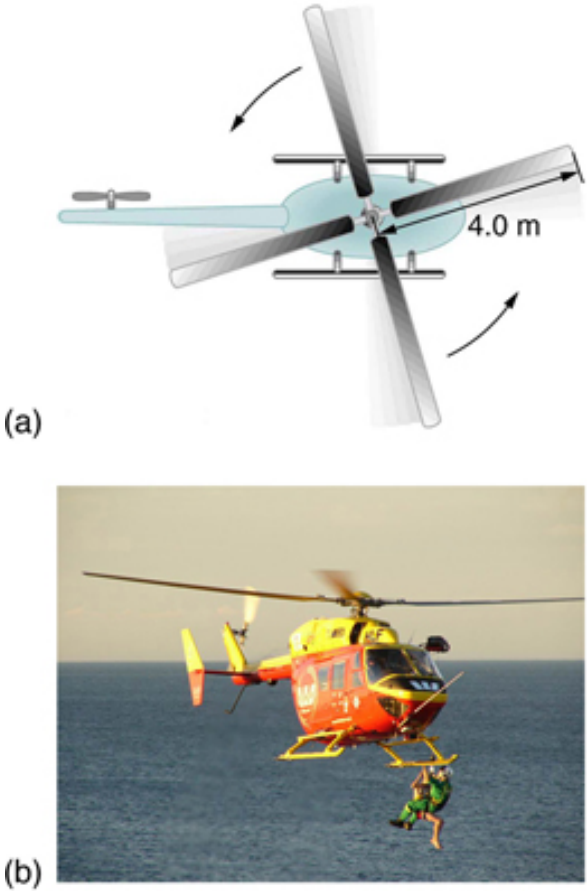


Figure 5. *The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. The second image shows a helicopter from the Auckland Westpac Rescue Helicopter Service. Over 50,000 lives have been saved since its operations beginning in 1973. Here, a water rescue operation is shown. (credit: 111 Emergency, Flickr)*

MAKING CONNECTIONS

Conservation of energy includes rotational motion, because rotational kinetic energy is another form of KE. Chapter 6 Work and Energy has a detailed treatment of conservation of energy.

How Thick Is the Soup? Or Why Don’t All Objects Roll Downhill at the Same

Rate?

One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest?

The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy PE_{grav}, which is converted entirely to KE, provided each rolls without slipping. KE, however, can take the form of KE_{trans} or KE_{rot}, and total KE is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation, leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can's original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in Figure 6.

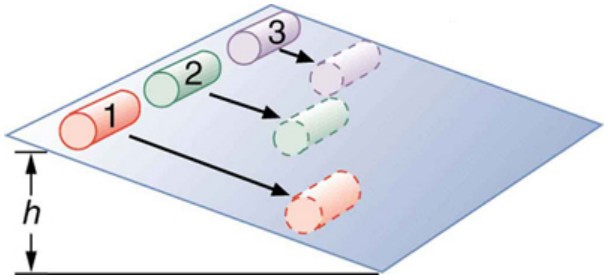


Figure 6. *Three cans of soup with identical masses race down an incline. The first can has a low friction coating and does not roll but just slides down the incline. It wins because it converts its entire PE into translational KE. The second and third cans both roll down the incline without slipping. The second can contains thin soup and comes in second because part of its initial PE goes into rotating the can (but not the thin soup). The third can contains thick soup. It comes in third because the soup rotates along with the can, taking even more of the initial PE for rotational KE, leaving less for translational KE.*

Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy gives

$$\text{PE}_{\text{i}} = \text{KE}_{\text{f}}.$$

More specifically,

$$\text{PE}_{\text{grav}} = \text{KE}_{\text{trans}} + \text{KE}_{\text{rot}}$$

or

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

So, the initial *mgh* is divided between translational kinetic energy and rotational kinetic energy; and the greater *I* is, the less energy goes into translation. If the can slides down without friction, then **ω = 0** and all the energy goes into translation; thus, the can goes faster.

TAKE-HOME EXPERIMENT

Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand.

Example 3: Calculating the Speed of a Cylinder Rolling Down an Incline

Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.

Strategy

We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with vv as the only unknown.

Solution

Conservation of energy for this situation is written as described above:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

Before we can solve for *v*, we must get an expression for *I* from Chapter 9.4 Making Connections. Because *v* and ω are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship ω = *v*/*R* into the expression. These substitutions yield

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}mR^2)(\frac{v^2}{R^2}).$$

Interestingly, the cylinder's radius *R* and mass *m* cancel, yielding

$$gh = \frac{1}{2}v^2 + \frac{1}{4}v^2 = \frac{3}{4}v^2.$$

Solving algebraically, the equation for the final velocity *v* gives

$$v = (\frac{4gh}{3})^{1/2}.$$

Substituting known values into the resulting expression yields

$$v = \frac{4(9.80\ \text{m/s}^2)(2.00\ \text{m})}{3}^{1/2} = 5.11\ \text{m/s}.$$

Discussion

Because *m* and *R* cancel, the result $v = (\frac{4}{3}gh)^{1/2}$ is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus, $\frac{1}{2}mv^2 = mgh$ and $v = (2gh)^{1/2}$, which is 22% greater than $(4gh/3)^{1/2}$. That is, the cylinder would go faster at the bottom.

Check Your Understanding

Analogy of Rotational and Translational Kinetic Energy

1: Is rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy.

PHET EXPLORATIONS: MY SOLAR SYSTEM

Build your own system of heavenly bodies and watch the gravitational ballet. With this orbit simulator, you can set initial positions, velocities, and masses of 2, 3, or 4 bodies, and then see them orbit each other.

Figure 7. *My Solar System*

Section Summary

- The rotational kinetic energy KE_{rot} for an object with a moment of inertia *I* and an angular

velocity ω is given by

$$\text{KE}_{\text{rot}} = \frac{1}{2} I\omega^2.$$

- Helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades.
- Work and energy in rotational motion are completely analogous to work and energy in translational motion.
- The equation for the **work-energy theorem** for rotational motion is,

$$\text{net } W = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0^2.$$

Glossary

work-energy theorem
: if one or more external forces act upon a rigid object, causing its kinetic energy to change from KE_1 to KE_2, then the work *W* done by the net force is equal to the change in kinetic energy

rotational kinetic energy
: the kinetic energy due to the rotation of an object. This is part of its total kinetic energy

Conceptual Questions

1: Describe the energy transformations involved when a yo-yo is thrown downward and then climbs back up its string to be caught in the user's hand.

2: What energy transformations are involved when a dragster engine is revved, its clutch let out rapidly, its tires spun, and it starts to accelerate forward? Describe the source and transformation of energy at each step.

3: The Earth has more rotational kinetic energy now than did the cloud of gas and dust from which it formed. Where did this energy come from?

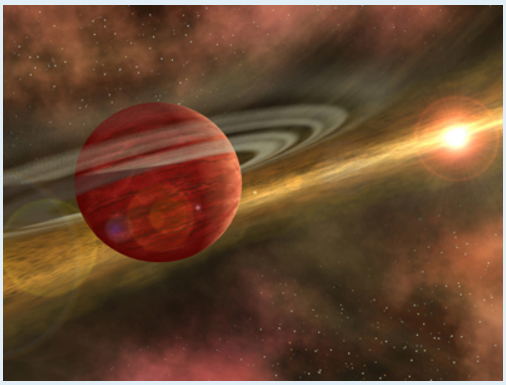


Figure 8. *An immense cloud of rotating gas and dust contracted under the influence of gravity to form the Earth and in the process rotational kinetic energy increased. (credit: NASA)*

Problems & Exercises

1: This problem considers energy and work aspects of Chapter 10.3 Example 1—use data from that example as needed. (a) Calculate the rotational kinetic energy in the merry-go-round plus child when they have an angular velocity of 20.0 rpm. (b) Using energy considerations, find the number of revolutions the father will have to push to achieve this angular velocity starting from rest. (c) Again, using energy considerations, calculate the force the father must exert to stop the merry-go-round in two revolutions

2: What is the final velocity of a hoop that rolls without slipping down a 5.00-m-high hill, starting from rest?

3: (a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?

4: Calculate the rotational kinetic energy in the motorcycle wheel (Figure 6) if its angular velocity is 120 rad/s. Assume M = 12.0 kg, R_1 = 0.280 m, and R_2 = 0.330 m.

5: A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is 0.500 kg·m^2, what is the rotational kinetic energy of the forearm?

6: While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is3.75 kg·m^2 and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter's shoe if it is 1.05 m from the hip joint? (c) Explain how the football can be given a velocity greater than the tip of the shoe (necessary for a decent kick distance).

7: A bus contains a 1500 kg flywheel (a disk that has a 0.600 m radius) and has a total mass of 10,000 kg. (a) Calculate the angular velocity the flywheel must have to contain enough energy to take the bus from rest to a speed of 20.0 m/s, assuming 90.0% of the rotational kinetic energy can be transformed into translational energy. (b) How high a hill can the bus climb with this stored energy and still have a speed of 3.00 m/s at the

top of the hill? Explicitly show how you follow the steps in the Problem-Solving Strategy for Rotational Energy.

8: A ball with an initial velocity of 8.00 m/s rolls up a hill without slipping. Treating the ball as a spherical shell, calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling.

9: While exercising in a fitness center, a man lies face down on a bench and lifts a weight with one lower leg by contacting the muscles in the back of the upper leg. (a) Find the angular acceleration produced given the mass lifted is 10.0 kg at a distance of 28.0 cm from the knee joint, the moment of inertia of the lower leg is 0.900 $kg \cdot m^2$, the muscle force is 1500 N, and its effective perpendicular lever arm is 3.00 cm. (b) How much work is done if the leg rotates through an angle of 20.0° with a constant force exerted by the muscle?

10: To develop muscle tone, a woman lifts a 2.00-kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of 60.0°. (a) What is the angular acceleration if the weight is 24.0 cm from the elbow joint, her forearm has a moment of inertia of 0.250 $kg \cdot m^2$, and the net force she exerts is 750 N at an effective perpendicular lever arm of 2.00 cm? (b) How much work does she do?

11: Consider two cylinders that start down identical inclines from rest except that one is frictionless. Thus one cylinder rolls without slipping, while the other slides frictionlessly without rolling. They both travel a short distance at the bottom and then start up another incline. (a) Show that they both reach the same height on the other incline, and that this height is equal to their original height. (b) Find the ratio of the time the rolling cylinder takes to reach the height on the second incline to the time the sliding cylinder takes to reach the height on the second incline. (c) Explain why the time for the rolling motion is greater than that for the sliding motion.

12: What is the moment of inertia of an object that rolls without slipping down a 2.00-m-high incline starting from rest, and has a final velocity of 6.00 m/s? Express the moment of inertia as a multiple of MR^2, where M is the mass of the object and R is its radius.

13: Suppose a 200-kg motorcycle has two wheels and is heading toward a hill at a speed of 30.0 m/s. (a) How high can it coast up the hill, if you neglect friction? (b) How much energy is lost to friction if the motorcycle only gains an altitude of 35.0 m before coming to rest?

14: In softball, the pitcher throws with the arm fully extended (straight at the elbow). In a fast pitch the ball leaves the hand with a speed of 139 km/h. (a) Find the rotational kinetic energy of the pitcher's arm given its moment of inertia is 0.720 $kg \cdot m^2$ and the ball leaves the hand at a distance of 0.600 m from the pivot at the shoulder. (b) What force did the muscles exert to cause the arm to rotate if their effective perpendicular lever arm is 4.00 cm and the ball is 0.156 kg?

15: Construct Your Own Problem

Consider the work done by a spinning skater pulling her arms in to increase her rate of spin. Construct a problem in which you calculate the work done with a "force multiplied by distance" calculation and compare it to the skater's increase in kinetic energy.

Solutions

Check Your Understanding

1: Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth.

Problems & Exercises

1: (a) 185 J (b) 0.0785 rev (c) 9.81 N

3: (a) $2.57 \times 10^{29}\ \text{J}$ (b) $\text{KE}_{\text{rot}} = 2.65 \times 10^{33}\ \text{J}$

5: $\text{KE}_{\text{rot}} = 434\ \text{J}$

7: (a) $128\ \text{rad/s}$ (b) $19.9\ \text{m}$

9: (a) $10.4\ \text{rad/s}^2$ (b) $\text{net}\ W = 6.11\ \text{J}$

14: (a) 1.49 kJ (b) $2.52 \times 10^4\ \text{N}$

9.6 Angular Momentum and Its Conservation

Summary

- Understand the analogy between angular momentum and linear momentum.
- Observe the relationship between torque and angular momentum.
- Apply the law of conservation of angular momentum.

Why does Earth keep on spinning? What started it spinning to begin with? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster? Questions like these have answers based in angular momentum, the rotational analog to linear momentum.

By now the pattern is clear—every rotational phenomenon has a direct translational analog. It seems quite reasonable, then, to define **angular momentum** L as

$$L = I\omega.$$

This equation is an analog to the definition of linear momentum as $p = mv$. Units for linear momentum are kg·m/s while units for angular momentum are kg·m^2/s. As we would expect, an object that has a large moment of inertia I, such as Earth, has a very large angular momentum. An object that has a large angular velocity ω, such as a centrifuge, also has a rather large angular momentum.

MAKING CONNECTIONS

Angular momentum is completely analogous to linear momentum. It has the same implications in terms of carrying rotation forward, and it is conserved when the net external torque is zero. Angular momentum, like linear momentum, is also a property of the atoms and subatomic particles.

Example 1: Calculating Angular Momentum of the Earth

Strategy

No information is given in the statement of the problem; so we must look up pertinent data before we

can calculate $L = I\omega$. First, according to Chapter 9.4 Making Connections, the formula for the moment of inertia of a sphere is

$$I = \frac{2MR^2}{5}$$

so that

$$L = I\omega = \frac{2MR^2\omega}{5}.$$

Earth's mass M is 5.979×10^{24} kg and its radius R is 6.376×10^6 m.The Earth's angular velocity ω is, of course, exactly one revolution per day, but we must covert ω to radians per second to do the calculation in SI units.

Solution

Substituting known information into the expression for L and converting ω to radians per second gives

$$\begin{aligned} L &= 0.4(5.979 \times 10^{24}\ \text{kg})(6.376 \times 10^6\ \text{m})^2\left(\frac{1\ \text{rev}}{d}\right) \\ &= 9.72 \times 10^{37}\ \text{kg}\cdot\text{m}^2\cdot\text{rev/d}. \end{aligned}$$

Substituting 2π rad for 1 rev and 8.64×10^4 s for 1 day gives

$$\begin{aligned} L &= (9.72 \times 10^{37}\ \text{kg}\cdot\text{m}^2)\left(\frac{2\pi\ \text{rad/rev}}{8.64 \times 10^4\ \text{s/d}}\right)(1\ \text{rev/d}) \\ &= 7.07 \times 10^{33}\ \text{kg}\cdot\text{m}^2/\text{s}. \end{aligned}$$

Discussion

This number is large, demonstrating that Earth, as expected, has a tremendous angular momentum. The answer is approximate, because we have assumed a constant density for Earth in order to estimate its moment of inertia.

When you push a merry-go-round, spin a bike wheel, or open a door, you exert a torque. If the torque you exert is greater than opposing torques, then the rotation accelerates, and angular momentum increases. The greater the net torque, the more rapid the increase in L. The relationship between torque and angular momentum is

$$\text{net}\ \tau = \frac{\Delta L}{\Delta t}.$$

This expression is exactly analogous to the relationship between force and linear momentum, $F = \Delta p/\Delta t$. The equation $\text{net}\ \tau = \dfrac{\Delta L}{\Delta t}$ is very fundamental and broadly applicable. It is, in fact, the rotational form of Newton's second law.

Example 2: Calculating the Torque Putting Angular Momentum Into a Lazy Susan

Figure 1 shows a Lazy Susan food tray being rotated by a person in quest of sustenance. Suppose the person exerts a 2.50 N force perpendicular to the lazy Susan's 0.260-m radius for 0.150 s. (a) What is the final angular momentum of the lazy Susan if it starts from rest, assuming friction is negligible? (b) What is the final angular velocity of the lazy Susan, given that its mass is 4.00 kg and assuming its moment of inertia is that of a disk?

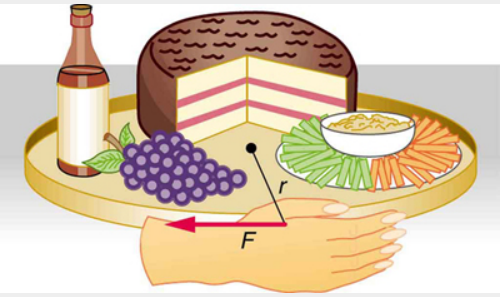


Figure 1. *A partygoer exerts a torque on a lazy Susan to make it rotate. The equation net τ = ΔL/Δt gives the relationship between torque and the angular momentum produced.*

Strategy

We can find the angular momentum by solving $\text{net } \tau = \frac{\Delta L}{\Delta t}$ for ΔL, and using the given information to calculate the torque. The final angular momentum equals the change in angular momentum, because the lazy Susan starts from rest. That is, $\Delta L = L$. To find the final velocity, we must calculate ω from the definition of L in $L = I\omega$.

Solution for (a)

Solving $\text{net } \tau = \frac{\Delta L}{\Delta t}$ for ΔL gives

$$\Delta L = \text{net } \tau \Delta t.$$

Because the force is perpendicular to r, we see that net τ = rF, so that

$$\begin{aligned} L &= \text{rF}\Delta t = (0.260 \text{ m})(2.50 \text{ N})(0.150 \text{ s}) \\ &= 9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}. \end{aligned}$$

Solution for (b)

The final angular velocity can be calculated from the definition of angular momentum,

$$L = I\omega.$$

Solving for ω and substituting the formula for the moment of inertia of a disk into the resulting equation gives

$$\omega = \frac{L}{I} = \frac{L}{\frac{1}{2}MR^2}.$$

And substituting known values into the preceding equation yields

$$\omega = \frac{9.75 \times 10^{-2}\ \text{kg} \cdot \text{m}^2/\text{s}}{(0.500)(4.00\ \text{kg})(0.260\ \text{m})} = 0.721\ \text{rad/s}.$$

Discussion

Note that the imparted angular momentum does not depend on any property of the object but only on torque and time. The final angular velocity is equivalent to one revolution in 8.71 s (determination of the time period is left as an exercise for the reader), which is about right for a lazy Susan.

Example 3: Calculating the Torque in a Kick

The person whose leg is shown in Figure 2 kicks his leg by exerting a 2000-N force with his upper leg muscle. The effective perpendicular lever arm is 2.20 cm. Given the moment of inertia of the lower leg is 1.25 kg·m^2, (a) find the angular acceleration of the leg. (b) Neglecting the gravitational force, what is the rotational kinetic energy of the leg after it has rotated through 57.3° (1.00 rad)?

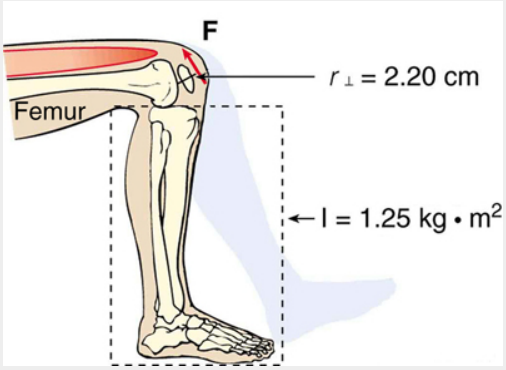


Figure 2. *The muscle in the upper leg gives the lower leg an angular acceleration and imparts rotational kinetic energy to it by exerting a torque about the knee. F is a vector that is perpendicular to r. This example examines the situation.*

Strategy

The angular acceleration can be found using the rotational analog to Newton's second law, or $\alpha = \text{net}\ \tau/I$. The moment of inertia *I* is given and the torque can be found easily from the given

force and perpendicular lever arm. Once the angular acceleration α is known, the final angular velocity and rotational kinetic energy can be calculated.

Solution to (a)

From the rotational analog to Newton's second law, the angular acceleration α is

$$\alpha = \frac{\text{net } \tau}{I}.$$

Because the force and the perpendicular lever arm are given and the leg is vertical so that its weight does not create a torque, the net torque is thus

$$\begin{aligned} \text{net } \tau &= r_{\perp} F \\ &= (0.0220 \text{ m})(2000 \text{ N}) \\ &= 44.0 \text{ N} \cdot \text{m}. \end{aligned}$$

Substituting this value for the torque and the given value for the moment of inertia into the expression for α gives

$$\alpha = \frac{44.0 \text{ N} \cdot \text{m}}{1.25 \text{ kg} \cdot \text{m}^2} = 35.2 \text{ rad/s}^2.$$

Solution to (b)

The final angular velocity can be calculated from the kinematic expression

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

or

$$\omega^2 = 2\alpha\theta$$

because the initial angular velocity is zero. The kinetic energy of rotation is

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2$$

so it is most convenient to use the value of ω^2 just found and the given value for the moment of inertia. The kinetic energy is then

$$\begin{aligned} \text{KE}_{\text{rot}} &= 0.5(1.25 \text{ kg} \cdot \text{m}^2)(70.4 \text{ rad}^2/\text{s}^2) \\ &= 44.0 \text{ J}. \end{aligned}$$

Discussion

These values are reasonable for a person kicking his leg starting from the position shown. The weight of the leg can be neglected in part (a) because it exerts no torque when the center of gravity of the lower leg is directly beneath the pivot in the knee. In part (b), the force exerted by the upper leg is so large that its torque is much greater than that created by the weight of the lower leg as it rotates. The rotational kinetic energy given to the lower leg is enough that it could give a ball a significant velocity by transferring some of this energy in a kick.

MAKING CONNECTIONS: CONSERVATION LAWS

Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

Conservation of Angular Momentum

We can now understand why Earth keeps on spinning. As we saw in the previous example, ΔL = (net τ)Δt. This equation means that, to change angular momentum, a torque must act over some period of time. Because Earth has a large angular momentum, a large torque acting over a long time is needed to change its rate of spin. So what external torques are there? Tidal friction exerts torque that is slowing Earth's rotation, but tens of millions of years must pass before the change is very significant. Recent research indicates the length of the day was 18 h some 900 million years ago. Only the tides exert significant retarding torques on Earth, and so it will continue to spin, although ever more slowly, for many billions of years.

What we have here is, in fact, another conservation law. If the net torque is *zero*, then angular momentum is constant or *conserved*. We can see this rigorously by considering $\text{net } \tau = \frac{\Delta L}{\Delta t}$ for the situation in which the net torque is zero. In that case,

$$\text{net}\tau = 0$$

implying that

$$\frac{\Delta L}{\Delta t} = 0.$$

If the change in angular momentum ΔL is zero, then the angular momentum is constant; thus,

$$L = \text{constant } (\text{net } \tau = 0)$$

or

$$L = L'(\text{net } \tau = 0).$$

These expressions are the **law of conservation of angular momentum**. Conservation laws are as scarce as they are important.

An example of conservation of angular momentum is seen in Figure 3, in which an ice skater is executing a spin. The net torque on her is very close to zero, because there is relatively little friction between her skates and the ice and because the friction is exerted very close to the pivot point. (Both F and r are small, and so τ is negligibly small.) Consequently, she can spin for quite some time. She can do something else, too. She can increase her rate of spin by pulling her arms and legs in. Why does pulling

her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

$$L = L'.$$

Expressing this equation in terms of the moment of inertia,

$$I\omega = I'\omega',$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because I' is smaller, the angular velocity ω' must increase to keep the angular momentum constant. The change can be dramatic, as the following example shows.

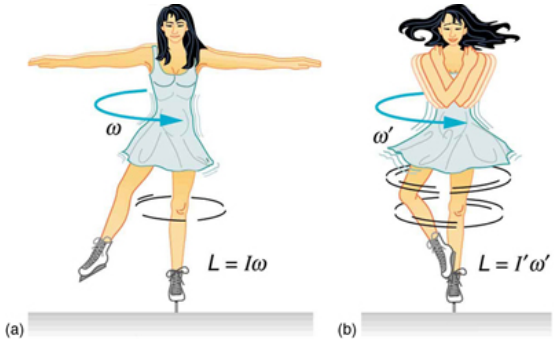


Figure 3. *(a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. In the next image, her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.*

Example 4: Calculating the Angular Momentum of a Spinning Skater

Suppose an ice skater, such as the one in Figure 3, is spinning at 0.800 rev/ s with her arms extended. She has a moment of inertia of 2.34 kg·m^2 with her arms extended and of 0.363 kg·m^2 with her arms close to her body. (These moments of inertia are based on reasonable assumptions about a 60.0-kg skater.) (a) What is her angular velocity in revolutions per second after she pulls in her arms? (b) What is her rotational kinetic energy before and after she does this?

Strategy

In the first part of the problem, we are looking for the skater's angular velocity ω' after she has pulled in her arms. To find this quantity, we use the conservation of angular momentum and note that the moments of inertia and initial angular velocity are given. To find the initial and final kinetic energies, we use the definition of rotational kinetic energy given by

$$\mathrm{KE}_{\mathrm{rot}} = \frac{1}{2}I\omega^2.$$

Solution for (a)

Because torque is negligible (as discussed above), the conservation of angular momentum given in $I\omega = I'\omega'$ is applicable. Thus,

$$L = L'$$

or

$$I\omega = I'\omega'$$

Solving for ω' and substituting known values into the resulting equation gives

$$\begin{aligned}\omega' &= \frac{I}{I'}\omega = \left(\frac{2.34\ \text{kg}\cdot\text{m}^2}{0.363\ \text{kg}\cdot\text{m}^2}\right)(0.800\ \text{rev/s}) \\ &= 5.16\ \text{rev/s}.\end{aligned}$$

Solution for (b)

Rotational kinetic energy is given by

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The initial value is found by substituting known values into the equation and converting the angular velocity to rad/s:

$$\begin{aligned}\text{KE}_{\text{rot}} &= (0.5)(2.34\ \text{kg}\cdot\text{m}^2)((0.800\ \text{rev/s})(2\pi\ \text{rad/rev}))^2 \\ &= 29.6\ \text{J}.\end{aligned}$$

The final rotational kinetic energy is

$$\text{KE}'_{\text{rot}} = \frac{1}{2}I'\omega'^2$$

Substituting known values into this equation gives

$$\begin{aligned}\text{KE}'_{\text{rot}} &= (0.5)(0.363\ \text{kg}\cdot\text{m}^2)[(5.16\ \text{rev/s})(2\pi\ \text{rad/rev})]^2 \\ &= 191\ \text{J}.\end{aligned}$$

Discussion

In both parts, there is an impressive increase. First, the final angular velocity is large, although most world-class skaters can achieve spin rates about this great. Second, the final kinetic energy is much greater than the initial kinetic energy. The increase in rotational kinetic energy comes from work done by the skater in pulling in her arms. This work is internal work that depletes some of the skater's food energy.

There are several other examples of objects that increase their rate of spin because something reduced their moment of inertia. Tornadoes are one example. Storm systems that create tornadoes are slowly rotating. When the radius of rotation narrows, even in a local region, angular velocity increases, sometimes to the furious level of a tornado. Earth is another example. Our planet was born from a huge cloud of gas and dust, the rotation of which came from turbulence in an even larger cloud. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result. (See Figure 4.)

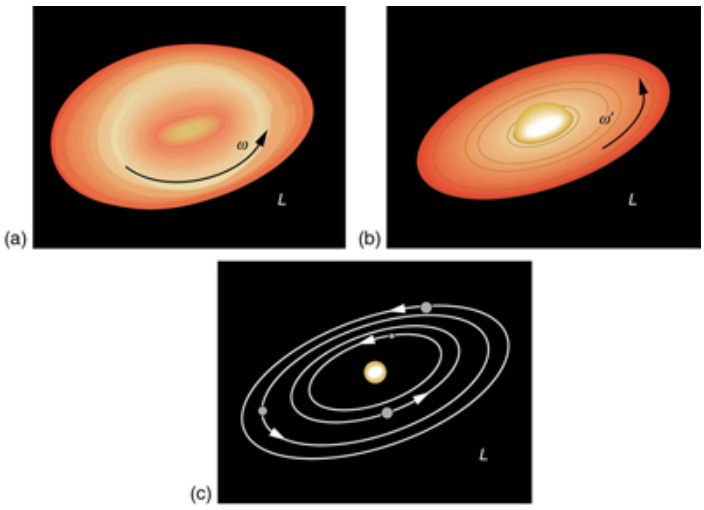


Figure 4. *The Solar System coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud.*

In case of human motion, one would not expect angular momentum to be conserved when a body interacts with the environment as its foot pushes off the ground. Astronauts floating in space aboard the International Space Station have no angular momentum relative to the inside of the ship if they are motionless. Their bodies will continue to have this zero value no matter how they twist about as long as they do not give themselves a push off the side of the vessel.

Check Your Understanding

1: Is angular momentum completely analogous to linear momentum? What, if any, are their differences?

Section Summary

- Every rotational phenomenon has a direct translational analog , likewise angular momentum L can be defined as $L = I\omega$.
- This equation is an analog to the definition of linear momentum as $p = mv$. The relationship between torque and angular momentum is $\text{net } \tau = \frac{\Delta L}{\Delta t}$.
- Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

Glossary

angular momentum
the product of moment of inertia and angular velocity

law of conservation of angular momentum
angular momentum is conserved, i.e., the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system

Conceptual Questions

1: When you start the engine of your car with the transmission in neutral, you notice that the car rocks in the opposite sense of the engine's rotation. Explain in terms of conservation of angular momentum. Is the angular momentum of the car conserved for long (for more than a few seconds)?

2: Suppose a child walks from the outer edge of a rotating merry-go round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer.

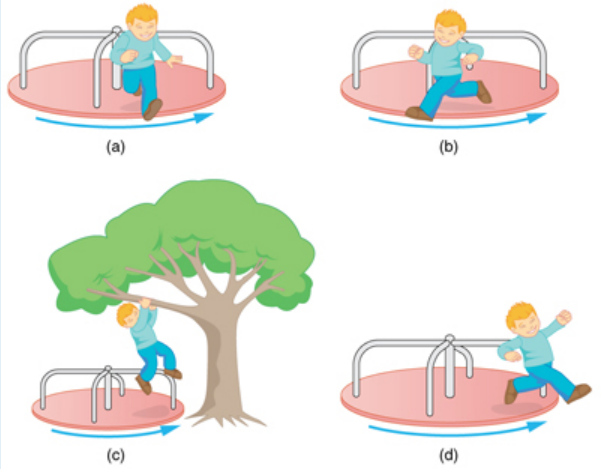


Figure 5. *A child may jump off a merry-go-round in a variety of directions.*

3: Suppose a child gets off a rotating merry-go-round. Does the angular velocity of the merry-go-round increase, decrease, or remain the same if: (a) He jumps off radially? (b) He jumps backward to land motionless? (c) He jumps straight up and hangs onto an overhead tree branch? (d) He jumps off forward, tangential to the edge? Explain your answers. (Refer to Figure 5).

4: Helicopters have a small propeller on their tail to keep them from rotating in the opposite direction of their main lifting blades. Explain in terms of Newton's third law why the helicopter body rotates in the opposite direction to the blades.

5: Whenever a helicopter has two sets of lifting blades, they rotate in opposite directions (and there will be no tail propeller). Explain why it is best to have the blades rotate in opposite directions.

6: Describe how work is done by a skater pulling in her arms during a spin. In particular, identify the force she exerts on each arm to pull it in and the distance each moves, noting that a component of the force is in the direction moved. Why is angular momentum not increased by this action?

7: When there is a global heating trend on Earth, the atmosphere expands and the length of the day increases very slightly. Explain why the length of a day increases.

8: Nearly all conventional piston engines have flywheels on them to smooth out engine vibrations caused by the thrust of individual piston firings. Why does the flywheel have this effect?

9: Jet turbines spin rapidly. They are designed to fly apart if something makes them seize suddenly, rather than transfer angular momentum to the plane's wing, possibly tearing it off. Explain how flying apart conserves angular momentum without transferring it to the wing.

10: An astronaut tightens a bolt on a satellite in orbit. He rotates in a direction opposite to that of the bolt, and the satellite rotates in the same direction as the bolt. Explain why. If a handhold is available on the satellite, can this counter-rotation be prevented? Explain your answer.

11: Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down. Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momenta.

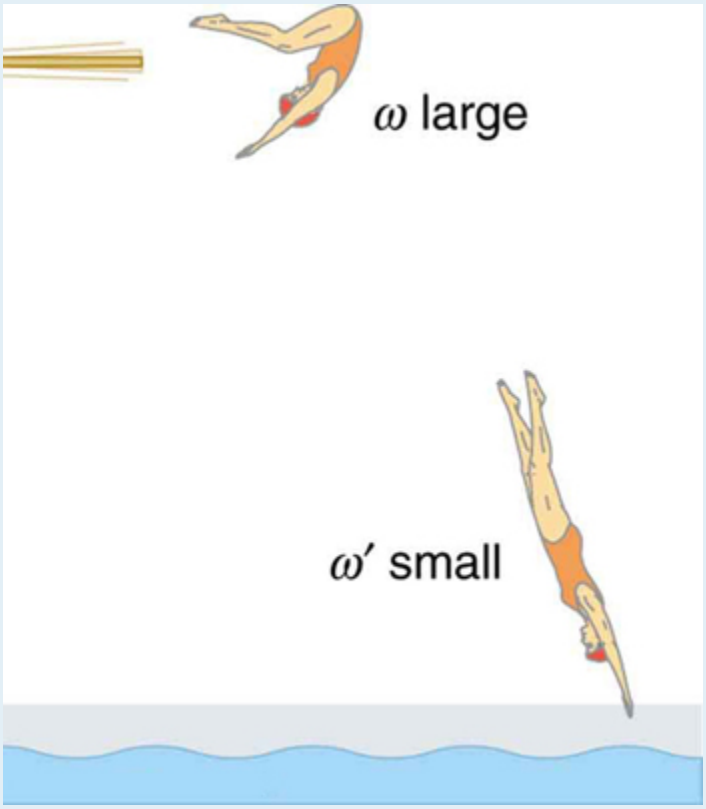


Figure 6. *The diver spins rapidly when curled up and slows when she extends her limbs before entering the water.*

12: Draw a free body diagram to show how a diver gains angular momentum when leaving the diving board.

13: In terms of angular momentum, what is the advantage of giving a football or a rifle bullet a spin when throwing or releasing it?

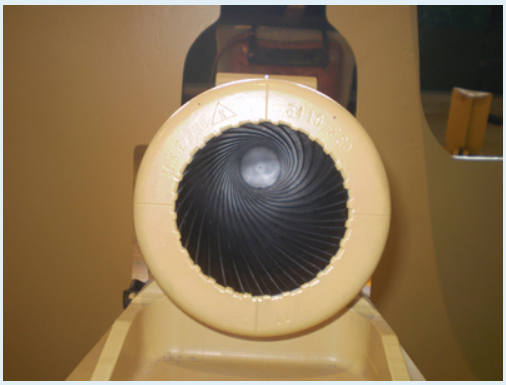


Figure 7. *The image shows a view down the barrel of a cannon, emphasizing its rifling. Rifling in the barrel of a canon causes the projectile to spin just as is the case for rifles (hence the name for the grooves in the barrel). (credit: Elsie esq., Flickr)*

Problems & Exercises

1: (a) Calculate the angular momentum of the Earth in its orbit around the Sun.

(b) Compare this angular momentum with the angular momentum of Earth on its axis.

2: (a) What is the angular momentum of the Moon in its orbit around Earth?

(b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.

(c) Discuss whether the values found in parts (a) and (b) seem consistent with the fact that tidal effects with Earth have caused the Moon to rotate with one side always facing Earth.

3: Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What angular momentum is given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque the entire time?

4: A playground merry-go-round has a mass of 120 kg and a radius of 1.80 m and it is rotating with an angular velocity of 0.500 rev/s. What is its angular velocity after a 22.0-kg child gets onto it by grabbing its outer edge? The child is initially at rest.

5: Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 kg moves to the center of the merry-go-round, what is the new angular velocity in rpm?

6: (a) Calculate the angular momentum of an ice skater spinning at 6.00 rev/s given his moment of inertia is 0.400 $kg \cdot m^2$. (b) He reduces his rate of spin (his angular velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to 1.25 rev/s. (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to 3.00 rev/s. What average torque was exerted if this takes 15.0 s?

7: Construct Your Own Problem

Consider the Earth-Moon system. Construct a problem in which you calculate the total angular momentum of the system including the spins of the Earth and the Moon on their axes and the orbital angular momentum of the Earth-Moon system in its nearly monthly rotation. Calculate what happens to the Moon's orbital radius if the Earth's rotation decreases due to tidal drag. Among the things to be considered are the amount by which the Earth's rotation slows and the fact that the Moon will continue to have one side always facing the Earth.

Solutions

Check Your Understanding

1: Yes, angular and linear momentums are completely analogous. While they are exact analogs they have different units and are not directly inter-convertible like forms of energy are.

Problems & Exercises

1: (a) $2.66 \times 10^{40}\ \text{kg}\cdot\text{m}^2/\text{s}$ (b) $7.07 \times 10^{33}\ \text{kg}\cdot\text{m}^2/\text{s}$

The angular momentum of the Earth in its orbit around the Sun is 3.77×10^6 times larger than the angular momentum of the Earth around its axis.

3: $22.5\ \text{kg}\cdot\text{m}^2/\text{s}$

5: 25.3 rpm

9.7 Gyroscopic Effects: Vector Aspects of Angular Momentum

Summary

- Describe the right-hand rule to find the direction of angular velocity, momentum, and torque.
- Explain the gyroscopic effect.
- Study how Earth acts like a gigantic gyroscope.

Angular momentum is a vector and, therefore, *has direction as well as magnitude*. Torque affects both the direction and the magnitude of angular momentum. What is the direction of the angular momentum of a rotating object like the disk in Figure 1? The figure shows the **right-hand rule** used to find the direction of both angular momentum and angular velocity. Both $\vec{\mathrm{L}}$ and $\vec{\omega}$ are vectors—each has direction and magnitude. Both can be represented by arrows. The right-hand rule defines both to be perpendicular to the plane of rotation in the direction shown. Because angular momentum is related to angular velocity by $\vec{\mathrm{L}} = I\vec{\omega}$, the direction of $\vec{\mathrm{L}}$ is the same as the direction of $\vec{\omega}$. Notice in the figure that both point along the axis of rotation.

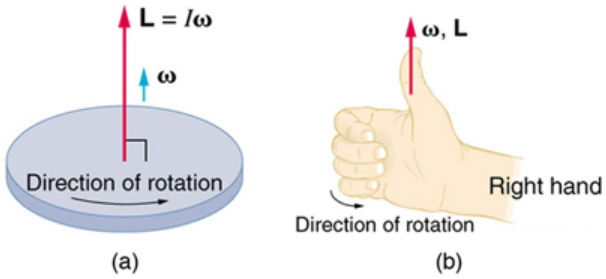


Figure 1. *Figure (a) shows a disk is rotating counterclockwise when viewed from above. Figure (b) shows the right-hand rule. The direction of angular velocity ω size and angular momentum L are defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation as shown.*

Now, recall that torque changes angular momentum as expressed by

$$\text{net } \tau = \frac{\Delta\vec{\mathrm{L}}}{\Delta t}.$$

This equation means that the direction of $\Delta\vec{\mathrm{L}}$ is the same as the direction of the torque τ that creates it. This result is illustrated in Figure 2, which shows the direction of torque and the angular momentum it creates.

Let us now consider a bicycle wheel with a couple of handles attached to it, as shown in Figure 3. (This device is popular in demonstrations among physicists, because it does unexpected things.) With the wheel rotating as shown, its angular momentum is to the woman's left. Suppose the person holding the wheel tries to rotate it as in the figure. Her natural expectation is that the wheel will rotate in the direction she pushes it—but what happens is quite different. The forces exerted create a torque that is horizontal toward the person, as shown in Figure 3(a). This torque creates a change in angular momentum $\vec{L}$ in the same direction, perpendicular to the original angular momentum $\vec{L}$, thus changing the direction of $\vec{L}$ but not the magnitude of $\vec{L}$. Figure 3 shows how $\Delta\vec{L}$ and $\vec{L}$ add, giving a new angular momentum with direction that is inclined more toward the person than before. The axis of the wheel has thus moved *perpendicular to the forces exerted on it*, instead of in the expected direction.

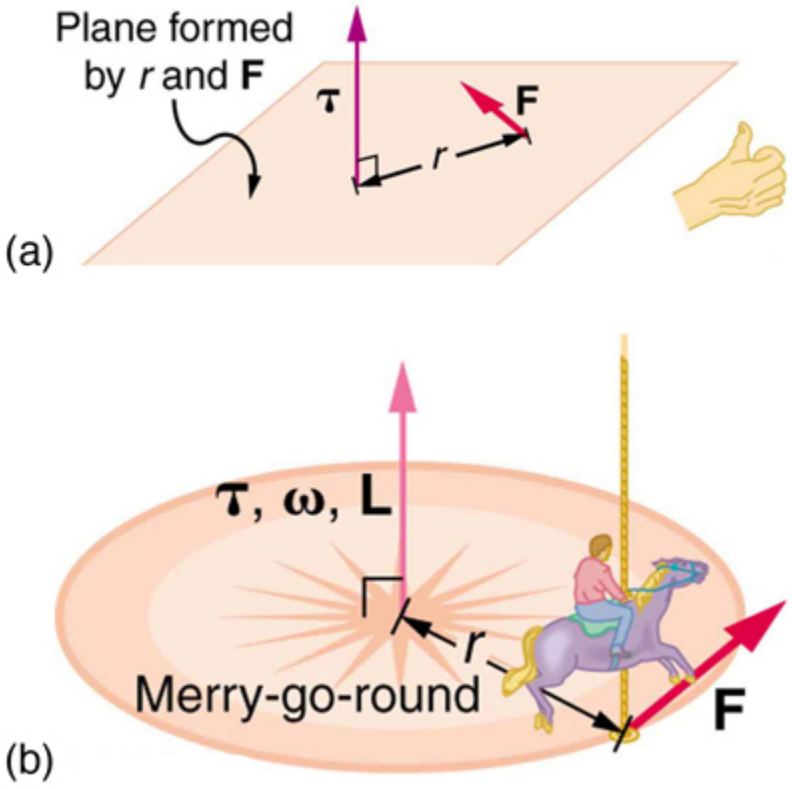


Figure 2. *In figure (a), the torque is perpendicular to the plane formed by r and F and is the direction your right thumb would point to if you curled your fingers in the direction of F. Figure (b) shows that the direction of the torque is the same as that of the angular momentum it produces.*

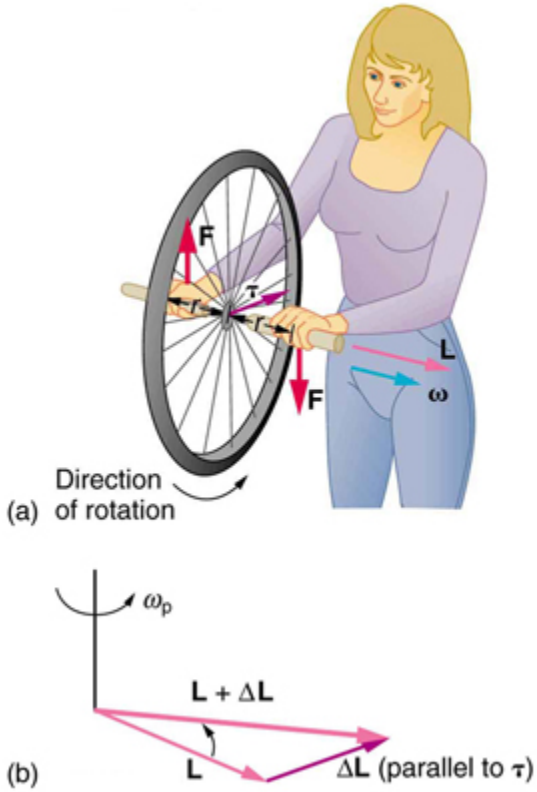


Figure 3. *In figure (a), a person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum ΔL in exactly the same direction. Figure (b) shows a vector diagram depicting how ΔL and L add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.*

This same logic explains the behavior of gyroscopes. Figure 4 shows the two forces acting on a spinning gyroscope. The torque produced is perpendicular to the angular momentum, thus the direction of the torque is changed, but not its magnitude. The gyroscope *precesses* around a vertical axis, since the torque is always horizontal and perpendicular to $\vec{\mathrm{L}}$. If the gyroscope is *not* spinning, it acquires angular momentum in the direction of the torque ($\vec{\mathrm{L}} = \Delta\vec{\mathrm{L}}$), and it rotates around a horizontal axis, falling over just as we would expect.

Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star. But Earth is slowly precessing (once in about 26,000 years) due to the torque of the Sun and the Moon on its nonspherical shape.

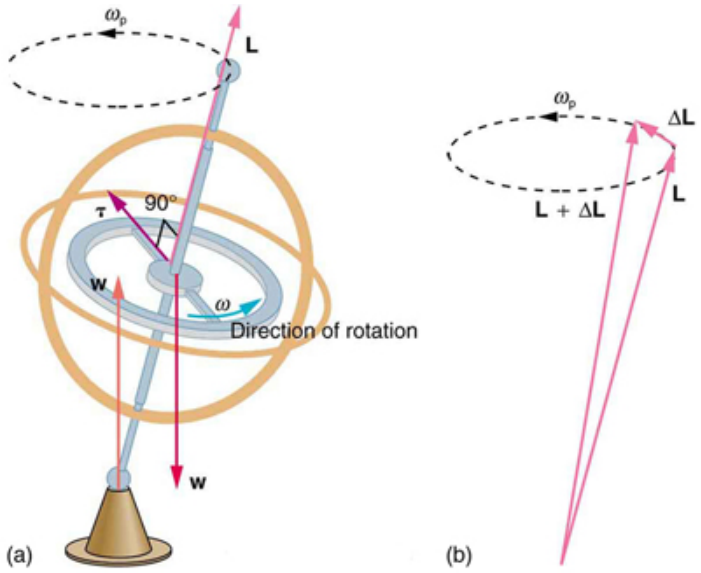


Figure 4. *As seen in figure (a), the forces on a spinning gyroscope are its weight and the supporting force from the stand. These forces create a horizontal torque on the gyroscope, which create a change in angular momentum ΔL that is also horizontal. In figure (b), ΔL and L add to produce a new angular momentum with the same magnitude, but different direction, so that the gyroscope precesses in the direction shown instead of falling over.*

Check Your Understanding

1: Rotational kinetic energy is associated with angular momentum? Does that mean that rotational kinetic energy is a vector?

Section Summary

- Torque is perpendicular to the plane formed by r and $\vec{\mathrm{F}}$ and is the direction your right thumb would point if you curled the fingers of your right hand in the direction of $\vec{\mathrm{F}}$. The direction of the torque is thus the same as that of the angular momentum it produces.
- The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to $\vec{\mathrm{L}}$. If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque ($\vec{\mathrm{L}} = \Delta\vec{\mathrm{L}}$), and it rotates about a horizontal axis, falling over just as we would expect.
- Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star.

Glossary

right-hand rule
direction of angular velocity ω and angular momentum L in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation

Conceptual Questions

1: While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

2: Gyroscopes used in guidance systems to indicate directions in space must have an angular momentum that does not change in direction. Yet they are often subjected to large forces and accelerations. How can the direction of their angular momentum be constant when they are accelerated?

Problems & Exercises

1: Integrated Concepts

The axis of Earth makes a 23.5° angle with a direction perpendicular to the plane of Earth's orbit. As shown in Figure 5, this axis precesses, making one complete rotation in 25,780 y.

(a) Calculate the change in angular momentum in half this time.

(b) What is the average torque producing this change in angular momentum?

(c) If this torque were created by a single force (it is not) acting at the most effective point on the equator, what would its magnitude be?

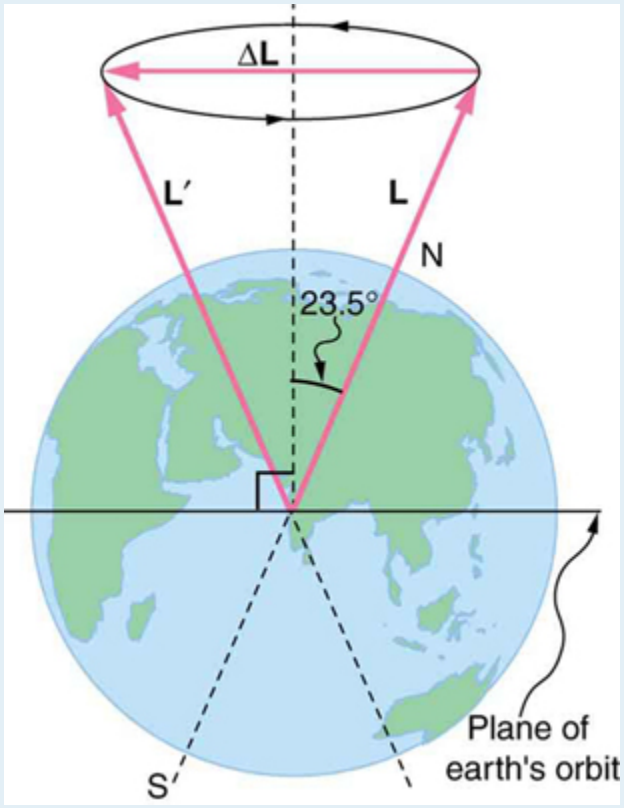


Figure 5. *The Earth's axis slowly precesses, always making an angle of 23.5° with the direction perpendicular to the plane of Earth's orbit. The change in angular momentum for the two shown positions is quite large, although the magnitude L is unchanged.*

Solutions

Check Your Understanding

1: No, energy is always a scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Problems & Exercises

1: (a) $5.64 \times 10^{33}\ \text{kg}\cdot\ \text{m}^2/\text{s}$ (b) $1.39 \times 10^{22}\ \text{N}\cdot\ \text{m}$ (c) $2.17 \times 10^{15}\ \text{N}$

9.8 Optional: Collisions of Extended Bodies in Two Dimensions

Summary

- Observe collisions of extended bodies in two dimensions.
- Examine collision at the point of percussion.

Bowling pins are sent flying and spinning when hit by a bowling ball—angular momentum as well as linear momentum and energy have been imparted to the pins. (See Figure 1). Many collisions involve angular momentum. Cars, for example, may spin and collide on ice or a wet surface. Baseball pitchers throw curves by putting spin on the baseball. A tennis player can put a lot of top spin on the tennis ball which causes it to dive down onto the court once it crosses the net. We now take a brief look at what happens when objects that can rotate collide.

Consider the relatively simple collision shown in Figure 2, in which a disk strikes and adheres to an initially motionless stick nailed at one end to a frictionless surface. After the collision, the two rotate about the nail. There is an unbalanced external force on the system at the nail. This force exerts no torque because its lever arm r is zero. Angular momentum is therefore conserved in the collision. Kinetic energy is not conserved, because the collision is inelastic. It is possible that momentum is not conserved either because the force at the nail may have a component in the direction of the disk's initial velocity. Let us examine a case of rotation in a collision in Example 1.

Figure 1. *The bowling ball causes the pins to fly, some of them spinning violently. (credit: Tinou Bao, Flickr)*

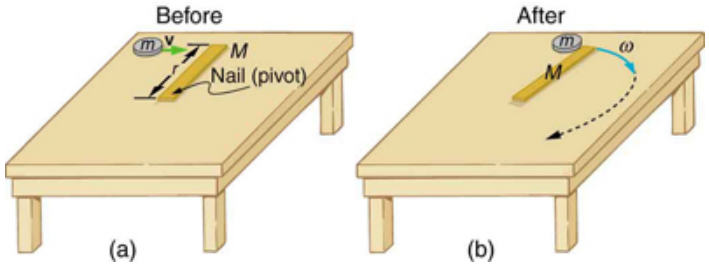


Figure 2. *(a) A disk slides toward a motionless stick on a frictionless surface. (b) The disk hits the stick at one end and adheres to it, and they rotate together, pivoting around the nail. Angular momentum is conserved for this inelastic collision because the surface is frictionless and the unbalanced external force at the nail exerts no torque.*

Example 1: Rotation in a Collision

Suppose the disk in Figure 2 has a mass of 50.0 g and an initial velocity of 30.0 m/s when it strikes the stick that is 1.20 m long and 2.00 kg.

(a) What is the angular velocity of the two after the collision?

(b) What is the kinetic energy before and after the collision?

(c) What is the total linear momentum before and after the collision?

Strategy for (a)

We can answer the first question using conservation of angular momentum as noted. Because angular momentum is $I\omega$, we can solve for angular velocity.

Solution for (a)

Conservation of angular momentum states

$$L = L',$$

where primed quantities stand for conditions after the collision and both momenta are calculated relative to the pivot point. The initial angular momentum of the system of stick-disk is that of the disk just before it strikes the stick. That is,

$$L = I\omega,$$

where I is the moment of inertia of the disk and ω is its angular velocity around the pivot point. Now, $I = mr^2$ (taking the disk to be approximately a point mass) and $\boldsymbol{\omega = v/r}$, so that

$$L = mr^2\frac{v}{r} = mvr.$$

After the collision,

$$L' = I'\omega'.$$

It is ω' that we wish to find. Conservation of angular momentum gives

$$I'\omega' = mvr.$$

Rearranging the equation yields

$$\omega' = \frac{mvr}{I'},$$

where I' is the moment of inertia of the stick and disk stuck together, which is the sum of their individual moments of inertia about the nail. The formula for a rod rotating around one end to be $I = Mr^{2/3}$. Thus,

$$I' = mr^2 + \frac{Mr^2}{3} = (m + \frac{M}{3})r^2.$$

Entering known values in this equation yields,

$$I' = (0.0500\ \text{kg} + 0.667\ \text{kg})(1.20\ \text{m})^2 = 1.032\ \text{kg}\cdot\text{m}^2.$$

The value of I' is now entered into the expression for ω′, which yields

$$\begin{aligned}\omega' = \frac{mvr}{I'} &= \frac{(0.0500\ \text{kg})(30.0\ \text{m/s})(1.20\ \text{m})}{1.032\ \text{kg}\cdot\text{m}^2} \\ &= 1.744\ \text{rad/s} \approx 1.74\ \text{rad/s}.\end{aligned}$$

Strategy for (b)

The kinetic energy before the collision is the incoming disk's translational kinetic energy, and after the collision, it is the rotational kinetic energy of the two stuck together.

Solution for (b)

First, we calculate the translational kinetic energy by entering given values for the mass and speed of the incoming disk.

$$\text{KE} = \frac{1}{2}mv^2 = (0.500)(0.0500\ \text{kg})(30.0\ \text{m/s})^2 = 22.5\ \text{J}$$

After the collision, the rotational kinetic energy can be found because we now know the final angular velocity and the final moment of inertia. Thus, entering the values into the rotational kinetic energy equation gives

$$\begin{aligned}\text{KE}' = \frac{1}{2}I'\omega'^2 &= (0.5)(1.032\ \text{kg}\cdot\text{m}^2)(1.744\frac{\text{rad}}{\text{s}})^2 \\ &= 1.57\ \text{J}.\end{aligned}$$

Strategy for (c)

The linear momentum before the collision is that of the disk. After the collision, it is the sum of the disk's momentum and that of the center of mass of the stick.

Solution of (c)

Before the collision, then, linear momentum is

$$p = mv = (0.0500\ \text{kg})(30.0\ \text{m/s}) = 1.50\ \text{kg}\cdot\text{m/s}.$$

After the collision, the disk and the stick's center of mass move in the same direction. The total linear momentum is that of the disk moving at a new velocity $\boldsymbol{v' = r\omega'}$ plus that of the stick's center of mass, which moves at half this speed because $v_{\text{CM}} = (\frac{r}{2})\omega' = \frac{v'}{2}$. Thus,

$$p' = mv' + Mv_{\text{CM}} = mv' + \frac{Mv'}{2}.$$

Gathering similar terms in the equation yields,

$$p' = (m + \frac{M}{2})v'$$

so that

$$p' = (m + \frac{M}{2})r\omega'.$$

Substituting known values into the equation,

$$p' = (1.050\ \text{kg})(1.20\ \text{m})(1.744\ \text{rad/s}) = 2.20\ \text{kg}\cdot\text{m/s}.$$

Discussion

First note that the kinetic energy is less after the collision, as predicted, because the collision is inelastic. More surprising is that the momentum after the collision is actually greater than before the collision. This result can be understood if you consider how the nail affects the stick and vice versa. Apparently, the stick pushes backward on the nail when first struck by the disk. The nail's reaction (consistent with Newton's third law) is to push forward on the stick, imparting momentum to it in the same direction in which the disk was initially moving, thereby increasing the momentum of the system.

The above example has other implications. For example, what would happen if the disk hit very close to the nail? Obviously, a force would be exerted on the nail in the forward direction. So, when the stick is struck at the end farthest from the nail, a backward force is exerted on the nail, and when it is hit at the end nearest the nail, a forward force is exerted on the nail. Thus, striking it at a certain point in between produces no force on the nail. This intermediate point is known as the *percussion point*.

An analogous situation occurs in tennis as seen in Figure 3. If you hit a ball with the end of your racquet, the handle is pulled away from your hand. If you hit a ball much farther down, for example, on the shaft of the racquet, the handle is pushed into your palm. And if you hit the ball at the racquet's percussion point (what some people call the "sweet spot"), then little or *no* force is exerted on your hand, and there is less vibration, reducing chances of a tennis elbow. The same effect occurs for a baseball bat.

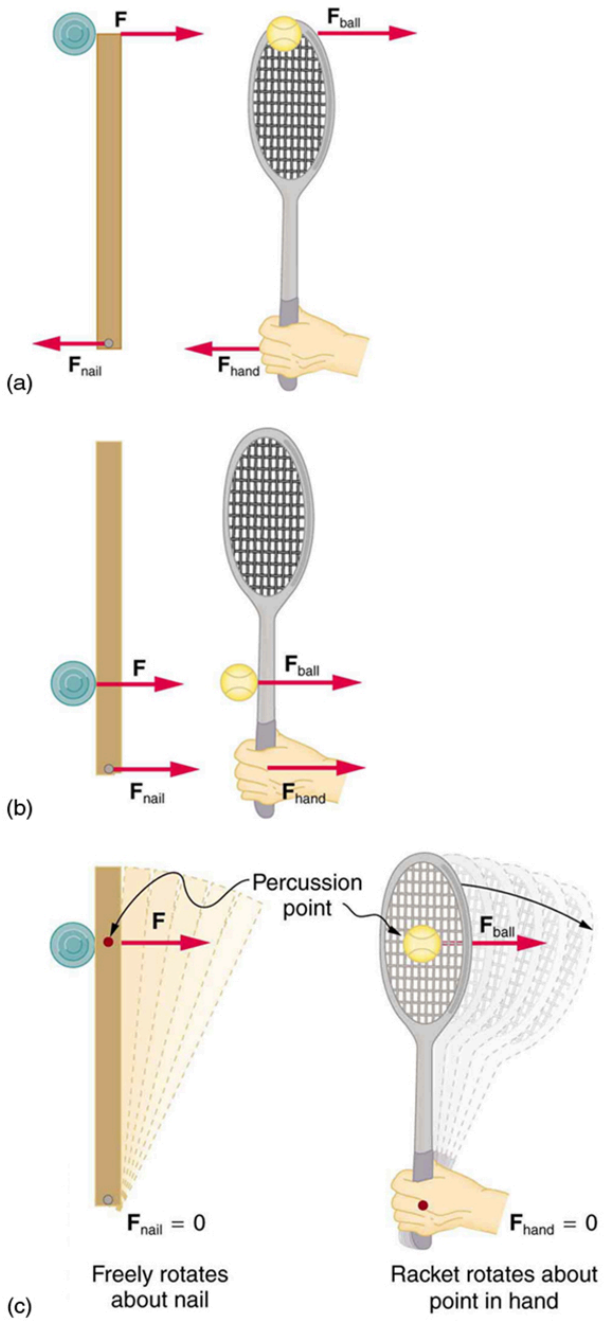


Figure 3. *A disk hitting a stick is compared to a tennis ball being hit by a racquet. (a) When the ball strikes the racquet near the end, a backward force is exerted on the hand. (b) When the racquet is struck much farther down, a forward force is exerted on the hand. (c) When the racquet is struck at the percussion point, no force is delivered to the hand.*

Check Your Understanding

1: Is rotational kinetic energy a vector? Justify your answer.

Section Summary

- Angular momentum L is analogous to linear momentum and is given by $L = I\omega$.
- Angular momentum is changed by torque, following the relationship $\text{net } \tau = \frac{\Delta L}{\Delta t}$.
- Angular momentum is conserved if the net torque is zero L = constant (net $\tau = 0$) or $L = L'$ (net $\tau=0$). This equation is known as the law of conservation of angular momentum, which may be conserved in collisions.

Conceptual Questions

1: Describe two different collisions—one in which angular momentum is conserved, and the other in which it is not. Which condition determines whether or not angular momentum is conserved in a collision?

2: Suppose an ice hockey puck strikes a hockey stick that lies flat on the ice and is free to move in any direction. Which quantities are likely to be conserved: angular momentum, linear momentum, or kinetic energy (assuming the puck and stick are very resilient)?

3: While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

Problems & Exercises

Problems & Exercises

1: Repeat Example 1 in which the disk strikes and adheres to the stick 0.100 m from the nail.

2: Repeat Example 1 in which the disk originally spins clockwise at 1000 rpm and has a radius of 1.50 cm.

3: Twin skaters approach one another as shown in Figure 4 and lock hands. (a) Calculate their final angular velocity, given each had an initial speed of 2.50 m/s relative to the ice. Each has a mass of 70.0 kg, and each has a center of mass located 0.800 m from their locked hands. You may approximate their moments of inertia to be that of point masses at this radius. (b) Compare the initial kinetic energy and final kinetic energy.

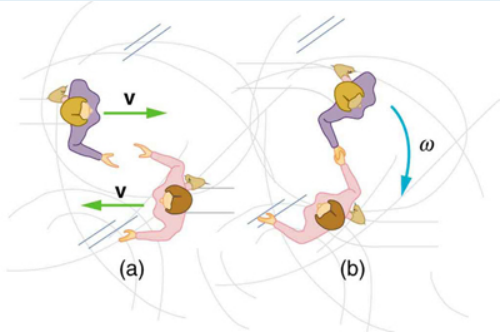


Figure 4. *Twin skaters approach each other with identical speeds. Then, the skaters lock hands and spin.*

4: Suppose a 0.250-kg ball is thrown at 15.0 m/s to a motionless person standing on ice who catches it with an outstretched arm as shown in Figure 5.

(a) Calculate the final linear velocity of the person, given his mass is 70.0 kg.

(b) What is his angular velocity if each arm is 5.00 kg? You may treat the ball as a point mass and treat the person's arms as uniform rods (each has a length of 0.900 m) and the rest of his body as a uniform cylinder of radius 0.180 m. Neglect the effect of the ball on his center of mass so that his center of mass remains in his geometrical center.

(c) Compare the initial and final total kinetic energies.

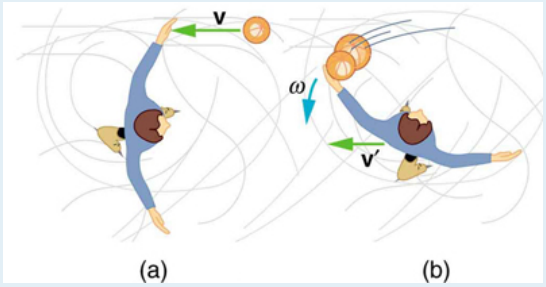


Figure 5. *The figure shows the overhead view of a person standing motionless on ice about to catch a ball. Both arms are outstretched. After catching the ball, the skater recoils and rotates.*

5: Repeat Example 1 in which the stick is free to have translational motion as well as rotational motion.

Solutions

Check Your Understanding

1: No, energy is always scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Problems & Exercises

1: (a) $0.156\ \text{rad/s}$ (b) $1.17 \times 10^{-2}\ \text{J}$ (c) $0.188\ \text{kg}\cdot\text{m/s}$

3: (a) 3.13 rad/s (b) $\text{Initial KE} = 438\ \text{J}, \text{final KE} = 438\ \text{J}$

5: (a) 1.70 rad/s (b) $\text{Initial KE} = 22.5\ \text{J}, \text{final KE} = 2.04\ \text{J}$ (c) $1.50\ \text{kg}\cdot\text{m/s}$

Chapter 10 Oscillatory Motion and Simple Harmonic Motion

Figure 1. *There are at least four types of waves in this picture—only the water waves are evident. There are also sound waves, light waves, and waves on the guitar strings. (credit: John Norton)*

What do an ocean buoy, a child in a swing, the cone inside a speaker, a guitar, atoms in a crystal, the motion of chest cavities, and the beating of hearts all have in common? They all **oscillate**—-that is, they move back and forth between two points. Many systems oscillate, and they have certain characteristics in common. All oscillations involve force and energy. You push a child in a swing to get the motion started. The energy of atoms vibrating in a crystal can be increased with heat. You put energy into a guitar string when you pluck it.

By studying oscillatory motion, we shall find that a small number of underlying principles describe all of them and that wave phenomena are more common than you have ever imagined. We begin by studying the type of force that underlies the simplest oscillations and waves. We will then expand our exploration of oscillatory motion and waves to include concepts such as simple harmonic motion, uniform circular motion, and damped harmonic motion.

Glossary

oscillate
moving back and forth regularly between two points

10.1 Hooke's Law and Oscillations

Summary

- Describe the restoration of force and displacement.
- Calculate the energy in Hook's Law of deformation, and the stored energy in a string.

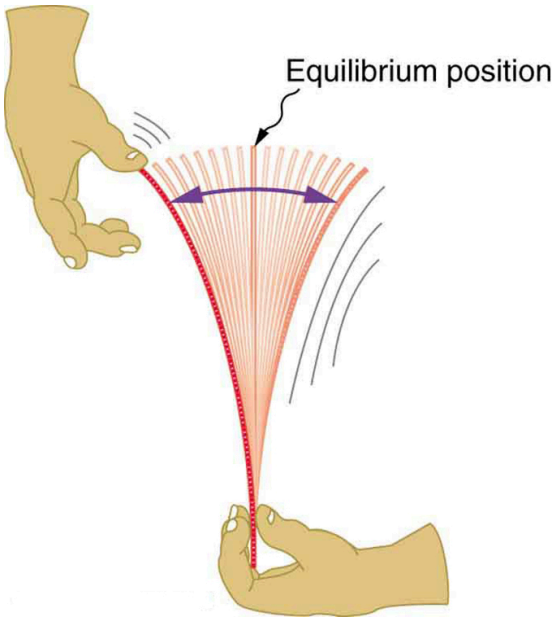


Figure 1. *When displaced from its vertical equilibrium position, this plastic ruler oscillates back and forth because of the restoring force opposing displacement. When the ruler is on the left, there is a force to the right, and vice versa.*

Newton's first law implies that an object oscillating back and forth is experiencing forces. Without force, the object would move in a straight line at a constant speed rather than oscillate. Consider, for example, plucking a plastic ruler to the left as shown in Figure 1. The deformation of the ruler creates a force in the opposite direction, known as a **restoring force**. Once released, the restoring force causes the ruler to move back toward its stable equilibrium position, where the net force on it is zero. However, by the time the ruler gets there, it gains momentum and continues to move to the right, producing the opposite deformation. It is then forced to the left, back through equilibrium, and the process is repeated until dissipative forces dampen the motion. These forces remove mechanical energy from the system, gradually reducing the motion until the ruler comes to rest.

The simplest oscillations occur when the restoring force is directly proportional to displacement. When

spring forces were covered in Chapter 4.5 Normal,Tension, and Spring Forces, the name was given to this relationship between force and displacement was Hooke's law:

$$\vec{F} = -k\vec{x}$$

Here, $\vec{F}$ is the restoring force, $\vec{x}$ is the displacement from equilibrium or **deformation**, and k is a constant related to the difficulty in deforming the system. The minus sign indicates the restoring force is in the direction opposite to the displacement.

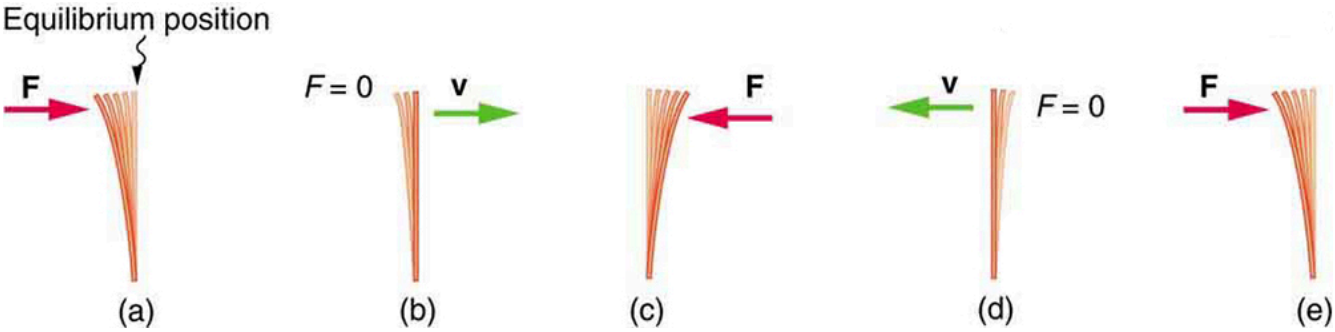
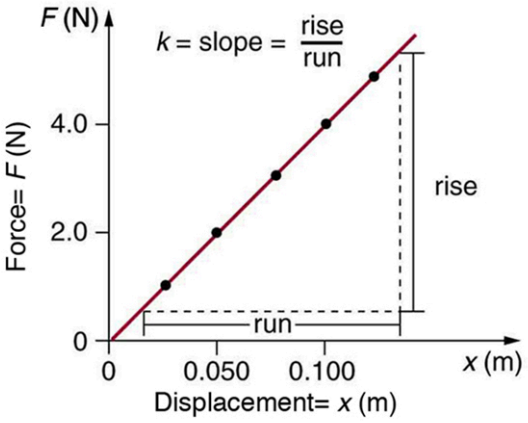


Figure 2. *(a) The plastic ruler has been released, and the restoring force is returning the ruler to its equilibrium position. (b) The net force is zero at the equilibrium position, but the ruler has momentum and continues to move to the right. (c) The restoring force is in the opposite direction. It stops the ruler and moves it back toward equilibrium again. (d) Now the ruler has momentum to the left. (e) In the absence of damping (caused by frictional forces), the ruler reaches its original position. From there, the motion will repeat itself.*

The **force constant** k is related to the rigidity (or stiffness) of a system—the larger the force constant, the greater the restoring force, and the stiffer the system. The units of k are newtons per meter (N/m). For example, k is directly related to Young's modulus when we stretch a string. Figure 3 shows a graph of the absolute value of the restoring force versus the displacement for a system that can be described by Hooke's law—a simple spring in this case. The slope of the graph equals the force constant k in newtons per meter. A common physics laboratory exercise is to measure restoring forces created by springs, determine if they follow Hooke's law, and calculate their force constants if they do.

a)



m (kg)	w (N)	x (m)
0.000	0.00	0.000
0.100	0.98	0.025
0.200	1.96	0.050
0.300	2.94	0.076
0.400	3.92	0.099
0.500	4.90	0.127

b)

m (kg)	w (N)	x (m)
0.000	0.00	0.000
0.100	0.98	0.025
0.200	1.96	0.050
0.300	2.94	0.076
0.400	3.92	0.099
0.500	4.90	0.127

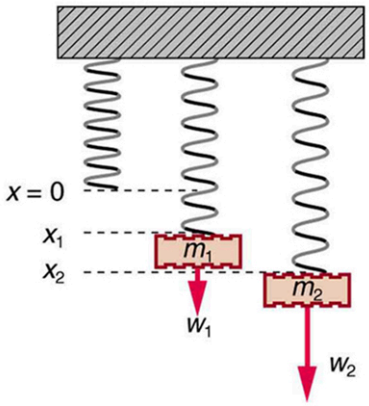


Figure 3. *(a) A graph of absolute value of the restoring force versus displacement is displayed. The fact that the graph is a straight line means that the system obeys Hooke's law. The slope of the graph is the force constant k. (b) The data in the graph were generated by measuring the displacement of a spring from equilibrium while supporting various weights. The restoring force equals the weight supported, if the mass is stationary.*

Example 1: How Still Are Car Springs?

What is the force constant for the suspension system of a car that settles 1.20 cm when an 80.0-kg person gets in?

Strategy

Consider the car to be in its equilibrium position $x = 0$ before the person gets in. The car then settles down 1.20 cm, which means it is displaced to a position $x = -1.20 \times 10^{-2}$ m. At that point, the springs supply a restoring force F equal to the person's weight $w = mg = (80.0\text{ kg})(9.80\text{ m/s}^2) = 784$ N. We take this force to be F in Hooke's law. Knowing F and x, we can then solve the force constant k.

Solution

1. Solve Hooke's law, $F = -kx$, for k:

$$k = -\frac{F}{x}.$$

2. Substitute known values and solve k:

$$\begin{aligned} k &= -\frac{784\ \text{N}}{-1.20 \times 10^{-2}\ \text{m}} \\ &= 6.53 \times 10^{4}\ \text{N/m}. \end{aligned}$$

Discussion

Note that F and x have opposite signs because they are in opposite directions—the restoring force is up, and the displacement is down. Also, note that the car would oscillate up and down when the person got in if it were not for damping (due to frictional forces) provided by shock absorbers. Bouncing cars are a sure sign of bad shock absorbers.

Energy in Hooke's Law of Deformation

In order to produce a deformation, work must be done. That is, a force must be exerted through a distance, whether you pluck a guitar string or compress a car spring. If the only result is deformation, and no work goes into thermal, sound, or kinetic energy, then all the work is initially stored in the deformed object as some form of potential energy. The potential energy stored in a spring is $\text{PE}_{\text{el}} = \frac{1}{2}kx^2$. Here, we generalize the idea to elastic potential energy for a deformation of any system that can be described by Hooke's law. Hence,

$$\text{PE}_{\text{el}} = \frac{1}{2}kx^2,$$

where PE_{el} is the **elastic potential energy** stored in any deformed system that obeys Hooke's law and has a displacement x from equilibrium and a force constant k.

It is possible to find the work done in deforming a system in order to find the energy stored. This work is performed by an applied force F_{app}. The applied force is exactly opposite to the restoring force (action-reaction), and so $F_{\text{app}}=kx$. Figure 5 shows a graph of the applied force versus deformation x for a system that can be described by Hooke's law. Work done on the system is force multiplied by distance, which equals the area under the curve or $(1/2)kx^2$ (Method A in the figure). Another way to determine the work is to note that the force increases linearly from 0 to kx, so that the average force is $(1/2)kx$, the distance moved is x, and thus $W = F_{\text{app}}d = [(1/2)kx](x) = (1/2)kx^2$ (Method B in the figure).

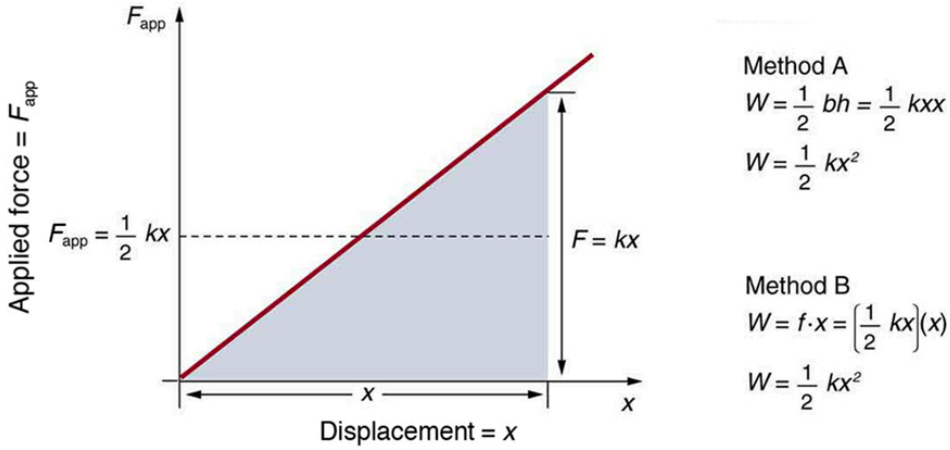


Figure 5. *A graph of applied force versus distance for the deformation of a system that can be described by Hooke's law is displayed. The work done on the system equals the area under the graph or the area of the triangle, which is half its base multiplied by its height, or* $W = (1/2)kx^2$.

Example 2: Calculating Stored Energy: A Tranquilizer Gun Spring

We can use a toy gun's spring mechanism to ask and answer two simple questions: (a) How much energy is stored in the spring of a tranquilizer gun that has a force constant of 50.0 N/m and is compressed 0.150 m? (b) If you neglect friction and the mass of the spring, at what speed will a 2.00-g projectile be ejected from the gun?

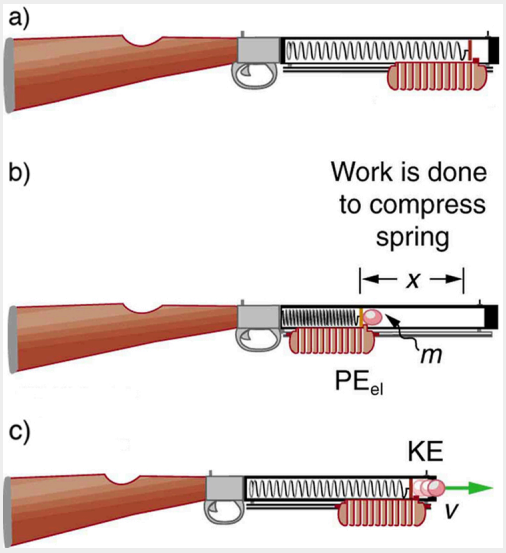


Figure 6. *(a) In this image of the gun, the spring is uncompressed before being cocked. (b) The spring has been compressed a distance x, and the projectile is in place. (c) When released, the spring converts elastic potential energy* PE_{el} *into kinetic energy.*

Strategy for a

(a): The energy stored in the spring can be found directly from elastic potential energy equation, because *k* and *x* are given.

Solution for a

Entering the given values for *k* and *x* yields

$$\begin{aligned} \text{PE}_{\text{el}} &= \frac{1}{2}kx^2 = \frac{1}{2}(50.0\ \text{N/m})(0.150\ \text{m})^2 = 0.563\ \text{N}\cdot\text{m} \\ &= 0.563\ \text{J} \end{aligned}$$

Strategy for b

Because there is no friction, the potential energy is converted entirely into kinetic energy. The expression for kinetic energy can be solved for the projectile's speed.

Solution for b

1. Identify known quantities:

$$\text{KE}_{\text{f}} = \text{PE}_{\text{el}} \text{ or } 1/2mv^2 = (1/2)kx^2 = \text{PE}_{\text{el}} = 0.563\ \text{J}$$

2. Solve for$v$

$$v = [\frac{2\text{PE}_{\text{el}}}{m}]^{1/2} = [\frac{2(0.563\ \text{J})}{0.002\ \text{kg}}]^{1/2} = 23.7(\text{J/kg})^{1/2}$$

3. Convert units: $23.7\ \text{m/s}$

Discussion

(a) and (b): This projectile speed is impressive for a tranquilizer gun (more than 80 km/h). The numbers in this problem seem reasonable. The force needed to compress the spring is small enough for an adult to manage, and the energy imparted to the dart is small enough to limit the damage it might do. Yet, the speed of the dart is great enough for it to travel an acceptable distance.

Check your Understanding 1

Envision holding the end of a ruler with one hand and deforming it with the other. When you let go, you can see the oscillations of the ruler. In what way could you modify this simple experiment to increase the rigidity of the system?

Check your Understanding 2

If you apply a deforming force on an object and let it come to equilibrium, what happened to the work you did on the system?

Section Summary

- An oscillation is a back and forth motion of an object between two points of deformation.
- An oscillation may create a wave, which is a disturbance that propagates from where it was created.
- The simplest type of oscillations and waves are related to systems that can be described by Hooke's law:

$$F = -kx,$$

  where F is the restoring force, x is the displacement from equilibrium or deformation, and k is the force constant of the system.
- Elastic potential energy PE_{el} stored in the deformation of a system that can be described by Hooke's law is given by

$$\text{PE}_{\text{el}} = (1/2)kx^2.$$

Glossary

deformation
displacement from equilibrium

elastic potential energy
potential energy stored as a result of deformation of an elastic object, such as the stretching of a spring

force constant
a constant related to the rigidity of a system: the larger the force constant, the more rigid the system; the force constant is represented by k

restoring force
force acting in opposition to the force caused by a deformation

Conceptual Questions

1: Describe a system in which elastic potential energy is stored.

Problems & Exercises

1: Fish are hung on a spring scale to determine their mass (most fishermen feel no obligation to truthfully report the mass).

(a) What is the force constant of the spring in such a scale if it the spring stretches 8.00 cm for a 10.0 kg load?

(b) What is the mass of a fish that stretches the spring 5.50 cm?

(c) How far apart are the half-kilogram marks on the scale?

2: It is weigh-in time for the local under-85-kg rugby team. The bathroom scale used to assess eligibility can be described by Hooke's law and is depressed 0.75 cm by its maximum load of 120 kg. (a) What is the spring's effective spring constant? (b) A player stands on the scales and depresses it by 0.48 cm. Is he eligible to play on this under-85 kg team?

3: One type of BB gun uses a spring-driven plunger to blow the BB from its barrel. (a) Calculate the force constant of its plunger's spring if you must compress it 0.150 m to drive the 0.0500-kg plunger to a top speed of 20.0 m/s. (b) What force must be exerted to compress the spring?

4: (a) The springs of a pickup truck act like a single spring with a force constant of 1.30×10^5 N/m. By how much will the truck be depressed by its maximum load of 1000 kg?

(b) If the pickup truck has four identical springs, what is the force constant of each?

5: When an 80.0-kg man stands on a pogo stick, the spring is compressed 0.120 m.

(a) What is the force constant of the spring? (b) Will the spring be compressed more when he hops down the road?

6: A spring has a length of 0.200 m when a 0.300-kg mass hangs from it, and a length of 0.750 m when a 1.95-kg mass hangs from it. (a) What is the force constant of the spring? (b) What is the unloaded length of the spring?

Solutions

Check Your Understanding 1

You could hold the ruler at its midpoint so that the part of the ruler that oscillates is half as long as in the original experiment.

Check Your Understanding 2

It was stored in the object as potential energy.

Problems & Exercises

1:(a) $1.23 \times 10^3\ \mathrm{N/m}$ (b) $6.88\ \mathrm{kg}$ (c) $4.00\ \mathrm{mm}$

3: (a) 889 N/m (b) 133 N

5:(a) $6.53 \times 10^3\ \mathrm{N/m}$ (b) Yes

10.2 Period and Frequency in Oscillations

Summary

- Observe the vibrations of a guitar string.
- Determine the frequency of oscillations.

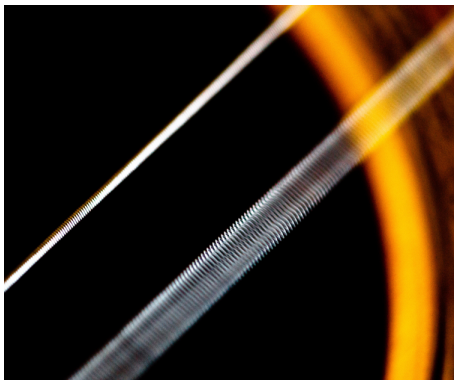


Figure 1. *The strings on this guitar vibrate at regular time intervals. (credit: JAR)*

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time. Each successive vibration of the string takes the same time as the previous one. We define **periodic motion** to be a motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by an object on a spring moving up and down. The time to complete one oscillation remains constant and is called the **period** *T*. Its units are usually seconds, but may be any convenient unit of time. The word period refers to the time for some event whether repetitive or not; but we shall be primarily interested in periodic motion, which is by definition repetitive. A concept closely related to period is the frequency of an event. For example, if you get a paycheck twice a month, the frequency of payment is two per month and the period between checks is half a month. **Frequency** *f* is defined to be the number of events per unit time. For periodic motion, frequency is the number of oscillations per unit time. The relationship between frequency and period is

$$f = \frac{1}{T}$$

The SI unit for frequency is the *cycle per second*, which is defined to be a *hertz* (Hz):

$$1 \text{ Hz} = 1\frac{\text{cycle}}{\text{sec}} \text{or } 1 \text{ Hz} = \frac{1}{\text{s}}$$

A cycle is one complete oscillation. Note that a vibration can be a single or multiple event, whereas oscillations are usually repetitive for a significant number of cycles.

In addition to the frequency and period, the timescale of periodic motion can also be described by the angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T},$$

which has units of rad/s.

Example 1: Determine the Frequency of Two Oscillations: Medical Ultrasound and the Period Middle C

We can use the formulas presented in this module to determine both the frequency based on known oscillations and the oscillation based on a known frequency. Let's try one example of each. (a) A medical imaging device produces ultrasound by oscillating with a period of 0.400 μs. What is the frequency of this oscillation? (b) The frequency of middle C on a typical musical instrument is 264 Hz. What is the time for one complete oscillation?

Strategy

Both questions (a) and (b) can be answered using the relationship between period and frequency. In question (a), the period *T* is given and we are asked to find frequency *f*. In question (b), the frequency *f* is given and we are asked to find the period *T*.

Solution a

1. Substitute 0.400 μs for *T* in $f = \frac{1}{T}$

$$f = \frac{1}{T} = \frac{1}{0.400 \times 10^{-6}\ \text{s}}.$$

Solve to find

$$f = 2.50 \times 10^{6}\ \text{Hz}.$$

Discussion a

The frequency of sound found in (a) is much higher than the highest frequency that humans can hear and, therefore, is called ultrasound. Appropriate oscillations at this frequency generate ultrasound used for noninvasive medical diagnoses, such as observations of a fetus in the womb.

Solution b

1. Identify the known values:

The time for one complete oscillation is the period *T*:

$$f = \frac{1}{T}.$$

2. Solve for T:
$$T = \frac{1}{f}.$$
3. Substitute the given value for the frequency into the resulting expression:
$$T = \frac{1}{f} = \frac{1}{264\ \text{Hz}} = \frac{1}{264\ \text{cycles/s}} = 3.79 \times 10^{-3}\ \text{s} = 3.79\ \text{ms}.$$

Discussion

The period found in (b) is the time per cycle, but this value is often quoted as simply the time in convenient units (ms or milliseconds in this case).

Check your Understanding

1: Identify an event in your life (such as receiving a paycheck) that occurs regularly. Identify both the period and frequency of this event.

Section Summary

- Periodic motion is a repetitious oscillation.
- The time for one oscillation is the period T.
- The number of oscillations per unit time is the frequency f.
- These quantities are related by
$$f = \frac{1}{T}.$$

Glossary

period
: time it takes to complete one oscillation

periodic motion
: motion that repeats itself at regular time intervals

frequency
: number of events per unit of time

Problems & Exercises

1: What is the period of 60.0 Hz electrical power?

2: If your heart rate is 150 beats per minute during strenuous exercise, what is the time per beat in units of seconds?

3: Find the frequency of a tuning fork that takes 2.50×10^{-3} s to complete one oscillation.

4: A stroboscope is set to flash every 8.00×10^{-5} s. What is the frequency of the flashes?

5: A tire has a tread pattern with a crevice every 2.00 cm. Each crevice makes a single vibration as the tire moves. What is the frequency of these vibrations if the car moves at 30.0 m/s?

6: Engineering Application

Each piston of an engine makes a sharp sound every other revolution of the engine. (a) How fast is a race car going if its eight-cylinder engine emits a sound of frequency 750 Hz, given that the engine makes 2000 revolutions per kilometer? (b) At how many revolutions per minute is the engine rotating?

Solutions

Check Your Understanding

1: I visit my parents for dinner every other Sunday. The frequency of my visits is 26 per calendar year. The period is two weeks.

Problems & Exercises

1: 16.7 ms

2: 0.400 s/beats

3: 400 Hz

4: 2,500 Hz

5: 1.50 kHz

6: (a) 93.8 m/s (b) 11.3 x 10^{3} rev/min

10.3 Uniform Circular Motion and Simple Harmonic Motion

Summary

- Compare simple harmonic motion with uniform circular motion.

The oscillations of a system in which the net force can be described by Hooke's law are of special importance, because they are very common. They are also the simplest oscillatory systems. **Simple Harmonic Motion** (SHM) is the name given to oscillatory motion for a system where the net force can be described by Hooke's law, and such a system is called a **simple harmonic oscillator**.

There is an easy way to produce simple harmonic motion by using uniform circular motion. Figure 2 shows one way of using this method. A ball is attached to a uniformly rotating vertical turntable, and its shadow is projected on the floor as shown. The shadow undergoes simple harmonic motion. Hooke's law usually describes uniform circular motions (ω constant) rather than systems that have large visible displacements. So observing the projection of uniform circular motion, as in Figure 2, is often easier than observing a precise large-scale simple harmonic oscillator. If studied in sufficient depth, simple harmonic motion produced in this manner can give considerable insight into many aspects of oscillations and waves and is very useful mathematically. In our brief treatment, we shall indicate some of the major features of this relationship and how they might be useful.

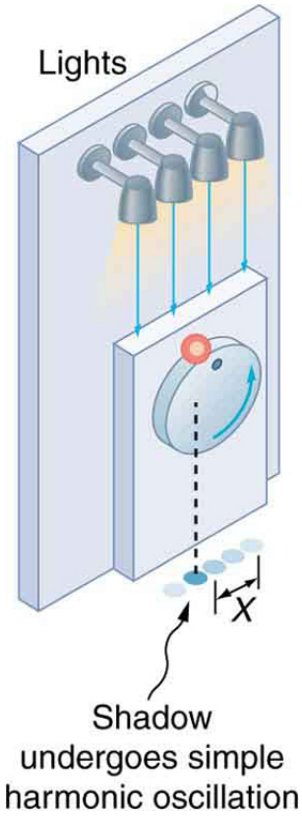


Figure 2. *The shadow of a ball rotating at constant angular velocity ω on a turntable goes back and forth in precise simple harmonic motion.*

Figure 3 shows the basic relationship between uniform circular motion and simple harmonic motion. The point P travels around the circle at constant angular velocity ω. The point P is analogous to an object on the merry-go-round. The projection of the position of P onto a fixed axis undergoes simple harmonic motion and is analogous to the shadow of the object. At the time shown in the figure, the projection has position *x* and moves to the left with velocity *v*. The velocity of the point P around the circle equals $\overline{v}_{\max}$. The projection of $\overline{v}_{\max}$ on the *x*-axis is the velocity *v* of the simple harmonic motion along the *x*-axis.

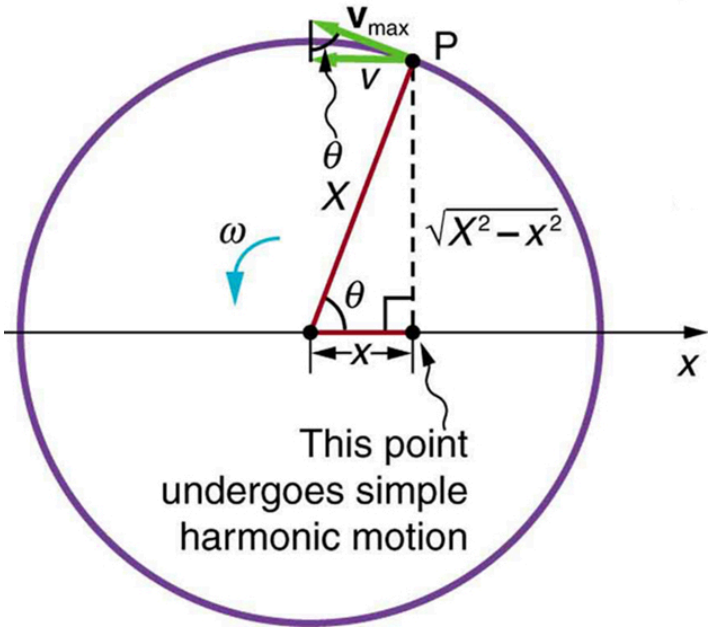


Figure 3. *A point P moving on a circular path with a constant angular velocity ω is undergoing uniform circular motion. Its projection on the x-axis undergoes simple harmonic motion. Also shown is the velocity of this point around the circle, $\bar{v}_{\max}$, and its projection, which is v. Note that these velocities form a similar triangle to the displacement triangle.*

To see that the projection undergoes simple harmonic motion, note that its position x is given by

$$x = X\cos\theta,$$

where $\theta = \omega t$, ω is the constant angular velocity, and X is the radius of the circular path. Thus,

$$x = X\cos\omega t.$$

The angular velocity ω is in radians per unit time; in this case 2π radians is the time for one revolution T. That is, $\omega = 2\pi T$. Substituting this expression for ω, we see that the position x is given by:

$$x(t) = \cos(\frac{2\pi t}{T}).$$

If we make a graph of position versus time as in Figure 4, we see again the wavelike character (typical of simple harmonic motion) of the projection of uniform circular motion onto the x-axis.

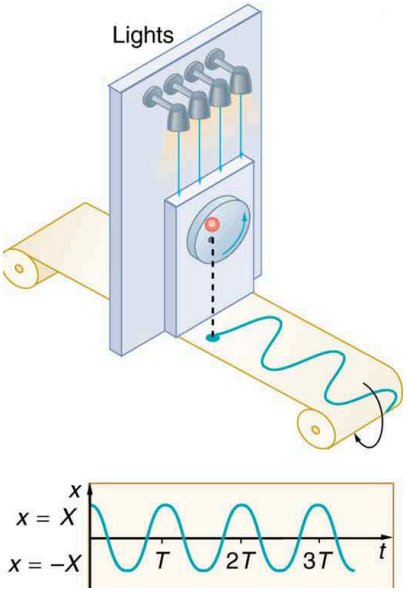


Figure 4. *The position of the projection of uniform circular motion performs simple harmonic motion, as this wavelike graph of x versus t indicates.*

Now let us use Figure 3 to do some further analysis of uniform circular motion as it relates to simple harmonic motion. The triangle formed by the velocities in the figure and the triangle formed by the displacements (X, x, and $\sqrt{X^2 - x^2}$) are similar right triangles. Taking ratios of similar sides, we see that

$$\frac{v}{v_{\max}} = \frac{\sqrt{X^2 - x^2}}{X} = \sqrt{1 - \frac{x^2}{X^2}}.$$

We can solve this equation for the speed$v$or

$$v = v_{\max}\sqrt{1 - \frac{x^2}{X^2}}.$$

You can begin to see that it is possible to get all of the characteristics of simple harmonic motion from an analysis of the projection of uniform circular motion.

Finally, let us consider the period T of the motion of the projection. This period is the time it takes the point P to complete one revolution. That time is the circumference of the circle $2\pi X$ divided by the velocity around the circle, $v_{\max}$. Thus, the period T is

$$T = \frac{2\pi X}{v_{\max}}.$$

We know from conservation of energy considerations that

$$v_{\text{max}} = \sqrt{\frac{k}{m}}X.$$

Solving this equation for X/v_{max} gives

$$\frac{X}{v_{\text{max}}} = \sqrt{\frac{m}{k}}.$$

Substituting this expression into the equation for T yields

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Thus, the period of the motion is the same as for a simple harmonic oscillator. We have determined the period for any simple harmonic oscillator using the relationship between uniform circular motion and simple harmonic motion.

Some modules occasionally refer to the connection between uniform circular motion and simple harmonic motion. Moreover, if you carry your study of physics and its applications to greater depths, you will find this relationship useful. It can, for example, help to analyze how waves add when they are superimposed.

Check Your Understanding

1: Identify an object that undergoes uniform circular motion. Describe how you could trace the simple harmonic motion of this object as a wave.

Problems & Exercises

1: (a)What is the maximum velocity of an 85.0-kg person bouncing on a bathroom scale having a force constant of 1.50×10^6 N/m, if the amplitude of the bounce is 0.200 cm? (b)What is the maximum energy stored in the spring?

2: A novelty clock has a 0.0100-kg mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position? (b) How many joules of kinetic energy does the object have at its maximum velocity?

3: At what positions is the speed of a simple harmonic oscillator half its maximum? That is, what values of x/X give $v=\pm v_{\text{max}}/2$, where X is the amplitude of the motion?

4: A ladybug sits 12.0 cm from the center of a Beatles music album spinning at 33.33 rpm. What is the maximum velocity of its shadow on the wall behind the turntable, if illuminated parallel to the record by the parallel rays of the setting Sun?

Solutions

Check Your Understanding

1: A record player undergoes uniform circular motion. You could attach dowel rod to one point on the outside edge of the turntable and attach a pen to the other end of the dowel. As the record player turns, the pen will move. You can drag a long piece of paper under the pen, capturing its motion as a wave.

Problems & Exercises

1: (a).0.266 m/s (b). 3.00 J

3: $\pm\frac{\sqrt{3}}{2}$

10.4 Simple Harmonic Motion: A Special Periodic Motion

Summary

- Describe a simple harmonic oscillator.
- Explain the link between simple harmonic motion and waves.

A **simple harmonic oscillator** is a system where the net force can be described by Hooke's Law, which results in the system undergoing an oscillatory motion called **simple harmonic motion.** If the net force can be described by Hooke's law and there is no *damping* (by friction or other non-conservative forces), then a simple harmonic oscillator will oscillate with equal displacement on either side of the equilibrium position, as shown for an object on a spring in Figure 1. The maximum displacement from equilibrium is called the **amplitude** X. The units for amplitude and displacement are the same, but depend on the type of oscillation. For the object on the spring, the units of amplitude and displacement are meters; whereas for sound oscillations, they have units of pressure (and other types of oscillations have yet other units). Because amplitude is the maximum displacement, it is related to the energy in the oscillation.

TAKE-HOME EXPERIMENT: SHM AND MARBLE

Find a bowl or basin that is shaped like a hemisphere on the inside. Place a marble inside the bowl and tilt the bowl periodically so the marble rolls from the bottom of the bowl to equally high points on the sides of the bowl. Get a feel for the force required to maintain this periodic motion. What is the restoring force and what role does the force you apply play in the simple harmonic motion (SHM) of the marble?

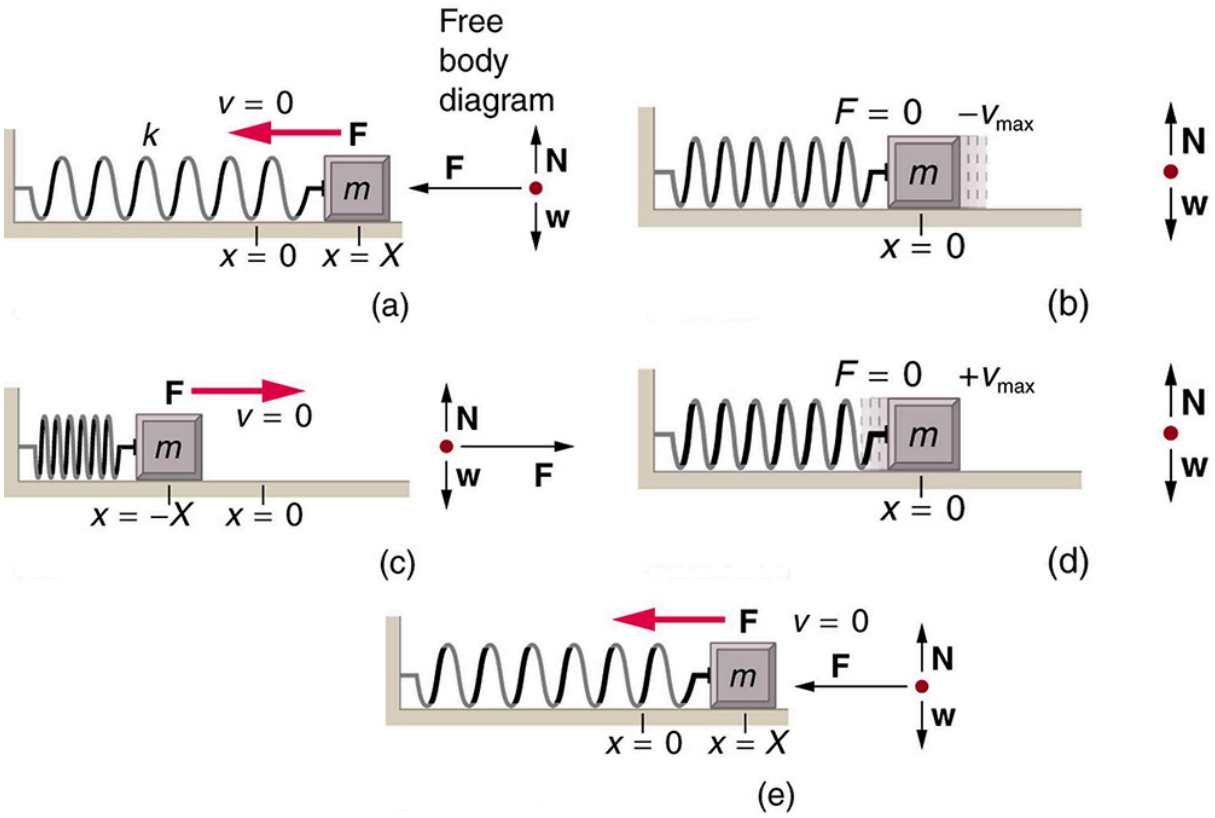


Figure 1. *An object attached to a spring sliding on a frictionless surface is an uncomplicated simple harmonic oscillator. When displaced from equilibrium, the object performs simple harmonic motion that has an amplitude X and a period T. The object's maximum speed occurs as it passes through equilibrium. The stiffer the spring is, the smaller the period T. The greater the mass of the object is, the greater the period T.*

What is so significant about simple harmonic motion? One special thing is that the period T and frequency a f of a simple harmonic oscillator are independent of amplitude. The string of a guitar, for example, will oscillate with the same frequency whether plucked gently or hard. Because the period is constant, a simple harmonic oscillator can be used as a clock.

Two important factors do affect the period of a simple harmonic oscillator. The period is related to how stiff the system is. A very stiff object has a large force constant k, which causes the system to have a smaller period. For example, you can adjust a diving board's stiffness—the stiffer it is, the faster it vibrates, and the shorter its period. Period also depends on the mass of the oscillating system. The more massive the system is, the longer the period. For example, a heavy person on a diving board bounces up and down more slowly than a light one.

In fact, the mass m and the force constant k are the *only* factors that affect the period and frequency of simple harmonic motion.

ANGULAR FREQUENCY, FREQUENCY, AND PERIOD OF SIMPLE HARMONIC OSCILLATOR

The *angular frequency* of a simple harmonic oscillator is given by

$$\omega = \sqrt{\frac{k}{m}}.$$

From this, the *frequency* of a simple harmonic oscillator is

\omega = 2\pi f \rightarrow f = \dfrac{\omega }{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}

and the *period* of a simple harmonic oscillator is given by

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

Note that none of these terms have any dependence on amplitude.

TAKE-HOME EXPERIMENT: MASS AND RULER OSCILLATIONS

Find two identical wooden or plastic rulers. Tape one end of each ruler firmly to the edge of a table so that the length of each ruler that protrudes from the table is the same. On the free end of one ruler tape a heavy object such as a few large coins. Pluck the ends of the rulers at the same time and observe which one undergoes more cycles in a time period, and measure the period of oscillation of each of the rulers.

Example 1: Calculate the Frequency and Period of Oscillations: Bad Shock Absorbers in a Car

If the shock absorbers in a car go bad, then the car will oscillate at the least provocation, such as when going over bumps in the road and after stopping (See Figure 2). Calculate the frequency and period of these oscillations for such a car if the car's mass (including its load) is 900 kg and the force constant *k*) of the suspension system is 6.53 ×10^4 N/m.

Strategy

The frequency of the car's oscillations will be that of a simple harmonic oscillator as given in the equation $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. The mass and the force constant are both given.

Solution

1. Enter the known values of *k* and *m*:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{6.53 \times 10^4 \text{ N/m}}{900 \text{ kg}}}.$$

- Calculate the frequency:

$$\frac{1}{2\pi}\sqrt{72.6/\text{s}^{-2}} = 1.3656/\text{s}^{-1} \approx 1.36/\text{s}^{-1} = 1.36 \text{ Hz}.$$

- You could use $T = 2\pi\sqrt{\frac{m}{k}}$ to calculate the period, but it is simpler to use the relationship $T = 1/f$ and substitute the value just found for f

$$T = \frac{1}{f} = \frac{1}{1.356 \text{ Hz}} = 0.738 \text{ s}.$$

Discussion

The values of *T* and *f* both seem about right for a bouncing car. You can observe these oscillations if you push down hard on the end of a car and let go.

The Link between Simple Harmonic Motion and Waves

If a time-exposure photograph of the bouncing car were taken as it drove by, the headlight would make a wavelike streak, as shown in Figure 2. Similarly, Figure 3 shows an object bouncing on a spring as it leaves a wavelike "trace of its position on a moving strip of paper. Both waves are sine functions. All simple harmonic motion is intimately related to sine and cosine waves.

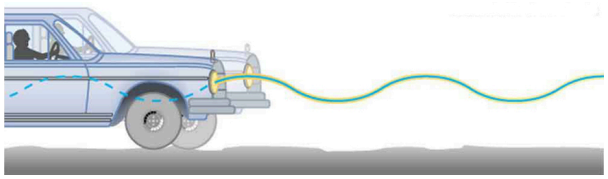


Figure 2. *The bouncing car makes a wavelike motion. If the restoring force in the suspension system can be described only by Hooke's law, then the wave is a sine function. (The wave is the trace produced by the headlight as the car moves to the right.)*

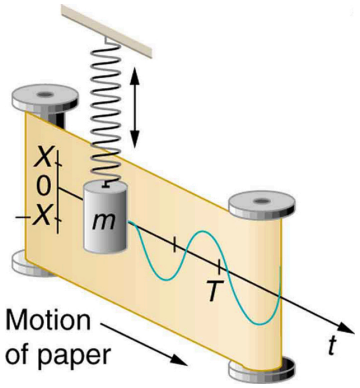


Figure 3. *The vertical position of an object bouncing on a spring is recorded on a strip of moving paper, leaving a sine wave.*

The displacement as a function of time *t* in any simple harmonic motion—that is, one in which the net restoring force can be described by Hooke's law, is given by

$$x(t) = X\cos(\omega t) = X\cos(\frac{2\pi t}{T})$$

where *X* is amplitude. At *t* = 0, the initial position is $x_0 = X$, and the displacement oscillates back and forth with a period *T*. When *t* = *T*, we get *x* = *X* again because cos 2π = 1.) Furthermore, from this expression for *x*, the velocity *v* as a function of time is given by:

$$v(t) = -v_{\max}\sin(\omega t) = -v_{\max}\sin(\frac{2\pi t}{T})$$

where $v_{\max} = X\omega = 2\pi X/T = X\sqrt{k/m}$ is the magnitude of the maximum velocity of the object.

The object has zero velocity at maximum displacement—for example, *v* = 0 when *t* = 0, and at that time *x* = *X*. The minus sign in the first equation for *v*(*t*) gives the correct direction for the velocity. Just after the start of the motion, for instance, the velocity is negative because the system is moving back toward the equilibrium point. Finally, we can get an expression for acceleration using Newton's second law. [Then we have *x(t)*, *v(t)*, *t*, and *a(t)*, the quantities needed for kinematics and a description of simple harmonic motion.] According to Newton's second law, the acceleration is *a* = *F*/*m* = *kx*/*m*. So, *a(t)* is also a cosine function:

$$v(t) = -a_{\max}\cos(\omega t) = -a_{\max}\cos(\frac{2\pi t}{T})$$

where $a_{\max} = X\omega^2 = 4\pi^2 X/T^2 = Xk/m$ is the magnitude of the maximum acceleration of the object. Hence, *a(t)* is directly proportional to and in the opposite direction to *x(t)*.

Vertical Springs

In a horizontal mass-spring system (like that shown in Figure 1) the equilibrium position of the oscillator is at the unstretched length of the spring. However, in a vertical mass-spring system (like that shown in Figure 4) there is an additional force acting on the mass – gravity! At the unstretched length of the spring, *the net force is* ***not*** *zero*. However, as the spring is stretched, the restoring force increases in size and acts in the upwards direction, opposite to the force of gravity. This means there is a point at a distance *d* below the unstretched spring length where the net force is zero and is therefore the equilibrium position for the simple harmonic motion. To find this distance, consider the net force acting on the mass at this point

$$\sum F = kd - mg = 0 \rightarrow d = \frac{mg}{k}$$

Other than a shift in the equilibrium position, everything that was introduced for horizontal mass-spring systems applies to vertical mass-spring systems. Figure 4 shows the simple harmonic motion of an object on a spring and presents graphs of *x(t)*, *v(t)*, and *a(t)* versus time. The most important point here is that these equations are mathematically straightforward and are valid for all simple harmonic motion.

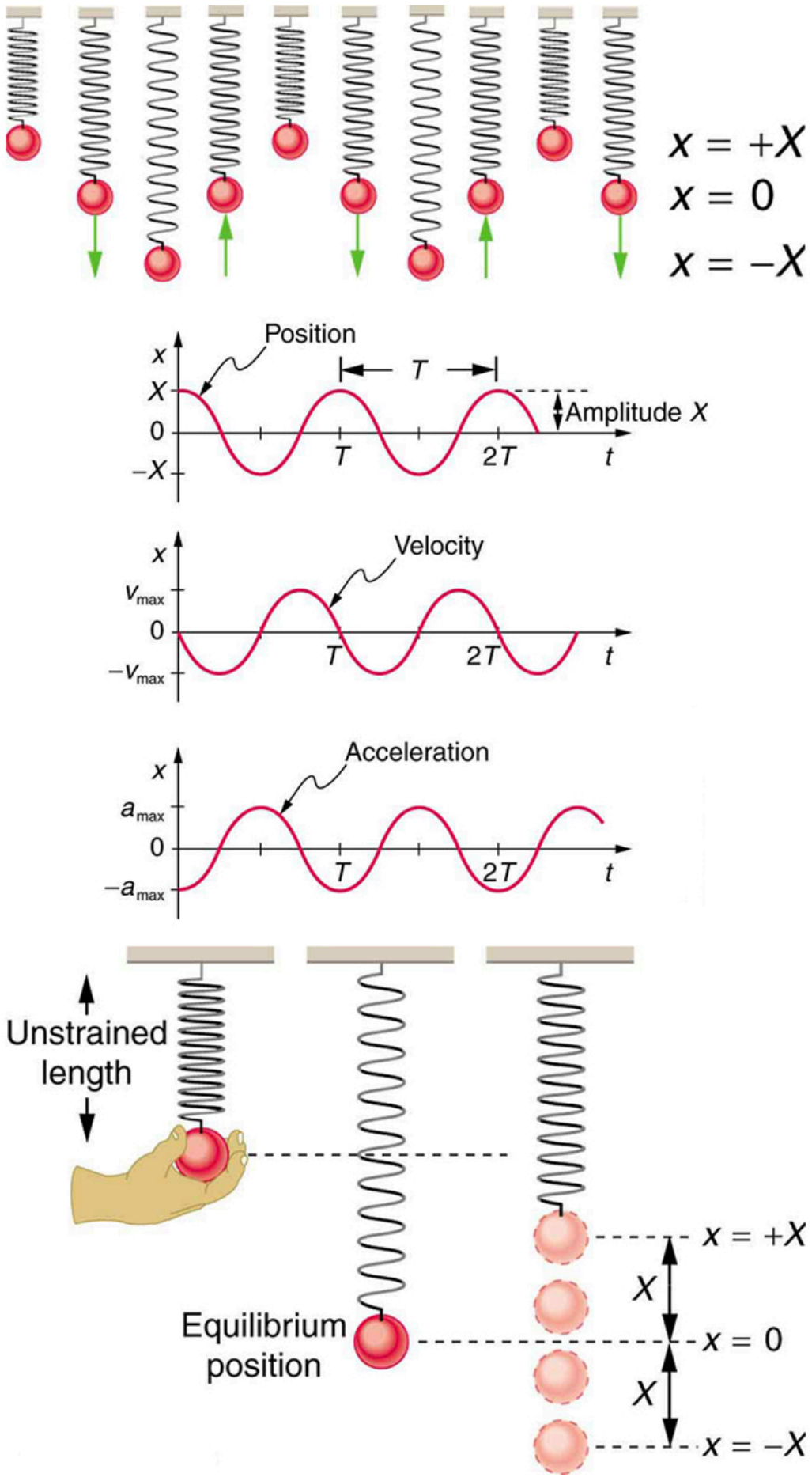

x = +X
x = 0
x = −X
Position
x
X
0
−X
T
Amplitude X
T
2T
t
Velocity
x
v_max
0
−v_max
T
2T
t
Acceleration
x
a_max
0
−a_max
T
2T
t
Unstrained length
Equilibrium position
x = +X
X
x = 0
X
x = −X

Figure 4. *Graphs of x(t), v(t), and a(t) versus t for the motion of an object on a spring. The net force on the object can be described by Hooke's law, and so the object undergoes simple harmonic motion. Note that the initial position has the vertical displacement at its maximum value X; v is initially zero and then negative as the object moves down; and the initial acceleration is negative, back toward the equilibrium position and becomes zero at that point.*

Optional: Damped Harmonic Motion (A More Realistic Case)

A guitar string stops oscillating a few seconds after being plucked. To keep a child happy on a swing, you must keep pushing. Although we can often make friction and other non-conservative forces negligibly small, completely undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.

For a system that has a small amount of damping, the period and frequency are nearly the same as for simple harmonic motion, but the amplitude gradually decreases as shown in Figure 5. This occurs because the non-conservative damping force removes energy from the system, usually in the form of thermal energy. For more details on damped harmonic motion see Chapter 10.7.

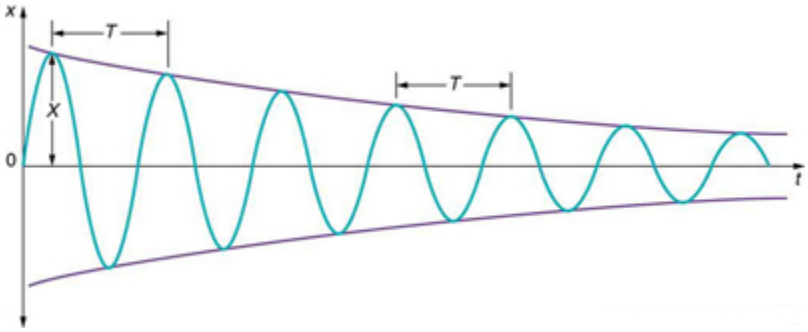


Figure 5. *In this graph of displacement versus time for a harmonic oscillator with a small amount of damping, the amplitude slowly decreases, but the period and frequency are nearly the same as if the system were completely undamped.*

Check Your Understanding 1

Suppose you pluck a banjo string. You hear a single note that starts out loud and slowly quiets over time. Describe what happens to the sound waves in terms of period, frequency and amplitude as the sound decreases in volume.

Check Your Understanding 2

A babysitter is pushing a child on a swing. At the point where the swing reaches *x*, where would the corresponding point on a wave of this motion be located?

PHET EXPLORATIONS: MASSES AND SPRING

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energy for each spring.

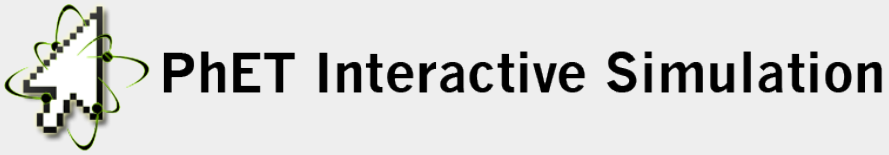


Figure 5. *Masses and Springs*

Section Summary

- Simple harmonic motion is oscillatory motion for a system that can be described only by Hooke's law. Such a system is also called a simple harmonic oscillator.
- Maximum displacement is the amplitude *X*.
- The angular frequency $\omega]$, frequency *f*, and period *T* of a simple harmonic oscillator are given by

$$\omega = sqrtkm \text{ and } f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \text{ and } T = 2\pi\sqrt{\frac{m}{k}}$$

  where *m* is the mass of the system.
- Displacement in simple harmonic motion as a function of time is given by

$$x(t) = X\cos(\omega t) = X\cos(\frac{2\pi t}{T})$$

- The velocity is given by

$$v(t) = -v_{\max}\sin(\omega t) = -v_{\max}\sin(\frac{2\pi t}{T})$$

where $v_{\max} = X\omega = 2\pi X/T = X\sqrt{k/m}$ is the magnitude of the maximum velocity of the object.

- The acceleration is given by

$$v(t) = -a_{\max}\cos(\omega t) = -a_{\max}\cos(\frac{2\pi t}{T})$$

where $a_{\max} = X\omega^2 = 4\pi^2 X/T^2 = Xk/m$ is the magnitude of the maximum acceleration of the object.

Glossary

amplitude
: the maximum displacement from the equilibrium position of an object oscillating around the equilibrium position

simple harmonic motion
: the oscillatory motion in a system where the net force can be described by Hooke's law

simple harmonic oscillator
: a device that implements Hooke's law, such as a mass that is attached to a spring, with the other end of the spring being connected to a rigid support such as a wall

Conceptual Questions

1: What conditions must be met to produce simple harmonic motion?

2: (a) If frequency is not constant for some oscillation, can the oscillation be simple harmonic motion?

(b) Can you think of any examples of harmonic motion where the frequency may depend on the amplitude?

3: Give an example of a simple harmonic oscillator, specifically noting how its frequency is independent of amplitude.

4: Explain why you expect an object made of a stiff material to vibrate at a higher frequency than a similar object made of a spongy material.

5: As you pass a freight truck with a trailer on a highway, you notice that its trailer is bouncing up and down slowly. Is it more likely that the trailer is heavily loaded or nearly empty? Explain your answer.

6: Some people modify cars to be much closer to the ground than when manufactured. Should they install stiffer springs? Explain your answer.

Problems & Exercises

1: A type of cuckoo clock keeps time by having a mass bouncing on a spring, usually something cute like a cherub in a chair. What force constant is needed to produce a period of 0.500 s for a 0.0150-kg mass?

2: If the spring constant of a simple harmonic oscillator is doubled, by what factor will the mass of the system need to change in order for the frequency of the motion to remain the same?

3: A 0.500-kg mass suspended from a spring oscillates with a period of 1.50 s. How much mass must be added to the object to change the period to 2.00 s?

4: By how much leeway (both percentage and mass) would you have in the selection of the mass of the object in the previous problem if you did not wish the new period to be greater than 2.01 s or less than 1.99 s?

5: Suppose you attach the object with mass *m* to a vertical spring originally at rest, and let it bounce up and down. You release the object from rest at the spring's original rest length. (a) Show that the spring exerts an upward force of 2.00 mg on the object at its lowest point. (b) If the spring has a force constant of 10.0 N/m and a 0.25-kg-mass object is set in motion as described, find the amplitude of the oscillations. (c) Find the maximum velocity.

6: A diver on a diving board is undergoing simple harmonic motion. Her mass is 55.0 kg and the period of her motion is 0.800 s. The next diver is a male whose period of simple harmonic oscillation is 1.05 s. What is his mass if the mass of the board is negligible?

7: Suppose a diving board with no one on it bounces up and down in a simple harmonic motion with a frequency of 4.00 Hz. The board has an effective mass of 10.0 kg. What is the frequency of the simple harmonic motion of a 75.0-kg diver on the board?

8:

Figure 6. *This child's toy relies on springs to keep infants entertained. (credit: By Humboldthead, Flickr)*

The device pictured in Figure 6 entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring constant.

(a) If the spring stretches 0.250 m while supporting an 8.0-kg child, what is its spring constant?

(b) What is the time for one complete bounce of this child? (c) What is the child's maximum velocity if the amplitude of her bounce is 0.200 m?

9: A 90.0-kg skydiver hanging from a parachute bounces up and down with a period of 1.50 s. What is the

new period of oscillation when a second skydiver, whose mass is 60.0 kg, hangs from the legs of the first, as seen in Figure 7.

Figure 7. *The oscillations of one skydiver are about to be affected by a second skydiver. (credit: U.S. Army, www.army.mil)*

Solutions

Check Your Understanding 1

Frequency and period remain essentially unchanged. Only amplitude decreases as volume decreases.

Check Your Understanding 2

$x$is the maximum deformation, which corresponds to the amplitude of the wave. The point on the wave would either be at the very top or the very bottom of the curve.

Problems & Exercises

1: $2.37\ \text{N/m}$

3: $0.389\ \text{kg}$

6: $94.7\ \text{kg}$

9: $1.94\ \text{s}$

10.5 Energy and the Simple Harmonic Oscillator

Sumamry

- Determine the maximum speed of an oscillating system.

To study the energy of a simple harmonic oscillator, we first consider all the forms of energy it can have We know that the energy stored in the deformation of a simple harmonic oscillator is a form of potential energy given by:

$$\text{PE}_{\text{el}} = \frac{1}{2}kx^2.$$

Because a simple harmonic oscillator has no dissipative forces, the other important form of energy is kinetic energy KE. Conservation of energy for these two forms is:

$$\text{KE} + \text{PE}_{\text{el}} = \text{ constant}$$

or

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{ constant}.$$

This statement of conservation of energy is valid for *all* simple harmonic oscillators, including ones where the gravitational force plays a role

Namely, for a simple pendulum we replace the velocity with $v = L\omega$, the spring constant with $k = mg/L$, and the displacement term with $x = L\theta$. Thus

$$\frac{1}{2}mL^2\omega^2 + \frac{1}{2}mgL\theta^2 = \text{ constant}.$$

In the case of undamped simple harmonic motion, the energy oscillates back and forth between kinetic and potential, going completely from one to the other as the system oscillates. So for the simple example of an object on a frictionless surface attached to a spring, as shown again in Figure 1, the motion starts with all of the energy stored in the spring. As the object starts to move, the elastic potential energy is converted to kinetic energy, becoming entirely kinetic energy at the equilibrium position. It is then converted back into elastic potential energy by the spring, the velocity becomes zero when the kinetic energy is completely converted, and so on. This concept provides extra insight here and in later applications of simple harmonic motion, such as alternating current circuits.

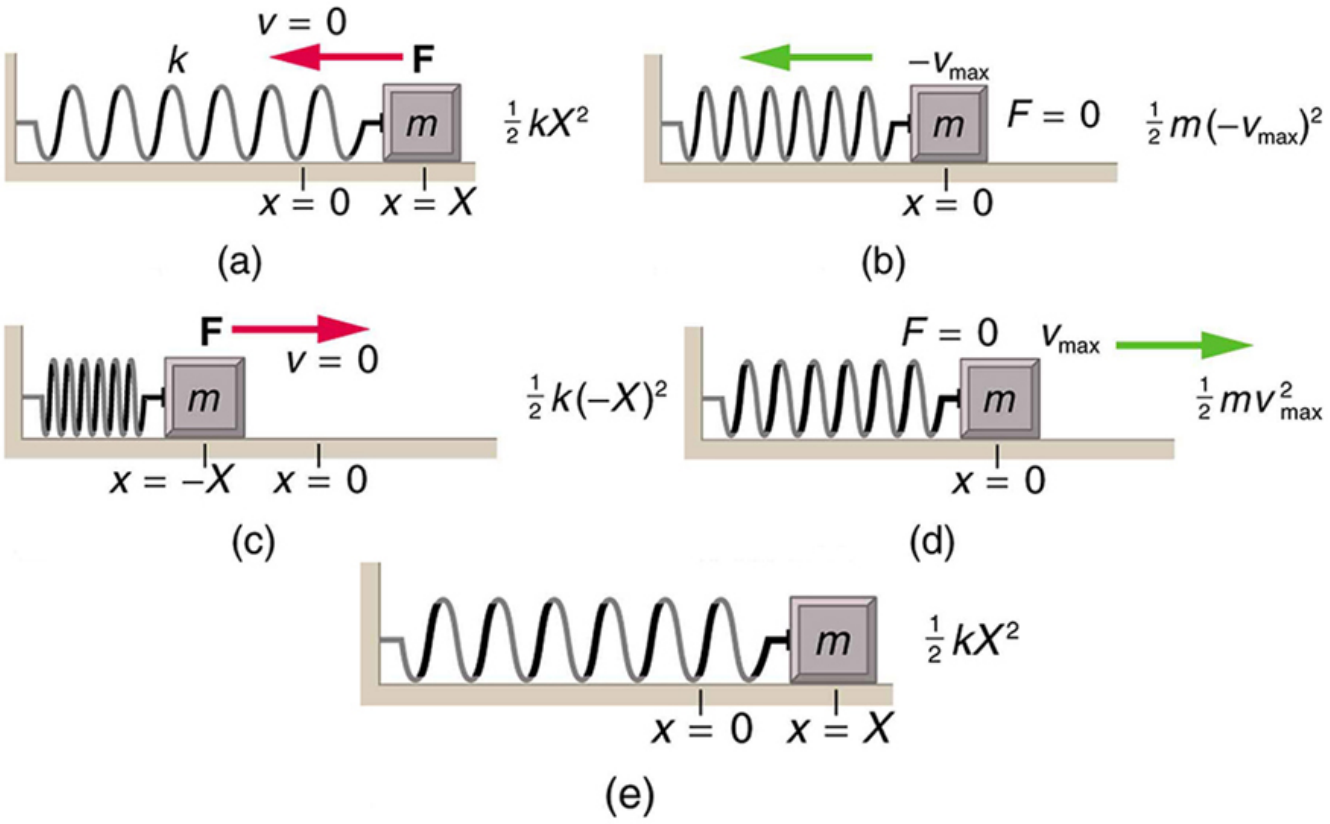


Figure 1. *The transformation of energy in simple harmonic motion is illustrated for an object attached to a spring on a frictionless surface.*

The conservation of energy principle can be used to derive an expression for velocity *v*. If we start our simple harmonic motion with zero velocity and maximum displacement ($x = X$), then the total energy is

$$\frac{1}{2}kX^2.$$

This total energy is constant and is shifted back and forth between kinetic energy and potential energy, at most times being shared by each. The conservation of energy for this system in equation form is thus:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kX^2.$$

Solving this equation for *v* yields:

$$v = \pm\sqrt{\frac{k}{m}(X^2 - x^2)}.$$

Manipulating this expression algebraically gives:

$$v = \pm\sqrt{\frac{k}{m}}X\sqrt{1 - \frac{x^2}{X^2}}$$

and so

$$v = \pm v_{\max}\sqrt{1 - \frac{x^2}{X^2}}$$

where

$$v_{\max} = \sqrt{\frac{k}{m}}X.$$

From this expression, we see that the velocity is a maximum ($v_{\max}$) at x = 0, as stated earlier in $v(t) = -v_{\max}\sin\frac{2\pi t}{T}$. Notice that the maximum velocity depends on three factors. Maximum velocity is directly proportional to amplitude. As you might guess, the greater the maximum displacement the greater the maximum velocity. Maximum velocity is also greater for stiffer systems, because they exert greater force for the same displacement. This observation is seen in the expression for $v_{\max}$; it is proportional to the square root of the force constant k. Finally, the maximum velocity is smaller for objects that have larger masses, because the maximum velocity is inversely proportional to the square root of m. For a given force, objects that have large masses accelerate more slowly.

A similar calculation for the simple pendulum produces a similar result, namely:

$$\omega_{\max} = \sqrt{\frac{g}{L}}\theta_{\max}$$

Example 1: Determine the Maximum Speed of an Oscillating System: A Bumpy Road

Suppose that a car is 900 kg and has a suspension system that has a force constant k=6.53 × 10^4 N/m. The car hits a bump and bounces with an amplitude of 0.100 m. What is its maximum vertical velocity if you assume no damping occurs?

Strategy

We can use the expression for $v_{\max}$ given in $v_{\max} = \sqrt{\frac{k}{m}}X$ to determine the maximum vertical velocity. The variables m and k are given in the problem statement, and the maximum displacement X is 0.100 m.

Solution

1. Identify known.
2. Substitute known values into $v_{\max} = \sqrt{\frac{k}{m}}X$

$$v_{\text{max}} = \sqrt{\frac{6.53 \times 10^4 \text{ N/m}}{900 \text{ kg}}}(0.100 \text{ m}).$$

3. Calculate to find v_{max} = 0.852 m/s.

Discussion

This answer seems reasonable for a bouncing car.

The small vertical displacement *y* of an oscillating simple pendulum, starting from its equilibrium position, is given as

$$y(t) = a \sin \omega t,$$

where *a* is the amplitude, ω is the angular velocity and *t* is the time taken. Substituting $\omega = \frac{2\pi}{T}$, we have

$$yt = a \sin(\frac{2\pi t}{T}).$$

Thus, the displacement of pendulum is a function of time as shown above.

Also the velocity of the pendulum is given by

$$v(t) = \frac{2a\pi}{T} \cos(\frac{2\pi t}{T}),$$

so the motion of the pendulum is a function of time.

Check Your Understanding 1

Why does it hurt more if your hand is snapped with a ruler than with a loose spring, even if the displacement of each system is equal?

Check Your Understanding 2

You are observing a simple harmonic oscillator. Identify one way you could decrease the maximum velocity of the system.

Section Summary

- Energy in the simple harmonic oscillator is shared between elastic potential energy and

kinetic energy, with the total being constant:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{ constant}.$$

- Maximum velocity depends on three factors: it is directly proportional to amplitude, it is greater for stiffer systems, and it is smaller for objects that have larger masses:

$$v_{\max} = \sqrt{\frac{k}{m}}X.$$

Conceptual Questions

1: Explain in terms of energy how dissipative forces such as friction reduce the amplitude of a harmonic oscillator. Also explain how a driving mechanism can compensate. (A pendulum clock is such a system.)

Problems & Exercises

1: The length of nylon rope from which a mountain climber is suspended has a force constant of 1.40×10^4 N/m.

(a) What is the frequency at which he bounces, given his mass plus and the mass of his equipment are 90.0 kg?

(b) How much would this rope stretch to break the climber's fall if he free-falls 2.00 m before the rope runs out of slack? Hint: Use conservation of energy.

(c) Repeat both parts of this problem in the situation where twice this length of nylon rope is used.

2: Engineering Application

Near the top of the Citigroup Center building in New York City, there is an object with mass of 4.00×10^5 kg on springs that have adjustable force constants. Its function is to dampen wind-driven oscillations of the building by oscillating at the same frequency as the building is being driven—the driving force is transferred to the object, which oscillates instead of the entire building. (a) What effective force constant should the springs have to make the object oscillate with a period of 2.00 s? (b) What energy is stored in the springs for a 2.00-m displacement from equilibrium?

Solutions

Check Your Understanding 1

The ruler is a stiffer system, which carries greater force for the same amount of displacement. The ruler snaps your hand with greater force, which hurts more.

Check Your Understanding 2

You could increase the mass of the object that is oscillating.

Problems & Exercises

1: (a) 1.99 Hz (b) 50.2 cm (c) 1.41 Hz and 35.5 cm

2: (a) $3.95 \times 10^6 \text{ N/m}$ (b) $7.90 \times 10^6 \text{ J}$

10.6 The Simple Pendulum

Summary

- Measure acceleration due to gravity.
- State what the period for a simple pendulum depends on.

Pendulums are in common usage. Some have crucial uses, such as in clocks; some are for fun, such as a child's swing; and some are just there, such as the sinker on a fishing line. For small displacements, a pendulum is a simple harmonic oscillator. A **simple pendulum** is defined to have an object that has a small mass, also known as the pendulum bob, which is suspended from a light wire or string, such as shown in Figure 1. Exploring the simple pendulum a bit further, we can discover the conditions under which it performs simple harmonic motion, and we can derive an interesting expression for its period.

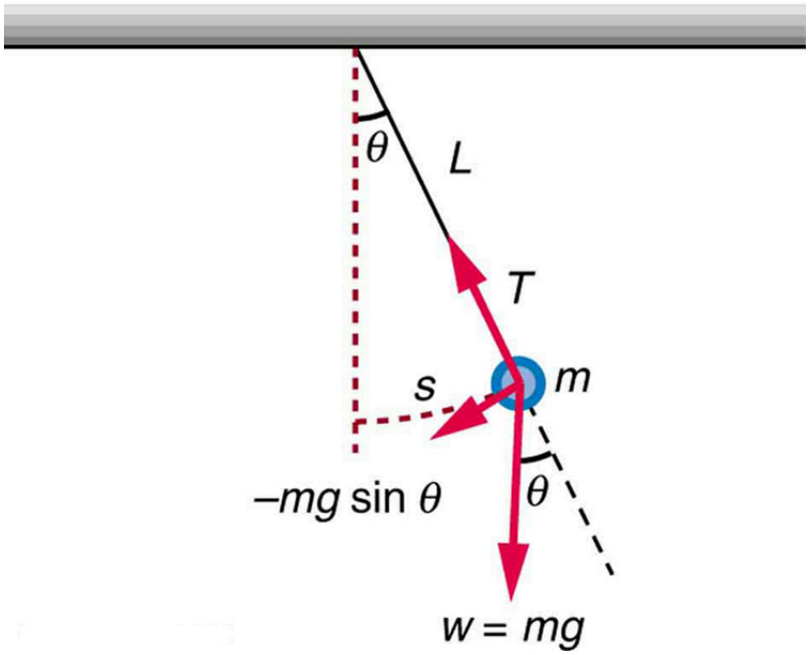


Figure 1. *A simple pendulum has a small-diameter bob and a string that has a very small mass but is strong enough not to stretch appreciably. The linear displacement from equilibrium is s, the length of the arc. Also shown are the forces on the bob, which result in a net force of −mg sinθ toward the equilibrium position—that is, a restoring force.*

We begin by defining the displacement to be the arc length s. We see from Figure 1 that the net force on the bob is tangent to the arc and equals $-mg \sin \theta$. (The weight mg has components $mg \cos \theta$ along

the string and *mg* sin θ tangent to the arc.) Tension in the string exactly cancels the component *mg* cos θ parallel to the string. This leaves a net restoring force back toward the equilibrium position at $\theta = 0$.

Now, if we can show that the restoring force is directly proportional to the displacement, then we have a simple harmonic oscillator. In trying to determine if we have a simple harmonic oscillator, we should note that for small angles (less than about 15°), sin θ *is almost equal to* θ (sin θ and θ differ by about 1% or less at smaller angles). Thus, for angles less than about 15°, the restoring force *F* is

$$F \approx -mg\theta.$$

The displacement *s* is directly proportional to θ. When θ is expressed in radians, the arc length in a circle is related to its radius (*L* in this instance) by:

$$s = L\theta,$$

so that

$$\theta = \frac{s}{L}.$$

For small angles, then, the expression for the restoring force is:

$$F \approx -\frac{mg}{L}s$$

This expression is of the form:

$$F = -kx,$$

where the force constant is given by $k = mg/L$ and the displacement is given by $x = s$. For angles less than about 15°, the restoring force is directly proportional to the displacement, and the simple pendulum is a simple harmonic oscillator.

Using this equation, we can find the period of a pendulum for amplitudes less than about 15°. For the simple pendulum:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}}.$$

Thus,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

for the period of a simple pendulum. This result is interesting because of its simplicity. The only things that affect the period of a simple pendulum are its length and the acceleration due to gravity. The period is completely independent of other factors, such as mass. As with simple harmonic oscillators, the period *T* for a pendulum is nearly independent of amplitude, especially if θ is less than about 15°. Even simple pendulum clocks can be finely adjusted and accurate.

Note the dependence of *T* on *g*. If the length of a pendulum is precisely known, it can actually be used to measure the acceleration due to gravity. Consider the following example.

Example 1: Measuring Acceleration due to Gravity: The Period of a Pendulum

What is the acceleration due to gravity in a region where a simple pendulum having a length 75.000 cm has a period of 1.7357 s?

Strategy

We are asked to find g given the period T and the length L of a pendulum. We can solve $T = 2\pi\sqrt{\frac{L}{g}}$ for g, assuming only that the angle of deflection is less than 15°.

Solution

Square $T = 2\pi\sqrt{\frac{L}{g}}$ and solve for g:

$$g = 4\pi^2 \frac{L}{T^2}.$$

Substitute known values into the new equation:

$$g = 4\pi^2 \frac{0.75000 \text{ m}}{(1.7357 \text{ s})^2}.$$

Calculate to find g:

$$g = 9.8281 \text{ m/s}^2.$$

Discussion

This method for determining g can be very accurate. This is why length and period are given to five digits in this example. For the precision of the approximation $\sin\theta$ almost equal to θ to be better than the precision of the pendulum length and period, the maximum displacement angle should be kept below about 0.5°.

MAKING CAREER CONNECTIONS

Knowing g can be important in geological exploration; for example, a map of g over large geographical regions aids the study of plate tectonics and helps in the search for oil fields and large mineral deposits.

TAKE-HOME EXPERIMENT: DETERMINING g

Use a simple pendulum to determine the acceleration due to gravity g in your own locale. Cut a piece of a string or dental floss so that it is about 1 m long. Attach a small object of high density to the end of the string (for example, a metal nut or a car key). Starting at an angle of less than 10°, allow the pendulum to swing and measure the pendulum's period for 10 oscillations using a stopwatch. Calculate g. How accurate is this measurement? How might it be improved?

Check Your Understanding

An engineer builds two simple pendula. Both are suspended from small wires secured to the ceiling of a room. Each pendulum hovers 2 cm above the floor. Pendulum 1 has a bob with a mass of 10 kg. Pendulum 2 has a bob with a mass of 100 kg. Describe how the motion of the pendula will differ if the bobs are both displaced by 12°.

PHET EXPLORATIONS: PENDULUM LAB

Play with one or two pendulums and discover how the period of a simple pendulum depends on the length of the string, the mass of the pendulum bob, and the amplitude of the swing. It's easy to measure the period using the photogate timer. You can vary friction and the strength of gravity. Use the pendulum to find the value of g on planet X. Notice the anharmonic behavior at large amplitude.

PhET Interactive Simulation

Figure 2. *Pendulum Lab*

Section Summary

- A mass m suspended by a wire of length L is a simple pendulum and undergoes simple harmonic motion for amplitudes less than about 15°.
- The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where L is the length of the string and g is the acceleration due to gravity.

Glossary

simple pendulum
an object with a small mass suspended from a light wire or string

Conceptual Questions

1: Pendulum clocks are made to run at the correct rate by adjusting the pendulum's length. Suppose you move from one city to another where the acceleration due to gravity is slightly greater, taking your pendulum clock with you, will you have to lengthen or shorten the pendulum to keep the correct time, other factors remaining constant? Explain your answer.

Problems & Exercises

As usual, the acceleration due to gravity in these problems is taken to be g = 9.80 m/s^2, unless otherwise specified.

1: What is the length of a pendulum that has a period of 0.500 s?

2: Some people think a pendulum with a period of 1.00 s can be driven with "mental energy" or psycho kinetically, because its period is the same as an average heartbeat. True or not, what is the length of such a pendulum?

3: What is the period of a 1.00-m-long pendulum?

4: How long does it take a child on a swing to complete one swing if her center of gravity is 4.00 m below the pivot?

5: The pendulum on a cuckoo clock is 5.00 cm long. What is its frequency?

6: Two parakeets sit on a swing with their combined center of mass 10.0 cm below the pivot. At what frequency do they swing?

7: (a) A pendulum that has a period of 3.00000 s and that is located where the acceleration due to gravity is 9.79 m/s^2 is moved to a location where it the acceleration due to gravity is 9.82 m/s^2. What is its new period? (b) Explain why so many digits are needed in the value for the period, based on the relation between the period and the acceleration due to gravity.

8: A pendulum with a period of 2.00000 s in one location (g=9.80 m/s^2) is moved to a new location where the period is now 1.99796 s. What is the acceleration due to gravity at its new location?

9: (a) What is the effect on the period of a pendulum if you double its length?

(b) What is the effect on the period of a pendulum if you decrease its length by 5.00%?

10: Find the ratio of the new/old periods of a pendulum if the pendulum were transported from Earth to the Moon, where the acceleration due to gravity is 1.63 m/s^2.

11: At what rate will a pendulum clock run on the Moon, where the acceleration due to gravity is 1.63 m/s^2, if it keeps time accurately on Earth? That is, find the time (in hours) it takes the clock's hour hand to make one revolution on the Moon.

12: Suppose the length of a clock's pendulum is changed by 1.000%, exactly at noon one day. What time will it read 24.00 hours later, assuming it the pendulum has kept perfect time before the change? Note that there are two answers, and perform the calculation to four-digit precision.

13: If a pendulum-driven clock gains 5.00 s/day, what fractional change in pendulum length must be made for it to keep perfect time?

Solutions

Check Your Understanding

1: The movement of the pendula will not differ at all because the mass of the bob has no effect on the motion of a simple pendulum. The pendula are only affected by the period (which is related to the pendulum's length) and by the acceleration due to gravity.

Problems & Exercises

1: 6.21 cm

3: 2.01 cm

5: 2.23 Hz

7: (a) 2.99541 s (b) Since the period is related to the square root of the acceleration of gravity, when the acceleration changes by 1% the period changes by $(0.01)^2 = 0.01\%$ so it is necessary to have at least 4 digits after the decimal to see the changes.

9: (a) Period increases by a factor of 1.41 ($\sqrt{2}$) (b) Period decreases to 97.5% of old period

11: Slow by a factor of 2.45

13: length must increase by 0.0116%.

10.7 Optional: Damped Harmonic Motion

Summary

- Compare and discuss underdamped and overdamped oscillating systems.
- Explain critically damped system.

Figure 1. *In order to counteract dampening forces, this dad needs to keep pushing the swing. (credit: Erik A. Johnson, Flickr)*

A guitar string stops oscillating a few seconds after being plucked. To keep a child happy on a swing, you must keep pushing. Although we can often make friction and other non-conservative forces negligibly small, completely undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.

For a system that has a small amount of damping, the period and frequency are nearly the same as for simple harmonic motion, but the amplitude gradually decreases as shown in Figure 2. This occurs because the non-conservative damping force removes energy from the system, usually in the form of thermal energy. In general, energy removal by non-conservative forces is described as

$$W_{\text{nc}} = \Delta(\text{KE} + \text{PE}),$$

where W_{nc} is work done by a non-conservative force (here the damping force). For a damped harmonic oscillator, $W_{\mathbf{nc}}$ is negative because it removes mechanical energy (KE + PE) from the system.

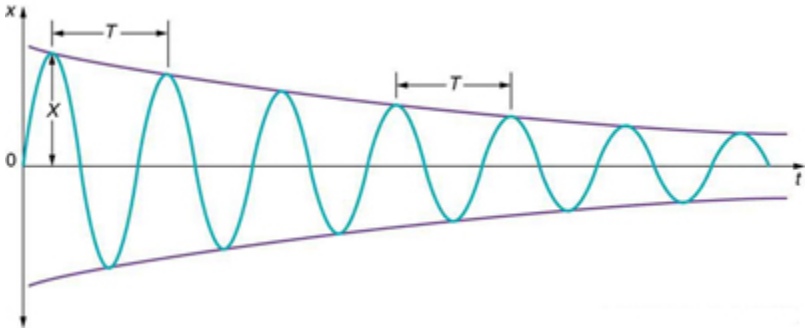


Figure 2. *In this graph of displacement versus time for a harmonic oscillator with a small amount of damping, the amplitude slowly decreases, but the period and frequency are nearly the same as if the system were completely undamped.*

If you gradually *increase* the amount of damping in a system, the period and frequency begin to be affected, because damping opposes and hence slows the back and forth motion. (The net force is smaller in both directions.) If there is very large damping, the system does not even oscillate—it slowly moves toward equilibrium. Figure 3 shows the displacement of a harmonic oscillator for different amounts of damping. When we want to damp out oscillations, such as in the suspension of a car, we may want the system to return to equilibrium as quickly as possible **Critical damping** is defined as the condition in which the damping of an oscillator results in it returning as quickly as possible to its equilibrium position The critically damped system may overshoot the equilibrium position, but if it does, it will do so only once. Critical damping is represented by Curve A in Figure 3. With less-than critical damping, the system will return to equilibrium faster but will overshoot and cross over one or more times. Such a system is **underdamped**; its displacement is represented by the curve in Figure 2. Curve B in Figure 3 represents an **overdamped** system. As with critical damping, it too may overshoot the equilibrium position, but will reach equilibrium over a longer period of time.

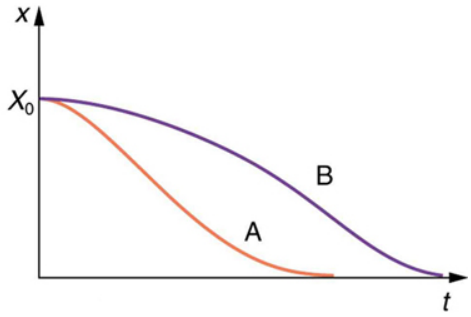


Figure 3. *Displacement versus time for a critically damped harmonic oscillator (A) and an overdamped harmonic oscillator (B). The critically damped oscillator returns to equilibrium at X = 0 in the smallest time possible without overshooting.*

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position. For example, when you stand on bathroom scales that have a needle gauge, the needle moves to its equilibrium position without oscillating. It would be quite inconvenient if the

needle oscillated about the new equilibrium position for a long time before settling. Damping forces can vary greatly in character. Friction, for example, is sometimes independent of velocity (as assumed in most places in this text). But many damping forces depend on velocity—sometimes in complex ways, sometimes simply being proportional to velocity.

Example 1: Damping an Oscillatory Motion: Friction on an Object Connected to a Spring

Damping oscillatory motion is important in many systems, and the ability to control the damping is even more so. This is generally attained using non-conservative forces such as the friction between surfaces, and viscosity for objects moving through fluids. The following example considers friction. Suppose a 0.200-kg object is connected to a spring as shown in Figure 4, but there is simple friction between the object and the surface, and the coefficient of friction μ_k is equal to 0.0800. (a) What is the frictional force between the surfaces? (b) What total distance does the object travel if it is released 0.100 m from equilibrium, starting at v=0? The force constant of the spring is k = 50.0 N/m.

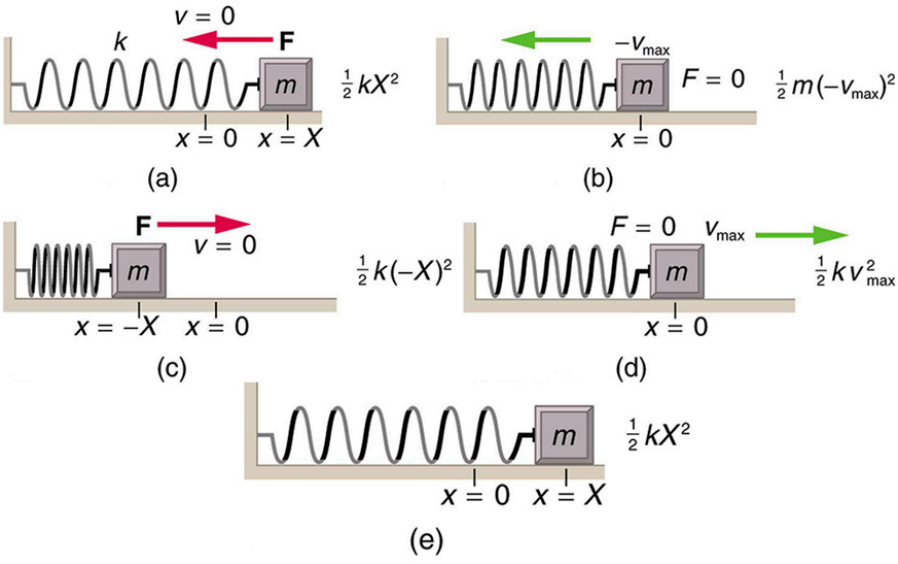


Figure 4. *The transformation of energy in simple harmonic motion is illustrated for an object attached to a spring on a frictionless surface.*

Strategy

This problem requires you to integrate your knowledge of various concepts regarding waves, oscillations, and damping. To solve an integrated concept problem, you must first identify the physical principles involved. Part (a) is about the frictional force. This is a topic involving the application of Newton's Laws. Part (b) requires an understanding of work and conservation of energy, as well as some understanding of horizontal oscillatory systems.

Now that we have identified the principles we must apply in order to solve the problems, we need to identify the knowns and unknowns for each part of the question, as well as the quantity that is constant in Part (a) and Part (b) of the question.

Solution a

1. Choose the proper equation: Friction is $f = \mu_k mg$.
2. Identify the known values.
3. Enter the known values into the equation:
$$f = (0.0800)(0.200\ \text{kg})(9.80\ \text{m/s}^2).$$
4. Calculate and convert units: $f = 0.157$ N.

Discussion a

The force here is small because the system and the coefficients are small.

Solution b

Identify the known:

- The system involves elastic potential energy as the spring compresses and expands, friction that is related to the work done, and the kinetic energy as the body speeds up and slows down.
- Energy is not conserved as the mass oscillates because friction is a non-conservative force.
- The motion is horizontal, so gravitational potential energy does not need to be considered.
- Because the motion starts from rest, the energy in the system is initially $\text{PE}_{\text{el,i}} = (1/2)kX^2$. This energy is removed by work done by friction $W_{\text{nc}} = -fd$, where d is the total distance traveled and $f = \mu_k mg$ is the force of friction. When the system stops moving, the friction force will balance the force exerted by the spring, so $\text{PE}_{\text{e1,f}} = (1/2)kx^2$ where x is the final position and is given by
$$\begin{aligned} F_{\text{el}} &= f \\ kx &= \mu_k mg \\ x &= \frac{\mu_k mg}{k}. \end{aligned}$$

1. By equating the work done to the energy removed, solve for the distance d.
2. The work done by the non-conservative forces equals the initial, stored elastic potential energy. Identify the correct equation to use:
$$W_{\text{nc}} = \Delta\text{KE} + \text{PE} = \text{PE}_{\text{el,f}} - \text{PE}_{\text{el,i}} = \frac{1}{2}k((\frac{\mu_k mg}{k})^2 - X^2).$$
3. Recall that $W_{\text{nc}} = -fd$.
4. Enter the friction as $f = \mu_k mg$ into $W_{\text{nc}} = -fd$, thus
$$W_{\text{nc}} = -\mu_k mgd.$$
5. Combine these two equations to find
$$\frac{1}{2}k((\frac{\mu_k mg}{k})^2 - X^2) = -\mu_k mgd$$

6. Solve the equation for d:
$$d = \frac{k}{2\mu_{\text{k}} mg}(X^2 - (\frac{\mu_{\text{k}} mg}{k})^2).$$
7. Enter the known values into the resulting equation:
$$d = \frac{50.0\ \text{N/m}}{2(0.0800)(0.200\ \text{kg})(9.80\ \text{m/s}^2)}(0.100\ \text{m}^2 - (\frac{(0.0800)(0.200\ \text{kg})(9.80\ \text{m/s}^2)}{50.0\ \text{N/m}})^2).$$
8. Calculate d and convert units:
$$d = 1.59\ \text{m}.$$

Discussion b

This is the total distance traveled back and forth across $x = 0$, which is the undamped equilibrium position. The number of oscillations about the equilibrium position will be more than d/X = (1.59 m)/(0.100 m) = 15.9 because the amplitude of the oscillations is decreasing with time. At the end of the motion, this system will not return to $x = 0$ for this type of damping force, because static friction will exceed the restoring force. This system is underdamped. In contrast, an overdamped system with a simple constant damping force would not cross the equilibrium position $x = 0$ a single time. For example, if this system had a damping force 20 times greater, it would only move 0.0484 m toward the equilibrium position from its original 0.100-m position.

This worked example illustrates how to apply problem-solving strategies to situations that integrate the different concepts you have learned. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknowns using familiar problem-solving strategies. These are found throughout the text, and many worked examples show how to use them for single topics. In this integrated concepts example, you can see how to apply them across several topics. You will find these techniques useful in applications of physics outside a physics course, such as in your profession, in other science disciplines, and in everyday life.

Check Your Understanding 1

Why are completely undamped harmonic oscillators so rare?

Check Your Understanding 2

Describe the difference between overdamping, underdamping, and critical damping.

Section Summary

- Damped harmonic oscillators have non-conservative forces that dissipate their energy.
- Critical damping returns the system to equilibrium as fast as possible without overshooting.
- An underdamped system will oscillate through the equilibrium position.
- An overdamped system moves more slowly toward equilibrium than one that is critically damped.

Glossary

critical damping
the condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

over damping
the condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system

under damping
the condition in which damping of an oscillator causes it to return to equilibrium with the amplitude gradually decreasing to zero; system returns to equilibrium faster but overshoots and crosses the equilibrium position one or more times

Conceptual Questions

1: Give an example of a damped harmonic oscillator. (They are more common than undamped or simple harmonic oscillators.)

2: How would a car bounce after a bump under each of these conditions?

- overdamping
- underdamping
- critical damping

3: Most harmonic oscillators are damped and, if undriven, eventually come to a stop. How is this observation related to the second law of thermodynamics?

Problems & Exercises

1: The amplitude of a lightly damped oscillator decreases by 3.0 % during each cycle. What percentage of the mechanical energy of the oscillator is lost in each cycle?

Solutions

Check Your Understanding 1

Friction often comes into play whenever an object is moving. Friction causes damping in a harmonic oscillator.

Check Your Understanding 2

An overdamped system moves slowly toward equilibrium. An underdamped system moves quickly to equilibrium, but will oscillate about the equilibrium point as it does so. A critically damped system moves as quickly as possible toward equilibrium without oscillating about the equilibrium.

10.8 Optional: Forced Oscillations and Resonance

Summary

- Observe resonance of a paddle ball on a string.
- Observe amplitude of a damped harmonic oscillator.

Figure 1. *You can cause the strings in a piano to vibrate simply by producing sound waves from your voice. (credit: Matt Billings, Flickr)*

Sit in front of a piano sometime and sing a loud brief note at it with the dampers off its strings. It will sing the same note back at you—the strings, having the same frequencies as your voice, are resonating in response to the forces from the sound waves that you sent to them. Your voice and a piano's strings is a good example of the fact that objects—in this case, piano strings—can be forced to oscillate but oscillate best at their natural frequency. In this section, we shall briefly explore applying a *periodic driving force* acting on a simple harmonic oscillator. The driving force puts energy into the system at a certain frequency, not necessarily the same as the natural frequency of the system. The **natural frequency** is the frequency at which a system would oscillate if there were no driving and no damping force.

Most of us have played with toys involving an object supported on an elastic band, something like the paddle ball suspended from a finger in Figure 2. Imagine the finger in the figure is your finger. At first you hold your finger steady, and the ball bounces up and down with a small amount of damping. If you move your finger up and down slowly, the ball will follow along without bouncing much on its own. As you increase the frequency at which you move your finger up and down, the ball will respond by oscillating with increasing amplitude. When you drive the ball at its natural frequency, the ball's oscillations increase in amplitude with each oscillation for as long as you drive it. The phenomenon of

driving a system with a frequency equal to its natural frequency is called **resonance**. A system being driven at its natural frequency is said to **resonate**. As the driving frequency gets progressively higher than the resonant or natural frequency, the amplitude of the oscillations becomes smaller, until the oscillations nearly disappear and your finger simply moves up and down with little effect on the ball.

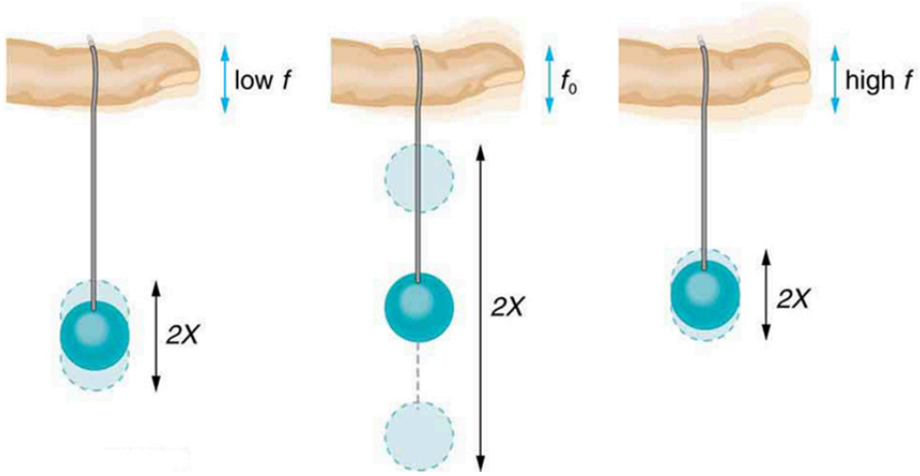


Figure 2. *The paddle ball on its rubber band moves in response to the finger supporting it. If the finger moves with the natural frequency f_0 of the ball on the rubber band, then a resonance is achieved, and the amplitude of the ball's oscillations increases dramatically. At higher and lower driving frequencies, energy is transferred to the ball less efficiently, and it responds with lower-amplitude oscillations.*

Figure 3 shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. There are three curves on the graph, each representing a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less energy is removed by the damping force.

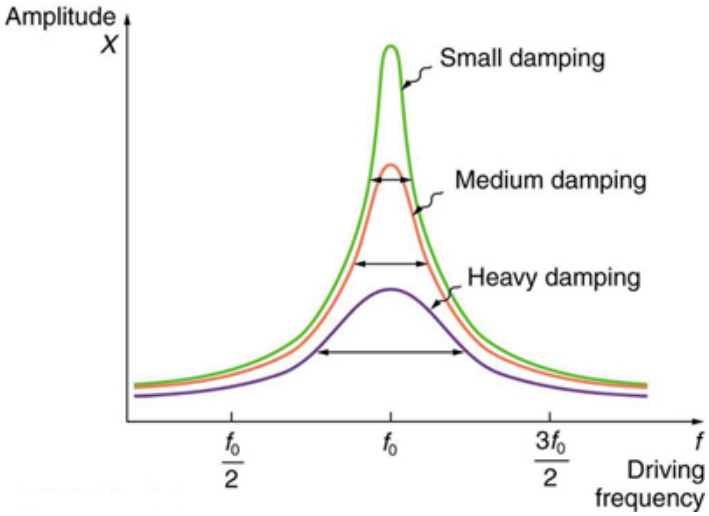


Figure 3. *Amplitude of a harmonic oscillator as a function of the frequency of the driving force. The curves represent the same oscillator with the same natural frequency but with different amounts of damping. Resonance occurs when the driving frequency equals the natural frequency, and the greatest response is for the least amount of damping. The narrowest response is also for the least damping.*

It is interesting that the widths of the resonance curves shown in Figure 3 depend on damping: the less the damping, the narrower the resonance. The message is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible. Little damping is the case for piano strings and many other musical instruments. Conversely, if you want small-amplitude oscillations, such as in a car's suspension system, then you want heavy damping. Heavy damping reduces the amplitude, but the tradeoff is that the system responds at more frequencies.

These features of driven harmonic oscillators apply to a huge variety of systems. When you tune a radio, for example, you are adjusting its resonant frequency so that it only oscillates to the desired station's broadcast (driving) frequency. The more selective the radio is in discriminating between stations, the smaller its damping. Magnetic resonance imaging (MRI) is a widely used medical diagnostic tool in which atomic nuclei (mostly hydrogen nuclei) are made to resonate by incoming radio waves (on the order of 100 MHz). A child on a swing is driven by a parent at the swing's natural frequency to achieve maximum amplitude. In all of these cases, the efficiency of energy transfer from the driving force into the oscillator is best at resonance. Speed bumps and gravel roads prove that even a car's suspension system is not immune to resonance. In spite of finely engineered shock absorbers, which ordinarily convert mechanical energy to thermal energy almost as fast as it comes in, speed bumps still cause a large-amplitude oscillation. On gravel roads that are corrugated, you may have noticed that if you travel at the "wrong" speed, the bumps are very noticeable whereas at other speeds you may hardly feel the bumps at all. Figure 4 shows a photograph of a famous example (the Tacoma Narrows Bridge) of the destructive effects of a driven harmonic oscillation. The Millennium Bridge in London was closed for a short period of time for the same reason while inspections were carried out.

In our bodies, the chest cavity is a clear example of a system at resonance. The diaphragm and chest wall drive the oscillations of the chest cavity which result in the lungs inflating and deflating. The system is

critically damped and the muscular diaphragm oscillates at the resonant value for the system, making it highly efficient.

Figure 4. *In 1940, the Tacoma Narrows Bridge in Washington state collapsed. Heavy cross winds drove the bridge into oscillations at its resonant frequency. Damping decreased when support cables broke loose and started to slip over the towers, allowing increasingly greater amplitudes until the structure failed (credit: PRI's Studio 360, via Flickr)*

Check Your Understanding

1: A famous magic trick involves a performer singing a note toward a crystal glass until the glass shatters. Explain why the trick works in terms of resonance and natural frequency.

Section Summary

- A system's natural frequency is the frequency at which the system will oscillate if not affected by driving or damping forces.
- A periodic force driving a harmonic oscillator at its natural frequency produces resonance. The system is said to resonate.
- The less damping a system has, the higher the amplitude of the forced oscillations near resonance. The more damping a system has, the broader response it has to varying driving frequencies.

Glossary

natural frequency
the frequency at which a system would oscillate if there were no driving and no damping forces

resonance
the phenomenon of driving a system with a frequency equal to the system's natural frequency

resonate

a system being driven at its natural frequency

Conceptual Questions

1: Why are soldiers in general ordered to “route step” (walk out of step) across a bridge?

Problems & Exercises

1: How much energy must the shock absorbers of a 1200-kg car dissipate in order to damp a bounce that initially has a velocity of 0.800 m/s at the equilibrium position? Assume the car returns to its original vertical position.

2: If a car has a suspension system with a force constant of 5.00×10^4 N/m, how much energy must the car’s shocks remove to dampen an oscillation starting with a maximum displacement of 0.0750 m?

3: (a) How much will a spring that has a force constant of 40.0 N/m be stretched by an object with a mass of 0.500 kg when hung motionless from the spring? (b) Calculate the decrease in gravitational potential energy of the 0.500-kg object when it descends this distance. (c) Part of this gravitational energy goes into the spring. Calculate the energy stored in the spring by this stretch, and compare it with the gravitational potential energy. Explain where the rest of the energy might go.

4: Suppose you have a 0.750-kg object on a horizontal surface connected to a spring that has a force constant of 150 N/m. There is simple friction between the object and surface with a static coefficient of friction μ_s=0.100. (a) How far can the spring be stretched without moving the mass? (b) If the object is set into oscillation with an amplitude twice the distance found in part (a), and the kinetic coefficient of friction is μ_k = 0.0850, what total distance does it travel before stopping? Assume it starts at the maximum amplitude.

5: Engineering Application: A suspension bridge oscillates with an effective force constant of 1.00×10^8 N/m. (a) How much energy is needed to make it oscillate with an amplitude of 0.100 m? (b) If soldiers march across the bridge with a cadence equal to the bridge’s natural frequency and impart 1.00×10^4 J of energy each second, how long does it take for the bridge’s oscillations to go from 0.100 m to 0.500 m amplitude?

Solutions

Check Your Understanding

1: The performer must be singing a note that corresponds to the natural frequency of the glass. As the sound wave is directed at the glass, the glass responds by resonating at the same frequency as the

sound wave. With enough energy introduced into the system, the glass begins to vibrate and eventually shatters.

Problems & Exercises

1: 384 J

3: (a). 0.123 m (b). -0.600 J (c). 0.300 J. The rest of the energy may go into heat caused by friction and other damping forces.

5: (a) $5.00 \times 10^5 \text{ J}$ (b) $1.20 \times 10^3 \text{ s}$

Chapter 11 Travelling Waves and Interference

In the chapter on Oscillatory Motion and Simple Harmonic Motion, you saw several examples of oscillations. In this chapter, you will see another example of an oscillation: **travelling waves**. A guitar creates sound waves. You can make water waves in a swimming pool by slapping the water with your hand. You can no doubt think of other types of waves. Some, such as water waves, are visible. Some, such as sound waves, are not. But every wave *is a disturbance that moves from its source and carries energy*. Other examples of waves include earthquakes and visible light. Even subatomic particles, such as electrons, can behave like waves. After introducing the idea of a travelling wave, we will explore what happens when two or more waves share the same space, in the phenomena known as superposition and interference.

11.1 Waves

Summary

- State the characteristics of a wave.
- State the mathematical description of a traveling wave
- Calculate the speed and determine the direction of wave propagation.

Figure 1. *Waves in the ocean behave similarly to all other types of waves. (credit: Steve Jurveston, Flickr)*

What do we mean when we say something is a wave? The most intuitive and easiest wave to imagine is the familiar water wave. More precisely, a **wave** is a disturbance that propagates, or moves from the place it was created. For water waves, the disturbance is in the surface of the water, perhaps created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker. For earthquakes, there are several types of disturbances, including disturbance of Earth's surface and pressure disturbances under the surface. Even radio waves are most easily understood using an analogy with water waves. Visualizing water waves is useful because there is more to it than just a mental image. Water waves exhibit characteristics common to all waves, such as amplitude, period, frequency and energy. All wave characteristics can be described by a small set of underlying principles.

A wave is a disturbance that propagates, or moves from the place it was created. The simplest waves repeat themselves for several cycles and are associated with simple harmonic motion. Let us start by considering the simplified water wave in Figure 2. The wave is an up and down disturbance of the water surface. It causes a sea gull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The time for one complete up and down motion is the wave's period T. The wave's frequency is $f = 1/T$, as usual. The wave itself moves to the right in the

figure. This movement of the wave is actually the disturbance moving to the right, not the water itself (or the bird would move to the right). We define **wave velocity** v_w to be the speed at which the disturbance moves. Wave velocity is sometimes also called the *propagation velocity or propagation speed*, because the disturbance propagates from one location to another.

MISCONCEPTION ALERT

Many people think that water waves push water from one direction to another. In fact, the particles of water tend to stay in one location, save for moving up and down due to the energy in the wave. The energy moves forward through the water, but the water stays in one place. If you feel yourself pushed in an ocean, what you feel is the energy of the wave, not a rush of water.

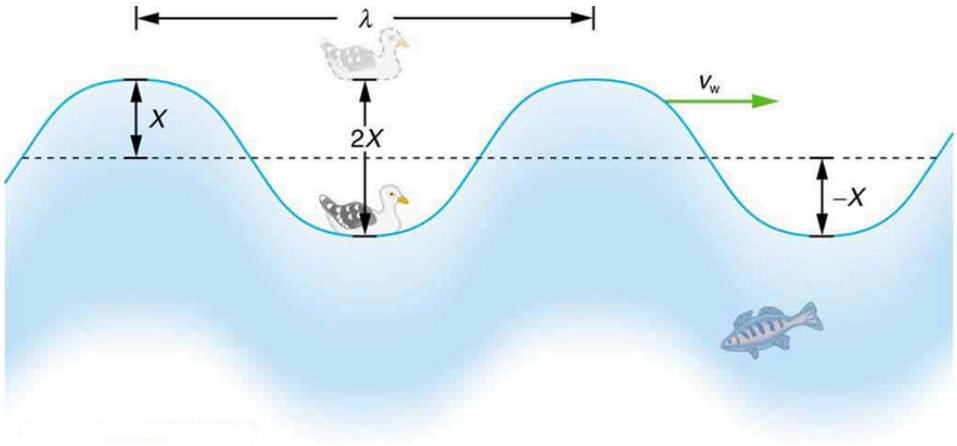


Figure 2. *An idealized ocean wave passes under a sea gull that bobs up and down in simple harmonic motion. The wave has a wavelength λ, which is the distance between adjacent identical parts of the wave. The up and down disturbance of the surface propagates parallel to the surface at a speed* v_w.

The water wave in the figure also has a length associated with it, called its **wavelength** $\lambda$the distance between adjacent identical parts of a wave. (λ is the distance parallel to the direction of propagation.) The speed of propagation v_w is the distance the wave travels in a given time, which is one wavelength in the time of one period. In equation form, that is

$$v_w = \frac{\lambda}{T} = \lambda f$$

This fundamental relationship holds for all types of waves. For example, for water waves, v_w is the speed of a surface wave; for sound, v_w is the speed of sound; and for visible light, v_w is the speed of light.

TAKE-HOME EXPERIMENT: WAVES IN A BOWL

Fill a large bowl or basin with water and wait for the water to settle so there are no ripples. Gently drop a cork into the middle of the bowl. Estimate the wavelength and period of oscillation of the water wave that propagates away from the cork. Remove the cork from the bowl and wait for the water to settle again. Gently drop the cork at a height that is different from the first drop. Does the wavelength depend upon how high above the water the cork is dropped?

Example 1: Calculate the Velocity of Wave Propagation: Gull in the Ocean

Calculate the wave velocity of the ocean wave in Figure 2 if the distance between wave crests is 10.0 m and the time for a sea gull to bob up and down is 5.00 s.

Strategy

We are asked to find v_{w}. The given information tells us that $\lambda = 10.0\ \text{m}$ and T = 5.00 s. Therefore, we can use $v_{\text{w}} = \dfrac{\lambda}{T}$ to find the wave velocity.

Solution

1. Enter the known values into $v_{\text{w}} = \dfrac{\lambda}{T}$
$$v_{\text{w}} = \frac{10.0\ \text{m}}{5.00\ \text{s}}.$$
2. Solve for v_{w} to find v_{w}= 2.00 m/s.

Discussion

This slow speed seems reasonable for an ocean wave. Note that the wave moves to the right in the figure at this speed, not the varying speed at which the sea gull moves up and down.

Transverse and Longitudinal Waves

A simple wave consists of a periodic disturbance that propagates from one place to another. The wave in Figure 3 propagates in the horizontal direction while the surface is disturbed in the vertical direction. Such a wave is called a **transverse wave** or shear wave; in such a wave, the disturbance is perpendicular to the direction of propagation. In contrast, in a **longitudinal wave** or compressional wave, the disturbance is parallel to the direction of propagation. Figure 4 shows an example of a

longitudinal wave. The size of the disturbance is its amplitude X and is completely independent of the speed of propagation $\boldsymbol{v_w}$.

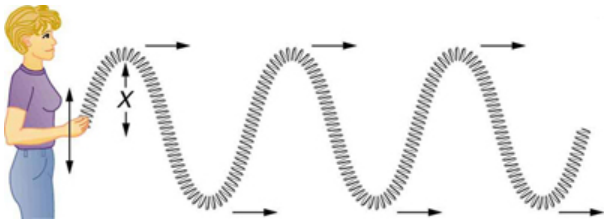


Figure 3. *In this example of a transverse wave, the wave propagates horizontally, and the disturbance in the cord is in the vertical direction.*

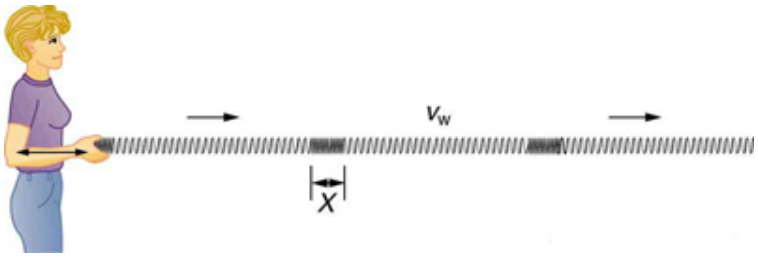


Figure 4. *In this example of a longitudinal wave, the wave propagates horizontally, and the disturbance in the cord is also in the horizontal direction.*

Waves may be transverse, longitudinal, or *a combination of the two*. (Water waves are actually a combination of transverse and longitudinal. The simplified water wave illustrated in Figure 2 shows no longitudinal motion of the bird.) The waves on the strings of musical instruments are transverse—so are electromagnetic waves, such as visible light.

Sound waves in air and water are longitudinal. Their disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and thus the sound waves in them must be longitudinal or compressional. Sound in solids can be both longitudinal and transverse.

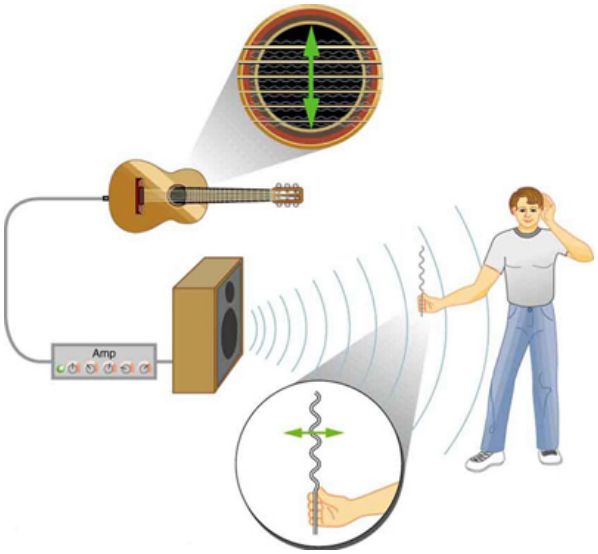


Figure 5. *The wave on a guitar string is transverse. The sound wave rattles a sheet of paper in a direction that shows the sound wave is longitudinal.*

Earthquake waves under Earth's surface also have both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). These components have important individual characteristics—they propagate at different speeds, for example. Earthquakes also have surface waves that are similar to surface waves on water.

Mathematical Description of Traveling Waves

The displacement of a traveling wave (for example, the distance of the guitar string from its equilibrium position, or the height of the water in a water wave) is a function of position **and** time. As an example, consider a water wave. If you float in the water at a fixed position, you would notice that you move up and down. This is due to a change in displacement of the wave as a function of time. On the other hand, if you take a picture of a lake on a windy day (which means you are seeing the all parts of the waves at the same time), you would find that the height of the water varies with position. Therefore, to describe the displacement of a traveling wave the displacement must depend on position and time. The mathematical description of a traveling wave is

$$D(x,t) = A\cos(\omega t - kx + \phi_0)$$

where D is the **displacement** of the wave, ω is the **angular frequency** and k is the **wave number**. The term ϕ_0 is called the **phase offset** which acts to shift the wave to the left or right by a fixed amount. The angular frequency is related to the frequency and period of the wave oscillation by

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

The wave number is related to the wavelength of the oscillation by

$$k = \frac{2\pi}{\lambda}.$$

Misconception Alert: The meaning of *k*

In the context of the mathematical description of traveling waves, *k* is not related to the spring constant that was used in simple harmonic motion.

Using the definition of k and ω the **speed** of a traveling wave can be written as

$$v_{\text{w}} = \lambda f = \frac{\omega}{k}.$$

The **direction** that the wave travels can be determined by the relative direction of the sign in front of k and ω:

- If k and ω have the same sign (either both positive or both negative) then the wave travels to the right.
- If k and ω have opposite signs (one positive and one negative) then the wave travels to the left.

Check Your Understanding

1: Why is it important to differentiate between longitudinal and transverse waves?

PHET EXPLORATIONS: WAVE ON A STRING

Watch a string vibrate in slow motion. Wiggle the end of the string and make waves, or adjust the frequency and amplitude of an oscillator. Adjust the damping and tension. The end can be fixed, loose, or open.

PhET Interactive Simulation

Figure 6. Wave on a String

Section Summary

- A wave is a disturbance that moves from the point of creation with a wave velocity v_{w}.
- A wave has a wavelength λ, which is the distance between adjacent identical parts of the wave.
- Wave velocity and wavelength are related to the wave's frequency and period by $v_{\text{w}} = \frac{\lambda}{T}$ or $v_{\text{w}} = f\lambda$.
- A transverse wave has a disturbance perpendicular to its direction of propagation, whereas a longitudinal wave has a disturbance parallel to its direction of propagation.

Glossary

longitudinal wave
: a wave in which the disturbance is parallel to the direction of propagation

transverse wave
: a wave in which the disturbance is perpendicular to the direction of propagation

wave velocity
: the speed at which the disturbance moves. Also called the propagation velocity or propagation speed

wavelength
: the distance between adjacent identical parts of a wave

Conceptual Questions

1: Give one example of a transverse wave and another of a longitudinal wave, being careful to note the relative directions of the disturbance and wave propagation in each.

2: What is the difference between propagation speed and the frequency of a wave? Does one or both affect wavelength? If so, how?

Problems & Exercises

1: Storms in the South Pacific can create waves that travel all the way to the California coast, which are 12,000 km away. How long does it take them if they travel at 15.0 m/s?

2: Waves on a swimming pool propagate at 0.750 m/s. You splash the water at one end of the pool and observe the wave go to the opposite end, reflect, and return in 30.0 s. How far away is the other end of the pool?

3: Wind gusts create ripples on the ocean that have a wavelength of 5.00 cm and propagate at 2.00 m/s. What is their frequency?

4: How many times a minute does a boat bob up and down on ocean waves that have a wavelength of 40.0 m and a propagation speed of 5.00 m/s?

5: Scouts at a camp shake the rope bridge they have just crossed and observe the wave crests to be 8.00 m apart. If they shake it the bridge twice per second, what is the propagation speed of the waves?

6: What is the wavelength of the waves you create in a swimming pool if you splash your hand at a rate of 2.00 Hz and the waves propagate at 0.800 m/s?

7: What is the wavelength of an earthquake that shakes you with a frequency of 10.0 Hz and gets to another city 84.0 km away in 12.0 s?

8: Radio waves transmitted through space at 3.00×10^8 m/s by the Voyager spacecraft have a wavelength of 0.120 m. What is their frequency?

9: Your ear is capable of differentiating sounds that arrive at the ear just 1.00 ms apart. What is the minimum distance between two speakers that produce sounds that arrive at noticeably different times on a day when the speed of sound is 340 m/s?

10: (a) Seismographs measure the arrival times of earthquakes with a precision of 0.100 s. To get the distance to the epicenter of the quake, they compare the arrival times of S- and P-waves, which travel at different speeds. Figure 7) If S- and P-waves travel at 4.00 and 7.20 km/s, respectively, in the region considered, how precisely can the distance to the source of the earthquake be determined? (b) Seismic waves from underground detonations of nuclear bombs can be used to locate the test site and detect violations of test bans. Discuss whether your answer to (a) implies a serious limit to such detection. (Note also that the uncertainty is greater if there is an uncertainty in the propagation speeds of the S- and P-waves.)

Figure 7. *A seismograph as described in above problem.(credit: Oleg Alexandrov)*

Solutions

Check Your Understanding

In the different types of waves, energy can propagate in a different direction relative to the motion of the wave. This is important to understand how different types of waves affect the materials around them.

Problems & Exercises

1: $t = 9.26\ \text{d}$

3: $f = 40.0\ \text{Hz}$

5: $v_{\text{w}} = 16.0\ \text{m/s}$

7: $\lambda = 700\ \text{m}$

9: $d = 34.0\ \text{cm}$

11.2 Superposition and Interference

Summary

- Explain standing waves.
- Describe the mathematical representation of overtones and beat frequency.

Figure 1. *These waves result from the superposition of several waves from different sources, producing a complex pattern. (credit: waterborough, Wikimedia Commons)*

Most waves do not look very simple. They look more like the waves in Figure 1 than like the simple water wave considered in Chapter 16.9 Waves. (Simple waves may be created by a simple harmonic oscillation, and thus have a sinusoidal shape). Complex waves are more interesting, even beautiful, but they look formidable. Most waves appear complex because they result from several simple waves adding together. Luckily, the rules for adding waves are quite simple.

When two or more waves arrive at the same point, they superimpose themselves on one another. More specifically, the disturbances of waves are superimposed when they come together—a phenomenon called **superposition**. Each disturbance corresponds to a force, and forces add. If the disturbances are along the same line, then the resulting wave is a simple addition of the disturbances of the individual waves—that is, their amplitudes add. Figure 2 and Figure 3 illustrate superposition in two special cases, both of which produce simple results.

Figure 2 shows two identical waves that arrive at the same point exactly in phase. The crests of the two waves are precisely aligned, as are the troughs. This superposition produces pure **constructive interference**. Because the disturbances add, pure constructive interference produces a wave that has twice the amplitude of the individual waves, but has the same wavelength.

Figure 3 shows two identical waves that arrive exactly out of phase—that is, precisely aligned crest

to trough—producing pure **destructive interference**. Because the disturbances are in the opposite direction for this superposition, the resulting amplitude is zero for pure destructive interference—the waves completely cancel.

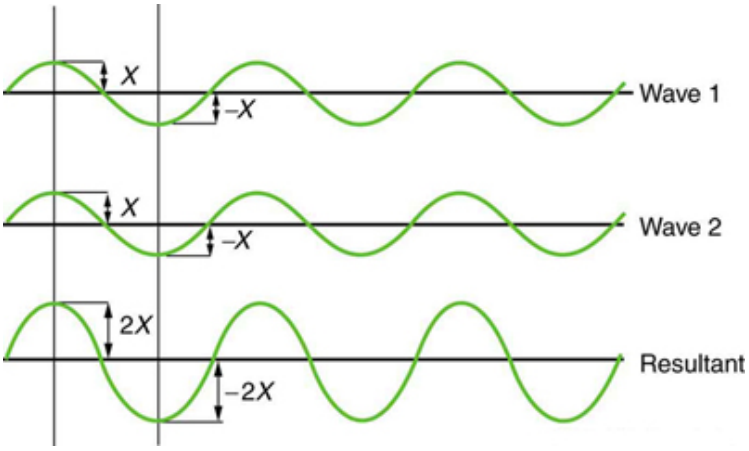


Figure 2. *Pure constructive interference of two identical waves produces one with twice the amplitude, but the same wavelength.*

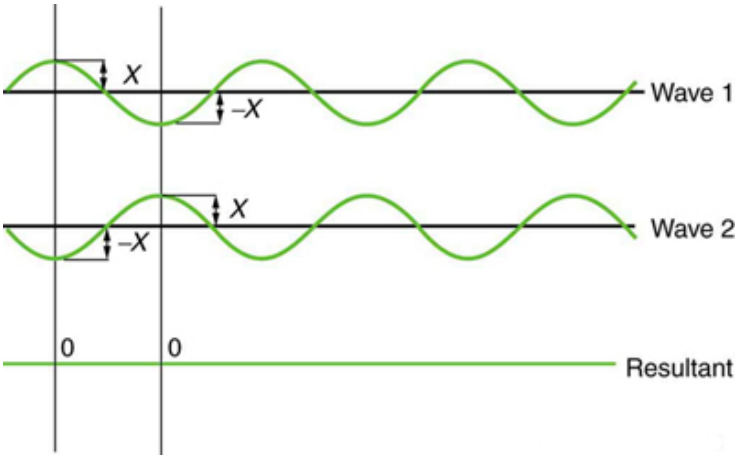


Figure 3. *Pure destructive interference of two identical waves produces zero amplitude, or complete cancellation.*

While pure constructive and pure destructive interference do occur, they require precisely aligned identical waves. The superposition of most waves produces a combination of constructive and destructive interference and can vary from place to place and time to time. Sound from a stereo, for example, can be loud in one spot and quiet in another. Varying loudness means the sound waves add partially constructively and partially destructively at different locations. A stereo has at least two speakers creating sound waves, and waves can reflect from walls. All these waves superimpose. An example of sounds that vary over time from constructive to destructive is found in the combined whine of airplane jets heard by a stationary passenger. The combined sound can fluctuate up and down in volume as the sound from the two engines varies in time from constructive to destructive. These examples are of waves that are similar.

An example of the superposition of two dissimilar waves is shown in Figure 4. Here again, the disturbances add and subtract, producing a more complicated looking wave.

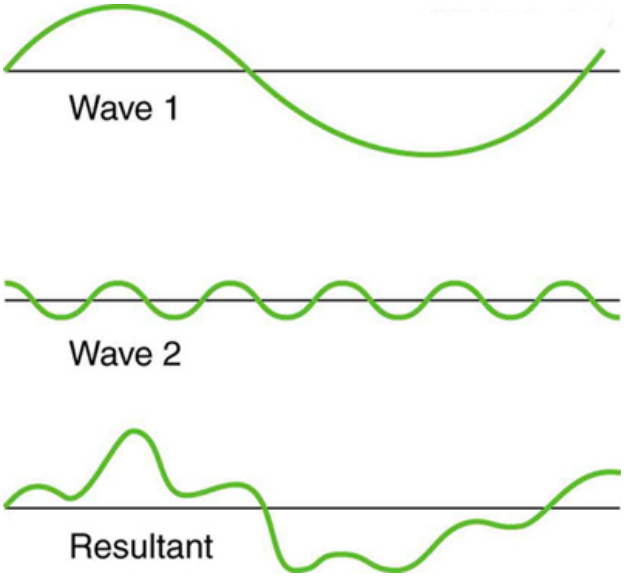


Figure 4. *Superposition of non-identical waves exhibits both constructive and destructive interference.*

Standing Waves

Sometimes waves do not seem to move; rather, they just vibrate in place. Unmoving waves can be seen on the surface of a glass of milk in a refrigerator, for example. Vibrations from the refrigerator motor create waves on the milk that oscillate up and down but do not seem to move across the surface. These waves are formed by the superposition of two or more moving waves, such as illustrated in Figure 5 for two identical waves moving in opposite directions. The waves move through each other with their disturbances adding as they go by. If the two waves have the same amplitude and wavelength, then they alternate between constructive and destructive interference. The resultant looks like a wave standing in place and, thus, is called a **standing wave**. Waves on the glass of milk are one example of standing waves. There are other standing waves, such as on guitar strings and in organ pipes. With the glass of milk, the two waves that produce standing waves may come from reflections from the side of the glass.

A closer look at earthquakes provides evidence for conditions appropriate for resonance, standing waves, and constructive and destructive interference. A building may be vibrated for several seconds with a driving frequency matching that of the natural frequency of vibration of the building—producing a resonance resulting in one building collapsing while neighboring buildings do not. Often buildings of a certain height are devastated while other taller buildings remain intact. The building height matches the condition for setting up a standing wave for that particular height. As the earthquake waves travel along the surface of Earth and reflect off denser rocks, constructive interference occurs at certain points. Often areas closer to the epicenter are not damaged while areas farther away are damaged.

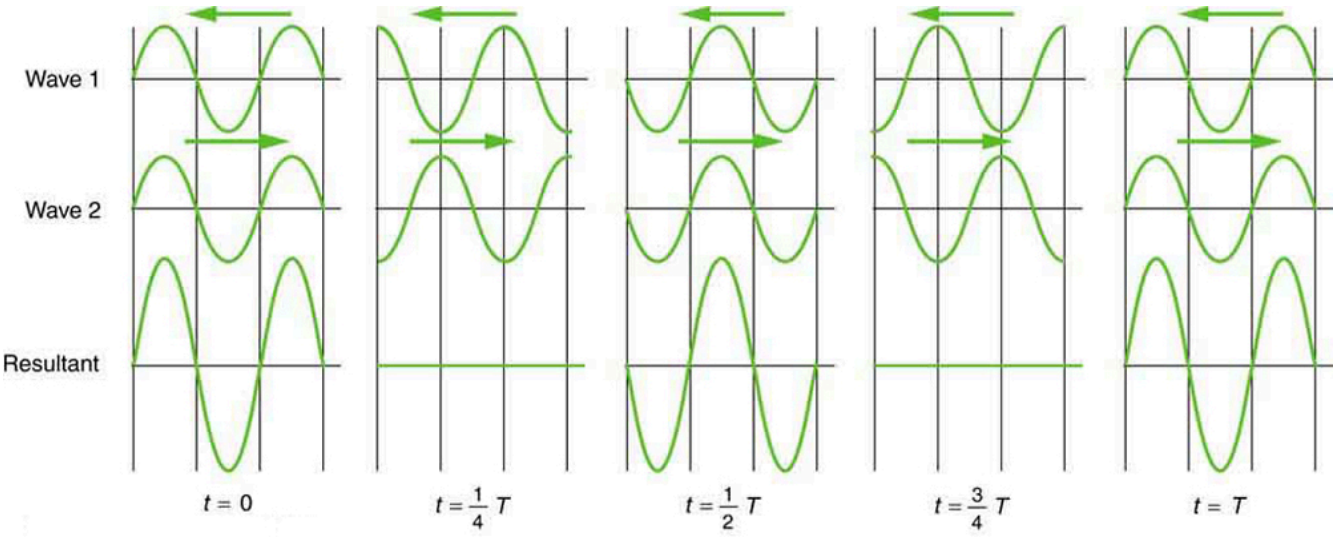


Figure 5. *Standing wave created by the superposition of two identical waves moving in opposite directions. The oscillations are at fixed locations in space and result from alternately constructive and destructive interference.*

Standing waves are also found on the strings of musical instruments and are due to reflections of waves from the ends of the string. Figure 6 and Figure 7 show three standing waves that can be created on a string that is fixed at both ends. **Nodes** are the points where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave. The fixed ends of strings must be nodes, too, because the string cannot move there. The word **antinode** is used to denote the location of maximum amplitude in standing waves. Standing waves on strings have a frequency that is related to the propagation speed v_{w} of the disturbance on the string. The wavelength λ is determined by the distance between the points where the string is fixed in place.

The lowest frequency, called the **fundamental frequency**, is thus for the longest wavelength, which is seen to be $\lambda_1 = 2L$. Therefore, the fundamental frequency is $f_1 = v_{\text{w}}/\lambda_1 = v_{\text{w}}/2L$. In this case, the **overtones** or harmonics are multiples of the fundamental frequency. As seen in Figure 7, the first harmonic can easily be calculated since $\lambda_2 = L$. Thus, $f_2 = v_{\text{w}}/\lambda_2 = v_{\text{w}}/2L = 2f_1$. Similarly, $f_3 = 3f_1$, and so on. All of these frequencies can be changed by adjusting the tension in the string. The greater the tension, the greater v_{w} is and the higher the frequencies. This observation is familiar to anyone who has ever observed a string instrument being tuned. We will see in later chapters that standing waves are crucial to many resonance phenomena, such as in sounding boxes on string instruments.

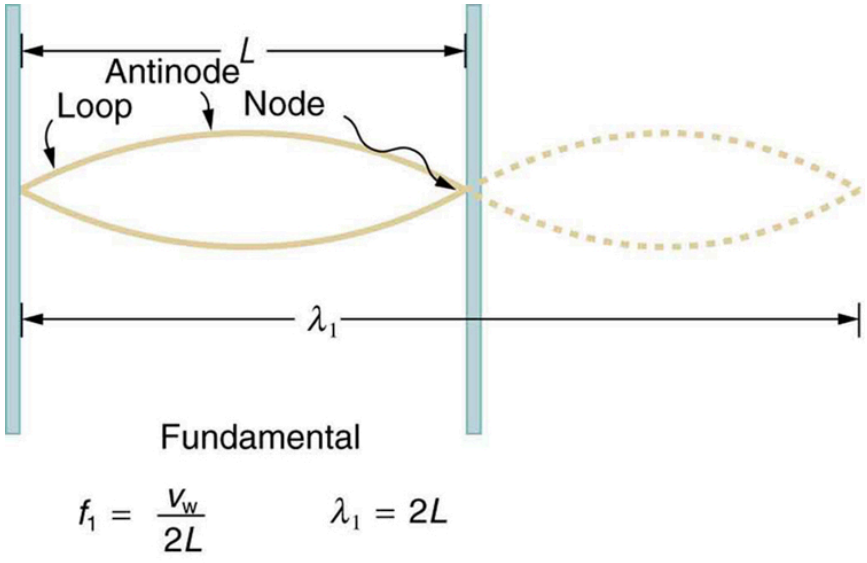


Figure 6. *The figure shows a string oscillating at its fundamental frequency.*

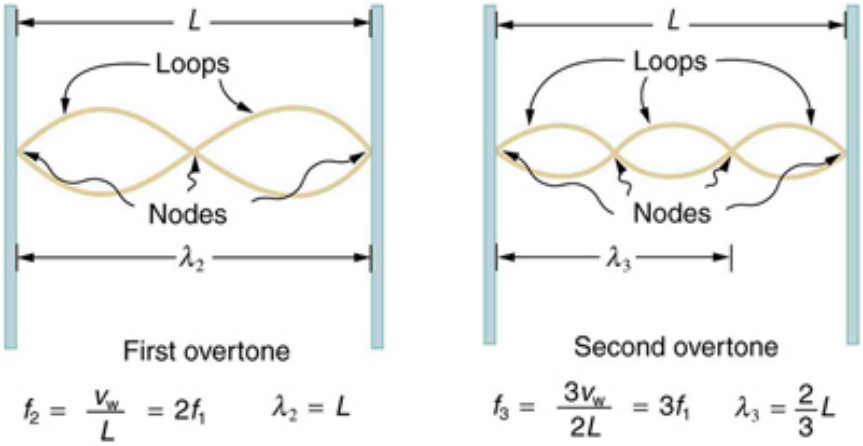


Figure 7. *First and second overtone frequencies (corresponding to the second and third harmonic frequencies) are shown.*

Beats

Striking two adjacent keys on a piano produces a warbling combination usually considered to be unpleasant. The superposition of two waves of similar but not identical frequencies is the culprit. Another example is often noticeable in jet aircraft, particularly the two-engine variety, while taxiing. The combined sound of the engines goes up and down in loudness. This varying loudness happens because the sound waves have similar but not identical frequencies. The discordant warbling of the piano and the fluctuating loudness of the jet engine noise are both due to alternately constructive and destructive interference as the two waves go in and out of phase. Figure 8 illustrates this graphically.

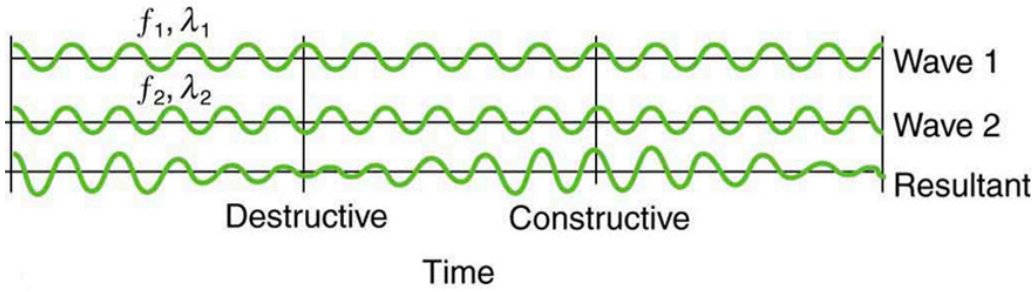


Figure 8. *Beats are produced by the superposition of two waves of slightly different frequencies but identical amplitudes. The waves alternate in time between constructive interference and destructive interference, giving the resulting wave a time-varying amplitude.*

The wave resulting from the superposition of two similar-frequency waves has a frequency that is the average of the two. This wave fluctuates in amplitude, or *beats*, with a frequency called the **beat frequency**. We can determine the beat frequency by adding two waves together mathematically. Note that a wave can be represented at one point in space as

$$x = X\cos(\frac{2\pi t}{T}) = X\cos(2\pi ft),$$

where $f = 1/T$ is the frequency of the wave. Adding two waves that have different frequencies but identical amplitudes produces a resultant

$$x = x_1 + x_2.$$

More specifically,

$$x = X\cos(2\pi f_1 t) + X\cos(2\pi f_2 t).$$

Using a trigonometric identity, it can be shown that

$$x = 2X\cos(\pi f_{\mathrm{B}} t)\cos(2\pi f_{\mathrm{ave}} t),$$

where

$$f_{\mathrm{B}} = |f_1 - f_2|$$

is the beat frequency, and f_{ave} is the average of f_1 and f_2. These results mean that the resultant wave has twice the amplitude and the average frequency of the two superimposed waves, but it also fluctuates in overall amplitude at the beat frequency f_{B}. The first cosine term in the expression effectively causes the amplitude to go up and down. The second cosine term is the wave with frequency f_{ave}. This result is valid for all types of waves. However, if it is a sound wave, providing the two frequencies are similar, then what we hear is an average frequency that gets louder and softer (or warbles) at the beat frequency.

MAKING CAREER CONNECTIONS

Piano tuners use beats routinely in their work. When comparing a note with a tuning fork, they listen for beats and adjust the string until the beats go away (to zero frequency). For example, if the tuning fork has a $256\ \text{Hz}$ frequency and two beats per second are heard, then the other frequency is either 254 or 258 Hz. Most keys hit multiple strings, and these strings are actually adjusted until they have nearly the same frequency and give a slow beat for richness. Twelve-string guitars and mandolins are also tuned using beats.

While beats may sometimes be annoying in audible sounds, we will find that beats have many applications. Observing beats is a very useful way to compare similar frequencies. There are applications of beats as apparently disparate as in ultrasonic imaging and radar speed traps.

Check Your Understanding 1

Imagine you are holding one end of a jump rope, and your friend holds the other. If your friend holds her end still, you can move your end up and down, creating a transverse wave. If your friend then begins to move her end up and down, generating a wave in the opposite direction, what resultant wave forms would you expect to see in the jump rope?

Check Your Understanding 2

Define nodes and antinodes.

Check Your Understanding 3

You hook up a stereo system. When you test the system, you notice that in one corner of the room, the sounds seem dull. In another area, the sounds seem excessively loud. Describe how the sound moving about the room could result in these effects.

PHET EXPLORATIONS: WAVE INTERFERENCE

Make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern.

Figure 9. *Wave Interference*

Section Summary

- Superposition is the combination of two waves at the same location.
- Constructive interference occurs when two identical waves are superimposed in phase.
- Destructive interference occurs when two identical waves are superimposed exactly out of phase.
- A standing wave is one in which two waves superimpose to produce a wave that varies in amplitude but does not propagate.
- Nodes are points of no motion in standing waves.
- An antinode is the location of maximum amplitude of a standing wave.
- Waves on a string are resonant standing waves with a fundamental frequency and can occur at higher multiples of the fundamental, called overtones or harmonics.
- Beats occur when waves of similar frequencies f_1 and f_2 are superimposed. The resulting amplitude oscillates with a beat frequency given by

$$f_{\text{B}} = |f_1 - f_2|.$$

Glossary

antinode
: the location of maximum amplitude in standing waves

beat frequency
: the frequency of the amplitude fluctuations of a wave

constructive interference
: when two waves arrive at the same point exactly in phase; that is, the crests of the two waves are precisely aligned, as are the troughs

destructive interference
: when two identical waves arrive at the same point exactly out of phase; that is, precisely aligned crest to trough

fundamental frequency
: the lowest frequency of a periodic waveform

nodes
: the points where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave

overtones
: multiples of the fundamental frequency of a sound

superposition
: the phenomenon that occurs when two or more waves arrive at the same point

Conceptual Questions

1: Speakers in stereo systems have two color-coded terminals to indicate how to hook up the wires. If the wires are reversed, the speaker moves in a direction opposite that of a properly connected speaker. Explain why it is important to have both speakers connected the same way.

Problems & Exercises

1: A car has two horns, one emitting a frequency of 199 Hz and the other emitting a frequency of 203 Hz. What beat frequency do they produce?

2: The middle-C hammer of a piano hits two strings, producing beats of 1.50 Hz. One of the strings is tuned to 260.00 Hz. What frequencies could the other string have?

3: Two tuning forks having frequencies of 460 and 464 Hz are struck simultaneously. What average frequency will you hear, and what will the beat frequency be?

4: Twin jet engines on an airplane are producing an average sound frequency of 4100 Hz with a beat frequency of 0.500 Hz. What are their individual frequencies?

5: A wave traveling on a Slinky® that is stretched to 4 m takes 2.4 s to travel the length of the Slinky and back again. (a) What is the speed of the wave? (b) Using the same Slinky stretched to the same length, a standing wave is created which consists of three antinodes and four nodes. At what frequency must the Slinky be oscillating?

6: Three adjacent keys on a piano (F, F-sharp, and G) are struck simultaneously, producing frequencies of 349, 370, and 392 Hz. What beat frequencies are produced by this discordant combination?

Solutions

Check Your Understanding 1

The rope would alternate between having waves with amplitudes two times the original amplitude and reaching equilibrium with no amplitude at all. The wavelengths will result in both constructive and destructive interference.

Check Your Understanding 2

Nodes are areas of wave interference where there is no motion. Antinodes are areas of wave interference where the motion is at its maximum point.

Check Your Understanding 3

With multiple speakers putting out sounds into the room, and these sounds bouncing off walls, there is bound to be some wave interference. In the dull areas, the interference is probably mostly destructive. In the louder areas, the interference is probably mostly constructive.

Problems & Exercises

1: $f = 4\ \text{Hz}$

3: 462 Hz, 4 Hz

5: (a) 3.33 m/s (b) 1.25 Hz

11.3 Energy in Waves: Intensity

Summary

- Calculate the intensity and the power of rays and waves.

Figure 1. *The destructive effect of an earthquake is palpable evidence of the energy carried in these waves. The Richter scale rating of earthquakes is related to both their amplitude and the energy they carry. (credit: Petty Officer 2nd Class Candice Villarreal, U.S. Navy)*

All waves carry energy. The energy of some waves can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls. Loud sounds pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

The amount of energy in a wave is related to its amplitude. Large-amplitude earthquakes produce large ground displacements. Loud sounds have higher pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. More quantitatively, a wave is a displacement that is resisted by a restoring force. The larger the displacement x, the larger the force $F = kx$ needed to create it. Because work W is related to force multiplied by distance (Fx) and energy is put into the wave by the work done to create it, the energy in a wave is related to amplitude. In fact, a wave's energy is directly proportional to its amplitude squared because

$$W \propto Fx = kx^2.$$

The energy effects of a wave depend on time as well as amplitude. For example, the longer deep-heat ultrasound is applied, the more energy it transfers. Waves can also be concentrated or spread out. Sunlight, for example, can be focused to burn wood. Earthquakes spread out, so they do less damage the farther they get from the source. In both cases, changing the area the waves cover has important effects. All these pertinent factors are included in the definition of **intensity** *I* as power per unit area:

$$I = \frac{P}{A}$$

where *P* is the power carried by the wave through area *A*. The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter (W/m^2). For example, infrared and visible energy from the Sun impinge on Earth at an intensity of 1300 W/m^2 just above the atmosphere. There are other intensity-related units, the most common is the decibel (see Chapter 12.3 Sound Intensity and Sound Level).

Example 1: Calculating intensity and power: How much energy is in a ray of Sunlight?

The average intensity of sunlight on Earth's surface is about 700 W/m^2.

(a) Calculate the amount of energy that falls on a solar collector having an area of 0.500 m^2 in 4.00 h.

(b) What intensity would such sunlight have if concentrated by a magnifying glass onto an area 200 times smaller than its own?

Strategy a

Because power is energy per unit time or $P = \frac{E}{t}$, the definition of intensity can be written as $I = \frac{P}{A} = \frac{E/t}{A}$, and this equation can be solved for E with the given information.

Solution a

1. Begin with the equation that states the definition of intensity:
$$I = \frac{P}{A}.$$
2. Replace *P* with its equivalent *E/t*:
$$I = \frac{E/t}{A}.$$
3. Solve for *E*:
$$E = IAt.$$
4. Substitute known values into the equation:
$$E = (700\ \text{W/m}^2)(0.500\ \text{m}^2)[(4.00\ \text{h})(3600\ \text{s/h})].$$
5. Calculate to find *E* and convert units:

$$5.04 \times 10^6 \text{ J},$$

Discussion a

The energy falling on the solar collector in 4 hours in part is enough to be useful—for example, for heating a significant amount of water.

Strategy b

Taking a ratio of new intensity to old intensity and using primes for the new quantities, we will find that it depends on the ratio of the areas. All other quantities will cancel.

Solution b

1. Take the ratio of intensities, which yields:
$$\frac{I'}{I} = \frac{P'/A'}{P/A} = \frac{A}{A'} \text{ (The powers cancel because } P' = P).$$
2. Identify the knowns:
$$A = 200A',$$
$$\frac{I'}{I} = 200.$$
3. Substitute known quantities:
$$I' = 200I = 200(700 \text{ W/m}^2).$$
4. Calculate to find I':
$$I' = 1.40 \times 10^5 \text{ W/m}^2.$$

Discussion b

Decreasing the area increases the intensity considerably. The intensity of the concentrated sunlight could even start a fire.

Example 2: Determine the combined intensity of two waves: Perfect constructive interference

If two identical waves, each having an intensity of 1.00 W/m^2, interfere perfectly constructively, what is the intensity of the resulting wave?

Strategy

We know from Chapter 11.2 Superposition and Interference that when two identical waves, which have equal amplitudes *X*, interfere perfectly constructively, the resulting wave has an amplitude of 2*X*.

Because a wave's intensity is proportional to amplitude squared, the intensity of the resulting wave is four times as great as in the individual waves.

Solution

1. Recall that intensity is proportional to amplitude squared.
2. Calculate the new amplitude:
$$I' \propto (X')^2 = (2X)^2 = 4X^2.$$
3. Recall that the intensity of the old amplitude was:
$$I \propto X^2.$$
4. Take the ratio of new intensity to the old intensity. This gives:
$$\frac{I'}{I} = 4.$$
5. Calculate to find I':
$$I' = 4I = 4.00 \text{ W/m}^2.$$

Discussion

The intensity goes up by a factor of 4 when the amplitude doubles. This answer is a little disquieting. The two individual waves each have intensities of 1.00 W/m^2, yet their sum has an intensity of 4.00 W/m^2, which may appear to violate conservation of energy. This violation, of course, cannot happen. What does happen is intriguing. The area over which the intensity is 4.00 W/m^2 is much less than the area covered by the two waves before they interfered. There are other areas where the intensity is zero. The addition of waves is not as simple as our first look in Chapter 11.2 Superposition and Interference suggested. We actually get a pattern of both constructive interference and destructive interference whenever two waves are added. For example, if we have two stereo speakers putting out 1.00 W/m^2 each, there will be places in the room where the intensity is 4.00 W/m^2, other places where the intensity is zero, and others in between. Figure 2 shows what this interference might look like. We will pursue interference patterns elsewhere in this text.

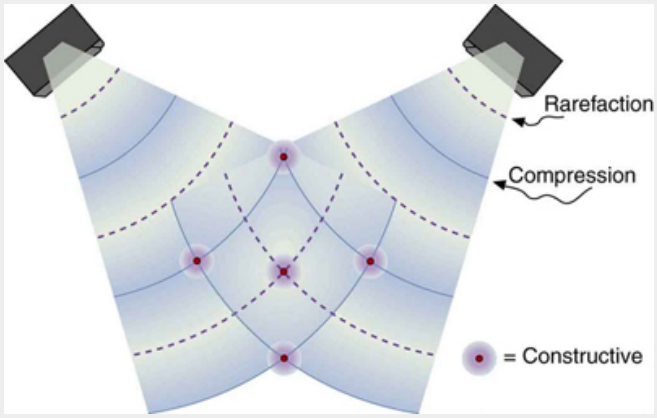


Figure 2. *These stereo speakers produce both constructive interference and destructive interference in the room, a property common to the superposition of all types of waves. The shading is proportional to intensity.*

Check Your Understanding

1: Which measurement of a wave is most important when determining the wave's intensity?

Section Summary

Intensity is defined to be the power per unit area:

$I = \frac{P}{A}$ and has units of W/m^2.

Glossary

intensity
power per unit area

Conceptual Questions

1: Two identical waves undergo pure constructive interference. Is the resultant intensity twice that of the individual waves? Explain your answer.

2: Circular water waves decrease in amplitude as they move away from where a rock is dropped. Explain why.

Exercises

1: Medical Application

Ultrasound of intensity 1.50×10^2 W/m^2 is produced by the rectangular head of a medical imaging device measuring 3.00 by 5.00 cm. What is its power output?

2: The low-frequency speaker of a stereo set has a surface area of 0.05 m^2 and produces 1W of acoustical power. What is the intensity at the speaker? If the speaker projects sound uniformly in all directions, at what distance from the speaker is the intensity 0.1 W/m^2?

3: To increase intensity of a wave by a factor of 50, by what factor should the amplitude be increased?

4: Engineering Application

A device called an insolation meter is used to measure the intensity of sunlight has an area of 100 cm^2 and registers 6.50 W. What is the intensity in W/m^2?

5: Astronomy Application

Energy from the Sun arrives at the top of the Earth's atmosphere with an intensity of 1.30 kW/m^2. How long does it take for 1.8×10^9 J to arrive on an area of 1.00 m^2?

6: Suppose you have a device that extracts energy from ocean breakers in direct proportion to their intensity. If the device produces 10.0 kW of power on a day when the breakers are 1.20 m high, how much will it produce when they are 0.600 m high?

7: Engineering Application

(a) A photovoltaic array of (solar cells) is 10.0% efficient in gathering solar energy and converting it to electricity. If the average intensity of sunlight on one day is 700 W/m^2, what area should your array have to gather energy at the rate of 100 W? (b) What is the maximum cost of the array if it must pay for itself in two years of operation averaging 10.0 hours per day? Assume that it earns money at the rate of 9.00 ¢ per kilowatt-hour.

8: A microphone receiving a pure sound tone feeds an oscilloscope, producing a wave on its screen. If the sound intensity is originally 2.00×10^{-5} W/m^2, but is turned up until the amplitude increases by 30.0%, what is the new intensity?

9: Medical Application

(a) What is the intensity in W/m^2 of a laser beam used to burn away cancerous tissue that, when 90.0% absorbed, puts 500 J of energy into a circular spot 2.00 mm in diameter in 4.00 s? (b) Discuss how this intensity compares to the average intensity of sunlight (about 700 W/m^2) and the implications that would have if the laser beam entered your eye. Note how your answer depends on the time duration of the exposure.

Solutions

Check Your Understanding

1: Amplitude, because a wave's energy is directly proportional to its amplitude squared.

Problems & Exercises

1: 0.225 W

3: 7.07

5: 16.0 days

6: 2.50 kW

8: $3.38 \times 10^{-5}\ \mathrm{W/m^2}$

Chapter 12 Physics of Hearing

Figure 1. *This tree fell some time ago. When it fell, atoms in the air were disturbed. Physicists would call this disturbance sound whether someone was around to hear it or not. (credit: B.A. Bowen Photography)*

If a tree falls in the forest and no one is there to hear it, does it make a sound? The answer to this old philosophical question depends on how you define sound. If sound only exists when someone is around to perceive it, then there was no sound. However, if we define sound in terms of physics; that is, a disturbance of the atoms in matter transmitted from its origin outward (in other words, a wave), then there *was* a sound, even if nobody was around to hear it.

Such a wave is the physical phenomenon we call *sound*. Its perception is hearing. Both the physical phenomenon and its perception are interesting and will be considered in this text. We shall explore both sound and hearing; they are related, but are not the same thing. We will also explore the many practical uses of sound waves, such as in medical imaging.

12.1 Sound

Summary

- Define sound and hearing.
- Describe sound as a longitudinal wave.

Figure 1. *This glass has been shattered by a high-intensity sound wave of the same frequency as the resonant frequency of the glass. While the sound is not visible, the effects of the sound prove its existence. (credit: ||read||, Flickr)*

Sound can be used as a familiar illustration of waves. Because hearing is one of our most important senses, it is interesting to see how the physical properties of sound correspond to our perceptions of it. **Hearing** is the perception of sound, just as vision is the perception of visible light. But sound has important applications beyond hearing. Ultrasound, for example, is not heard but can be employed to form medical images and is also used in treatment.

The physical phenomenon of **sound** is defined to be a disturbance of matter that is transmitted from its source outward. Sound is a wave. On the atomic scale, it is a disturbance of atoms that is far more ordered than their thermal motions. In many instances, sound is a periodic wave, and the atoms undergo simple harmonic motion. In this text, we shall explore such periodic sound waves.

A vibrating string produces a sound wave as illustrated in Figure 2, Figure 3, and Figure 4. As the string oscillates back and forth, it transfers energy to the air, mostly as thermal energy created by turbulence. But a small part of the string's energy goes into compressing and expanding the surrounding air, creating slightly higher and lower local pressures. These compressions (high pressure regions) and

rarefactions (low pressure regions) move out as longitudinal pressure waves having the same frequency as the string—they are the disturbance that is a sound wave. (Sound waves in air and most fluids are longitudinal, because fluids have almost no shear strength. In solids, sound waves can be both transverse and longitudinal.) Figure 4 shows a graph of gauge pressure versus distance from the vibrating string.

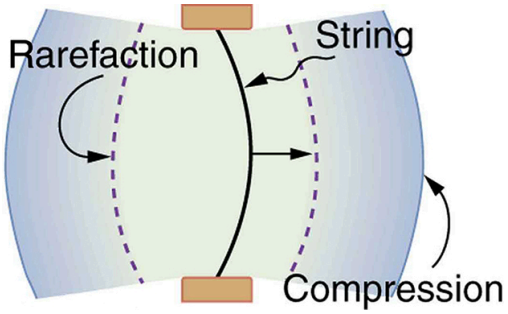


Figure 2. *A vibrating string moving to the right compresses the air in front of it and expands the air behind it.*

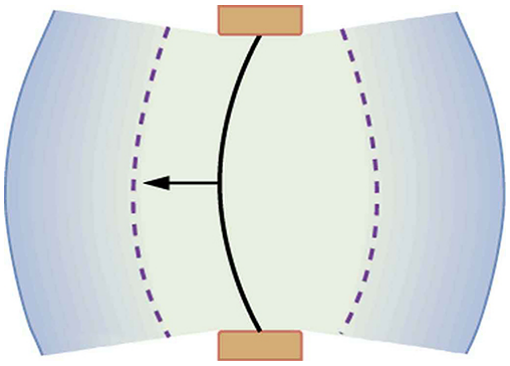


Figure 3. *As the string moves to the left, it creates another compression and rarefaction as the ones on the right move away from the string.*

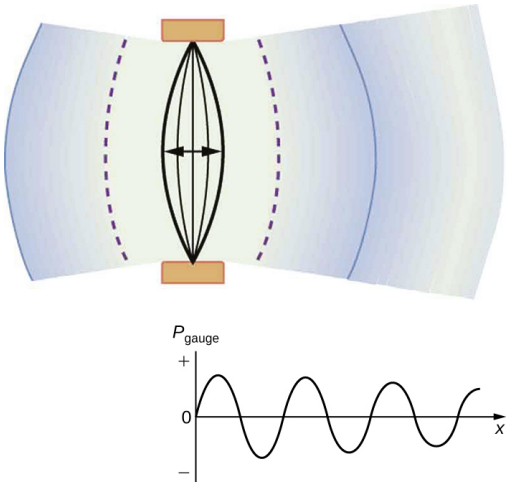


Figure 4. *After many vibrations, there are a series of compressions and rarefactions moving out from the string as a sound wave. The graph shows gauge pressure versus distance from the source. Pressures vary only slightly from atmospheric for ordinary sounds.*

The amplitude of a sound wave decreases with distance from its source, because the energy of the wave is spread over a larger and larger area. But it is also absorbed by objects, such as the eardrum in Figure 5, and converted to thermal energy by the viscosity of air. In addition, during each compression a little heat transfers to the air and during each rarefaction even less heat transfers from the air, so that the heat transfer reduces the organized disturbance into random thermal motions. Whether the heat transfer from compression to rarefaction is significant depends on how far apart they are—that is, it depends on wavelength. Wavelength, frequency, amplitude, and speed of propagation are important for sound, as they are for all waves.

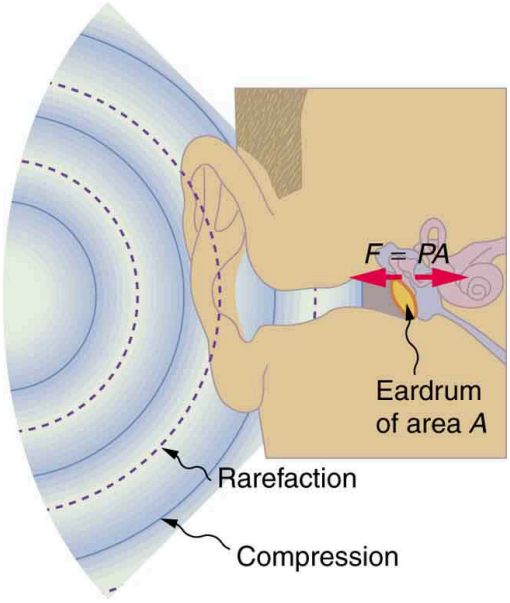


Figure 5. *Sound wave compressions and rarefactions travel up the ear canal and force the eardrum to vibrate. There is a net force on the eardrum, since the sound wave pressures differ from the atmospheric pressure found behind the eardrum. A complicated mechanism converts the vibrations to nerve impulses, which are perceived by the person.*

PhET Explorations: Wave Interference

Make waves with a dripping faucet, audio speaker, or laser. Add a second source or a pair of slits to create an interference pattern.

PhET Interactive Simulation

Figure 6. *Wave Interference*

Section Summary

- Sound is a disturbance of matter that is transmitted from its source outward.
- Sound is one type of wave.

- Hearing is the perception of sound.

Glossary

sound
a disturbance of matter that is transmitted from its source outward

hearing
the perception of sound

12.2 Speed of Sound, Frequency, and Wavelength

Summary

- Define pitch.
- Describe the relationship between the speed of sound, its frequency, and its wavelength.
- Describe the effects on the speed of sound as it travels through various media.
- Describe the effects of temperature on the speed of sound.

Figure 1. *When a firework explodes, the light energy is perceived before the sound energy. Sound travels more slowly than light does. (credit: Dominic Alves, Flickr)*

Sound, like all waves, travels at a certain speed and has the properties of frequency and wavelength. You can observe direct evidence of the speed of sound while watching a fireworks display. The flash of an explosion is seen well before its sound is heard, implying both that sound travels at a finite speed and that it is much slower than light. You can also directly sense the frequency of a sound. Perception of frequency is called **pitch**. The wavelength of sound is not directly sensed, but indirect evidence is found in the correlation of the size of musical instruments with their pitch. Small instruments, such as a piccolo, typically make high-pitch sounds, while large instruments, such as a tuba, typically make low-pitch sounds. High pitch means small wavelength, and the size of a musical instrument is directly related to the wavelengths of sound it produces. So a small instrument creates short-wavelength sounds. Similar arguments hold that a large instrument creates long-wavelength sounds.

The relationship of the speed of sound, its frequency, and wavelength is the same as for all waves:

$$v_{\text{w}} = f\lambda,$$

where v_w is the speed of sound, f is its frequency, and λ is its wavelength. The wavelength of a sound is the distance between adjacent identical parts of a wave—for example, between adjacent compressions as illustrated in Figure 2. The frequency is the same as that of the source and is the number of waves that pass a point per unit time.

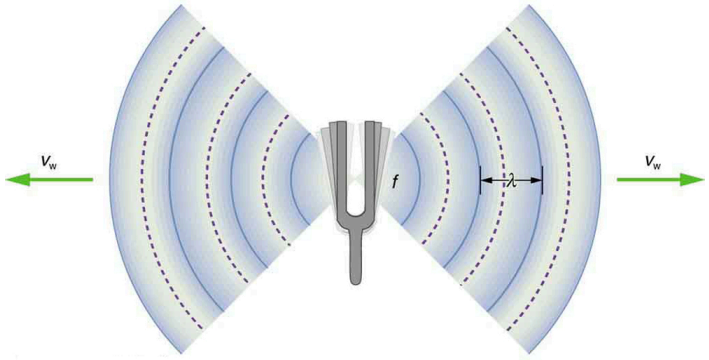


Figure 2. *A sound wave emanates from a source vibrating at a frequency f, propagates at v_w, and has a wavelength λ.*

Table 1 makes it apparent that the speed of sound varies greatly in different media. The speed of sound in a medium is determined by a combination of the medium's rigidity (or compressibility in gases) and its density. The more rigid (or less compressible) the medium, the faster the speed of sound. This observation is analogous to the fact that the frequency of a simple harmonic motion is directly proportional to the stiffness of the oscillating object. The greater the density of a medium, the slower the speed of sound. This observation is analogous to the fact that the frequency of a simple harmonic motion is inversely proportional to the mass of the oscillating object. The speed of sound in air is low, because air is compressible. Because liquids and solids are relatively rigid and very difficult to compress, the speed of sound in such media is generally greater than in gases.

Medium	v_w (m/s)
Gases at 0ºC	
Air	331
Carbon dioxide	259
Oxygen	316
Helium	965
Hydrogen	1290
Liquids at 20ºC	
Ethanol	1160
Mercury	1450
Water, fresh	1480
Sea water	1540
Human tissue	1540
Solids (longitudinal or bulk)	
Vulcanized rubber	54
Polyethylene	920
Marble	3810
Glass, Pyrex	5640
Lead	1960
Aluminum	5120
Steel	5960

Table 1. Speed of Sound in Various Media.

Earthquakes, essentially sound waves in Earth's crust, are an interesting example of how the speed of sound depends on the rigidity of the medium. Earthquakes have both longitudinal and transverse components, and these travel at different speeds. The bulk modulus of granite is greater than its shear modulus. For that reason, the speed of longitudinal or pressure waves (P-waves) in earthquakes in granite is significantly higher than the speed of transverse or shear waves (S-waves). Both components of earthquakes travel slower in less rigid material, such as sediments. P-waves have speeds of 4 to 7 km/s, and S-waves correspondingly range in speed from 2 to 5 km/s, both being faster in more rigid material. The P-wave gets progressively farther ahead of the S-wave as they travel through Earth's crust. The time between the P- and S-waves is routinely used to determine the distance to their source, the epicenter of the earthquake.

The speed of sound is affected by temperature in a given medium. For air at sea level, the speed of sound is given by

$$v_{\text{w}=(331\ \text{m/s})}\sqrt{\frac{T}{273\ \text{K}}},$$

where the temperature (denoted as T is in units of kelvin. The speed of sound in gases is related to the average speed of particles in the gas, v_{rms}, and that

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}},$$

where k is the Boltzmann constant $(1.38 \times 10^{-23}\ \text{J/K}$ and m is the mass of each (identical) particle in the gas. So, it is reasonable that the speed of sound in air and other gases should depend on the square root of temperature. While not negligible, this is not a strong dependence. At0°C, the speed of sound is 331 m/s, whereas a t20.0°C it is 343 m/s, less than a 4% increase. Figure 3 shows a use of the speed of sound by a bat to sense distances. Echoes are also used in medical imaging.

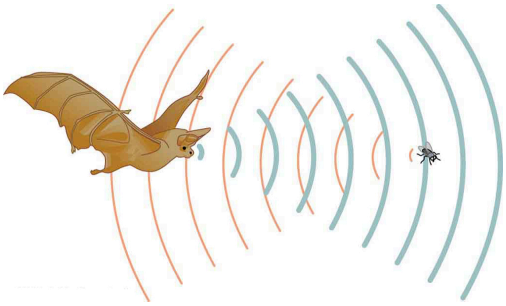


Figure 3. *A bat uses sound echoes to find its way about and to catch prey. The time for the echo to return is directly proportional to the distance.*

One of the more important properties of sound is that its speed is nearly independent of frequency. This independence is certainly true in open air for sounds in the audible range of 20 to 20,000 Hz. If this independence were not true, you would certainly notice it for music played by a marching band in a football stadium, for example. Suppose that high-frequency sounds traveled faster—then the farther you were from the band, the more the sound from the low-pitch instruments would lag that from the high-pitch ones. But the music from all instruments arrives in cadence independent of distance, and so all frequencies must travel at nearly the same speed. Recall that

$$v_{\text{w}} = f\lambda.$$

In a given medium under fixed conditions, v_{w} is constant, so that there is a relationship between f and λ the higher the frequency, the smaller the wavelength. See Figure 4 and consider the following example.

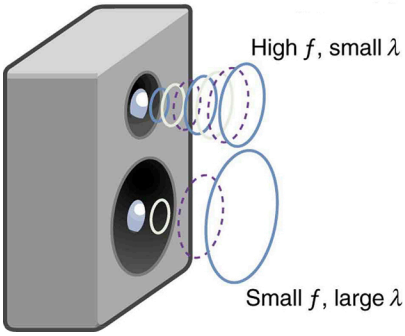


Figure 4. Because they travel at the same speed in a given medium, low-frequency sounds must have a greater wavelength than high-frequency sounds. Here, the lower-frequency sounds are emitted by the large speaker, called a woofer, while the higher-frequency sounds are emitted by the small speaker, called a tweeter.

Example 1: Calculating Wavelengths: What Are the Wavelengths of Audible Sounds?

Calculate the wavelengths of sounds at the extremes of the audible range, 20 and 20,000 Hz, in 30.0°C air. (Assume that the frequency values are accurate to two significant figures.)

Strategy

To find wavelength from frequency, we can use $v_\text{w} = f\lambda$.

Solution

1. Identify knowns. The value for v_w, is given by
$$v_\text{w} = (331\ \text{m/s})\sqrt{\frac{T}{273\ \text{K}}}.$$
2. Convert the temperature into kelvin and then enter the temperature into the equation
$$v_\text{w} = (331\ \text{m/s})\sqrt{\frac{303\ \text{K}}{273\ \text{K}}} = 348.7\ \text{m/s}.$$
3. Solve the relationship between speed and wavelength for λ
$$\lambda = \frac{v_\text{w}}{f}.$$
4. Enter the speed and the minimum frequency to give the maximum wavelength:
$$\lambda_\text{max} = \frac{348.7\ \text{m/s}}{20\ \text{Hz}} = 17\ \text{m}.$$

5. Enter the speed and the maximum frequency to give the minimum wavelength:

$$\lambda_{\text{min}} = \frac{348.7 \text{ m/s}}{20,000 \text{ Hz}} = 0.017 \text{ m} = 1.7 \text{ cm}.$$

Discussion

Because the product of f multiplied by λ equals a constant, the smaller f is, the larger λ must be, and vice versa.

The speed of sound can change when sound travels from one medium to another. However, the frequency usually remains the same because it is like a driven oscillation and has the frequency of the original source. If v_{w} changes and $f$remains the same, then the wavelength λ must change. That is, because $v_{\text{w}} = f\lambda$the higher the speed of a sound, the greater its wavelength for a given frequency.

Making Connections: Take-Home Investigation—Voice as a Sound Wave

Suspend a sheet of paper so that the top edge of the paper is fixed and the bottom edge is free to move. You could tape the top edge of the paper to the edge of a table. Gently blow near the edge of the bottom of the sheet and note how the sheet moves. Speak softly and then louder such that the sounds hit the edge of the bottom of the paper, and note how the sheet moves. Explain the effects.

Check Your Understanding 1

Imagine you observe two fireworks explode. You hear the explosion of one as soon as you see it. However, you see the other firework for several milliseconds before you hear the explosion. Explain why this is so.

Check Your Understanding 2

You observe two musical instruments that you cannot identify. One plays high-pitch sounds and the other plays low-pitch sounds. How could you determine which is which without hearing either of them play?

Section Summary

The relationship of the speed of sound v_{w}, its frequency f, and its wavelength λ is given by

$$v_{\text{w}} = f\lambda,$$

which is the same relationship given for all waves.

In air, the speed of sound is related to air temperature T by

$$v_{\text{w}} = (331 \text{ m/s})\sqrt{\frac{T}{273 \text{ K}}}.$$

v_{w} is the same for all frequencies and wavelengths.

Glossary

pitch
the perception of the frequency of a sound

Conceptual Questions

1: How do sound vibrations of atoms differ from thermal motion?

2: When sound passes from one medium to another where its propagation speed is different, does its frequency or wavelength change? Explain your answer briefly.

Exercises

1: When poked by a spear, an operatic soprano lets out a 1200-Hz shriek. What is its wavelength if the speed of sound is 345 m/s?

2: What frequency sound has a 0.10-m wavelength when the speed of sound is 340 m/s?

3: Calculate the speed of sound on a day when a 1500 Hz frequency has a wavelength of 0.221 m.

4: (a) What is the speed of sound in a medium where a 100-kHz frequency produces a 5.96-cm wavelength? (b) Which substance in Table 1 is this likely to be?

5: Show that the speed of sound in 20.0 $^{\text{o}}$C air is 343 m/s, as claimed in the text.

6: Air temperature in the Sahara Desert can reach 56.0°C(about 134°F). What is the speed of sound in air at that temperature?

7: Dolphins make sounds in air and water. What is the ratio of the wavelength of a sound in air to its wavelength in seawater? Assume air temperature is 20.0 ^{0}C.

8: A sonar echo returns to a submarine 1.20 s after being emitted. What is the distance to the object creating the echo? (Assume that the submarine is in the ocean, not in fresh water.)

9: (a) If a submarine's sonar can measure echo times with a precision of 0.0100 s, what is the smallest difference in distances it can detect? (Assume that the submarine is in the ocean, not in fresh water.)

(b) Discuss the limits this time resolution imposes on the ability of the sonar system to detect the size and shape of the object creating the echo.

10: A physicist at a fireworks display times the lag between seeing an explosion and hearing its sound, and finds it to be 0.400 s. (a) How far away is the explosion if air temperature is 24.0 ^{o}C and if you neglect the time taken for light to reach the physicist? (b) Calculate the distance to the explosion taking the speed of light into account. Note that this distance is negligibly greater.

11: Suppose a bat uses sound echoes to locate its insect prey, 3.00 m away. (See Figure 3.) (a) Calculate the echo times for temperatures of 5.00 ^{o}C and 35.0 ^{o}C. b) What percent uncertainty does this cause for the bat in locating the insect? (c) Discuss the significance of this uncertainty and whether it could cause difficulties for the bat. (In practice, the bat continues to use sound as it closes in, eliminating most of any difficulties imposed by this and other effects, such as motion of the prey.)

Solutions

Check Your Understanding 1

Sound and light both travel at definite speeds. The speed of sound is slower than the speed of light. The first firework is probably very close by, so the speed difference is not noticeable. The second firework is farther away, so the light arrives at your eyes noticeably sooner than the sound wave arrives at your ears.

Check Your Understanding 2

Compare their sizes. High-pitch instruments are generally smaller than low-pitch instruments because they generate a smaller wavelength.

Problems & Exercises

1: 0.288 m

3: 332 m/s

5: $$v_{\text{w}} = (331 \text{ m/s})\sqrt{\frac{T}{273 \text{ K}}} = (331 \text{ m/s})\sqrt{\frac{293 \text{ K}}{273 \text{ K}}}$$
$$= 343 \text{ m/s}$$

7: 0.223

9: (a) 7.70 m (b) This means that sonar is good for spotting and locating large objects, but it isn't able to resolve smaller objects, or detect the detailed shapes of objects. Objects like ships or large pieces of airplanes can be found by sonar, while smaller pieces must be found by other means.

11: (a) 18.0 ms, 17.1 ms (b) 5.00% (c) This uncertainty could definitely cause difficulties for the bat, if it didn't continue to use sound as it closed in on its prey. A 5% uncertainty could be the difference between catching the prey around the neck or around the chest, which means that it could miss grabbing its prey.

12.3 Sound Intensity and Sound Level

Summary

- Define intensity, sound intensity, and sound pressure level.
- Calculate sound intensity levels in decibels (dB).

In a quiet forest, you can sometimes hear a single leaf fall to the ground. After settling into bed, you may hear your blood pulsing through your ears. But when a passing motorist has his stereo turned up, you cannot even hear what the person next to you in your car is saying. We are all very familiar with the loudness of sounds and aware that they are related to how energetically the source is vibrating. In cartoons depicting a screaming person (or an animal making a loud noise), the cartoonist often shows an open mouth with a vibrating uvula, the hanging tissue at the back of the mouth, to suggest a loud sound coming from the throat Figure 1. High noise exposure is hazardous to hearing, and it is common for musicians to have hearing losses that are sufficiently severe that they interfere with the musicians' abilities to perform. The relevant physical quantity is sound intensity, a concept that is valid for all sounds whether or not they are in the audible range.

Intensity is defined to be the power per unit area carried by a wave. Power is the rate at which energy is transferred by the wave. In equation form, **intensity** I is

$$I = \frac{P}{A},$$

where P is the power through an area A. The SI unit for I is $\mathrm{W/m^2}$. The intensity of a sound wave is related to its amplitude squared by the following relationship:

$$I = \frac{(\Delta p)^2}{2\rho v_{\mathrm{w}}}.$$

Here Δp is the pressure variation or pressure amplitude (half the difference between the maximum and minimum pressure in the sound wave) in units of pascals (Pa) or $\mathrm{N/m^2}$. (We are using a lower case pp for pressure to distinguish it from power, denoted by P above.) The energy (as kinetic energy $\frac{mv^2}{2}$ of an oscillating element of air due to a traveling sound wave is proportional to its amplitude squared. In this equation, ρ is the density of the material in which the sound wave travels, in units of $\mathrm{kg/m^3}$, and v_{w} is the speed of sound in the medium, in units of m/s. The pressure variation is proportional to the amplitude of the oscillation, and so I varies as $(\Delta p)^2$ (Figure 2). This relationship is consistent with the fact that the sound wave is produced by some vibration; the greater its pressure amplitude, the more the air is compressed in the sound it creates.

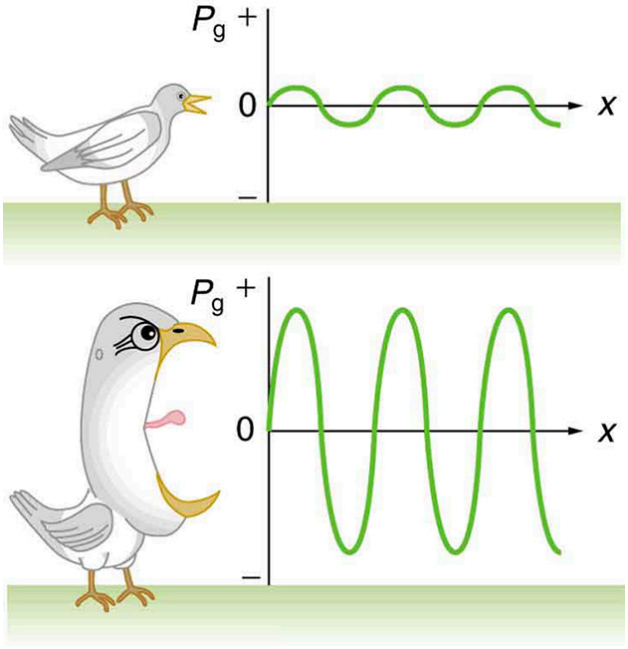


Figure 1. *Graphs of the gauge pressures in two sound waves of different intensities. The more intense sound is produced by a source that has larger-amplitude oscillations and has greater pressure maxima and minima. Because pressures are higher in the greater-intensity sound, it can exert larger forces on the objects it encounters.*

Sound intensity levels are quoted in decibels (dB) much more often than sound intensities in watts per meter squared. Decibels are the unit of choice in the scientific literature as well as in the popular media. The reasons for this choice of units are related to how we perceive sounds. How our ears perceive sound can be more accurately described by the logarithm of the intensity rather than directly to the intensity. The **sound intensity level** β in decibels of a sound having an intensity I in watts per meter squared is defined to be

$$\beta(\text{dB}) = 10 \log_{10}\left(\frac{I}{I_0}\right),$$

where $I_0 = 10^{-12}\ \text{W/m}^2$ is a reference intensity. In particular, I_0 is the lowest or threshold intensity of sound a person with normal hearing can perceive at a frequency of 1000 Hz. Sound intensity level is not the same as intensity. Because β is defined in terms of a ratio, it is a unitless quantity telling you the *level* of the sound relative to a fixed standard ($10^{-12}\ \text{W/m}^2$, in this case). The units of decibels (dB) are used to indicate this ratio is multiplied by 10 in its definition. The bel, upon which the decibel is based, is named for Alexander Graham Bell, the inventor of the telephone.

Sound intensity level β (dB)	Intensity I(W/m^2)	Example/effect
0	1×10^{-12}	Threshold of hearing at 1000 Hz
10	1×10^{-11}	Rustle of leaves
20	1×10^{-10}	Whisper at 1 m distance
30	1×10^{-9}	Quiet home
40	1×10^{-8}	Average home
50	1×10^{-7}	Average office, soft music
60	1×10^{-6}	Normal conversation
70	1×10^{-5}	Noisy office, busy traffic
80	1×10^{-4}	Loud radio, classroom lecture
90	1×10^{-3}	Inside a heavy truck; damage from prolonged exposure[1]
100	1×10^{-2}	Noisy factory, siren at 30 m; damage from 8 h per day exposure
110	1×10^{-1}	Damage from 30 min per day exposure
120	1	Loud rock concert, pneumatic chipper at 2 m; threshold of pain
140	1×10^{2}	Jet airplane at 30 m; severe pain, damage in seconds
160	1×10^{4}	Bursting of eardrums

Table 2. Sound Intensity Levels and Intensities.

The decibel level of a sound having the threshold intensity of $10^{-12}\ \mathrm{W/m^2}$ is $\beta = 0\mathrm{dB}$, because $\log_{10} 1 = 0$. That is, the threshold of hearing is 0 decibels. Table 2 gives levels in decibels and intensities in watts per meter squared for some familiar sounds.

One of the more striking things about the intensities in Table 2 is that the intensity in watts per meter squared is quite small for most sounds. The ear is sensitive to as little as a trillionth of a watt per meter squared—even more impressive when you realize that the area of the eardrum is only about $1\ \mathrm{cm^2}$, so that only 10^{-16} W falls on it at the threshold of hearing! Air molecules in a sound wave of this intensity vibrate over a distance of less than one molecular diameter, and the gauge pressures involved are less than 10^{-9} atm.

Another impressive feature of the sounds in Table 2 is their numerical range. Sound intensity varies by a factor of 10^{12} from threshold to a sound that causes damage in seconds. You are unaware of this

tremendous range in sound intensity because how your ears respond can be described approximately as the logarithm of intensity. Thus, sound intensity levels in decibels fit your experience better than intensities in watts per meter squared. The decibel scale is also easier to relate to because most people are more accustomed to dealing with numbers such as 0, 53, or 120 than numbers such as 1.00×10^{-11}.

One more observation readily verified by examining Table 2 or using $I = \dfrac{(\Delta p)^2}{2\rho v_{\text{w}}}$ is that each factor of 10 in intensity corresponds to 10 dB. For example, a 90 dB sound compared with a 60 dB sound is 30 dB greater, or three factors of 10 (that is, 10^3 times) as intense. Another example is that if one sound is 10^7 as intense as another, it is 70 dB higher. See Table 3.

I_2/I_1	$\beta_1 - \beta_2$
2.0	3.0 dB
5.0	7.0 dB
10.0	10.0 dB

Table 3. Ratios of Intensities and Corresponding Differences in Sound Intensity Levels.

Example 1: Calculating Sound Intensity Levels: Sound Waves

Calculate the sound intensity level in decibels for a sound wave traveling in air at 0°C and having a pressure amplitude of 0.656 Pa.

Strategy

We are given Δp, so we can calculate I using the equation $I = (\Delta p)^2/(2\rho v_{\text{w}})^2$. Using I, we can calculate β straight from its definition in $\beta(\text{dB}) = 10 \log_{10}(I/I_0)$.

Solution

(1) Identify knowns:

Sound travels at 331 m/s in air at 0°C.

Air has a density of 1.29 kg/m^3 at atmospheric pressure and 0°C.

(2) Enter these values and the pressure amplitude into $I = (\Delta p)^2/(2\rho v_{\text{w}})$

$$I = \frac{(\Delta p)^2}{2\rho v_{\text{w}}} = \frac{(0.656 \text{ Pa})^2}{2(1.29 \text{ kg/m}^3)(331 \text{ m/s})} = 5.04 \times 10^{-4} \text{ W/m}^2.$$

(3) Enter the value for I and the known value for I_0 into $\beta(\text{dB}) = 10 \log_{10}(I/I_0)$. Calculate to find the sound intensity level in decibels:

$$10 \log_{10}(5.04 \times 10^8) = 10(8.70) \text{ dB} = 87 \text{ dB}.$$

Discussion

This 87 dB sound has an intensity five times as great as an 80 dB sound. So a factor of five in intensity corresponds to a difference of 7 dB in sound intensity level. This value is true for any intensities differing by a factor of five.

Example 2: Change Intensity Levels of a Sound: What Happens to the Decibel Level?

Show that if one sound is twice as intense as another, it has a sound level about 3 dB higher.

Strategy

You are given that the ratio of two intensities is 2 to 1, and are then asked to find the difference in their sound levels in decibels. You can solve this problem using of the properties of logarithms.

Solution

(1) Identify knowns:

The ratio of the two intensities is 2 to 1, or:

$$\frac{I_2}{I_1} = 2.00.$$

We wish to show that the difference in sound levels is about 3 dB. That is, we want to show:

$$\beta_2 - \beta_1 = 3 \text{ dB}.$$

Note that:

$$\log_{10} b - \log_{10} a = \log_{10}(\frac{b}{a}).$$

(2) Use the definition of β to get:

$$\beta_2 - \beta_1 = 10 \log_{10}(\frac{I_2}{I_1}) = 10 \log_{10} 2.00 = 10(0.301) \text{ dB}.$$

Thus,

$$\beta_2 - \beta_1 = 3.01 \text{ dB}.$$

Discussion

This means that the two sound intensity levels differ by 3.01 dB, or about 3 dB, as advertised. Note that because only the ratio I_2 / I_1 is given (and not the actual intensities), this result is true for any intensities that differ by a factor of two. For example, a 56.0 dB sound is twice as intense as a 53.0 dB sound, a 97.0 dB sound is half as intense as a 100 dB sound, and so on.

It should be noted at this point that there is another decibel scale in use, called the **sound pressure level**,

based on the ratio of the pressure amplitude to a reference pressure. This scale is used particularly in applications where sound travels in water. It is beyond the scope of most introductory texts to treat this scale because it is not commonly used for sounds in air, but it is important to note that very different decibel levels may be encountered when sound pressure levels are quoted. For example, ocean noise pollution produced by ships may be as great as 200 dB expressed in the sound pressure level, where the more familiar sound intensity level we use here would be something under 140 dB for the same sound.

Take-Home Investigation: Feeling Sound

Find a CD player and a CD that has rock music. Place the player on a light table, insert the CD into the player, and start playing the CD. Place your hand gently on the table next to the speakers. Increase the volume and note the level when the table just begins to vibrate as the rock music plays. Increase the reading on the volume control until it doubles. What has happened to the vibrations?

Check Your Understanding 1

Describe how amplitude is related to the loudness of a sound.

Check Your Understanding 2

Identify common sounds at the levels of 10 dB, 50 dB, and 100 dB.

Section Summary

- Intensity is the same for a sound wave as was defined for all waves; it is

$$I = \frac{P}{A},$$

where P is the power crossing area A. The SI unit for I is watts per meter squared. The intensity of a sound wave is also related to the pressure amplitude Δp

$$I = \frac{(\Delta p)^2}{2\rho v_{\text{w}}},$$

where ρ is the density of the medium in which the sound wave travels and v_{w} is the speed of sound in the medium.

- Sound intensity level in units of decibels (dB) is

$$\beta(\text{dB}) = 10 \log_{10}(\frac{I}{I_0}),$$

where $I_0 = 10^{-12}\ \text{W/m}^2$ is the threshold intensity of hearing.

Glossary

intensity
: the power per unit area carried by a wave

sound intensity level
: a unitless quantity telling you the level of the sound relative to a fixed standard

sound pressure level
: the ratio of the pressure amplitude to a reference pressure

Conceptual Questions

1: Six members of a synchronized swim team wear earplugs to protect themselves against water pressure at depths, but they can still hear the music and perform the combinations in the water perfectly. One day, they were asked to leave the pool so the dive team could practice a few dives, and they tried to practice on a mat, but seemed to have a lot more difficulty. Why might this be?

2: A community is concerned about a plan to bring train service to their downtown from the town's outskirts. The current sound intensity level, even though the rail yard is blocks away, is 70 dB downtown. The mayor assures the public that there will be a difference of only 30 dB in sound in the downtown area. Should the townspeople be concerned? Why?

Problems & Exercises

1: What is the intensity in watts per meter squared of 85.0-dB sound?

2: The warning tag on a lawn mower states that it produces noise at a level of 91.0 dB. What is this in watts per meter squared?

3: A sound wave traveling in 20°C air has a pressure amplitude of 0.5 Pa. What is the intensity of the wave?

4: What intensity level does the sound in the preceding problem correspond to?

5: What sound intensity level in dB is produced by earphones that create an intensity of $4.00 \times 10^{-2}\ \text{W/m}^2$

6: Show that an intensity of $10^{-12}\ \mathrm{W/m^2}$ is the same as $10^{-16}\mathrm{W/cm^2}$.

7: (a) What is the decibel level of a sound that is twice as intense as a 90.0-dB sound? (b) What is the decibel level of a sound that is one-fifth as intense as a 90.0-dB sound?

8: (a) What is the intensity of a sound that has a level 7.00 dB lower than a $4.00 \times 10^{-9}\ \mathrm{W/m^2}$ sound? (b) What is the intensity of a sound that is 3.00 dB higher than a $4.00 \times 10^{-9}\ \mathrm{W/m^2}$ sound?

9: (a) How much more intense is a sound that has a level 17.0 dB higher than another? (b) If one sound has a level 23.0 dB less than another, what is the ratio of their intensities?

10: People with good hearing can perceive sounds as low in level as $-8.00\ \mathrm{dB}$ at a frequency of 3000 Hz. What is the intensity of this sound in watts per meter squared?

11: If a large housefly 3.0 m away from you makes a noise of 40.0 dB, what is the noise level of 1000 flies at that distance, assuming interference has a negligible effect?

12: Ten cars in a circle at a boom box competition produce a 120-dB sound intensity level at the center of the circle. What is the average sound intensity level produced there by each stereo, assuming interference effects can be neglected?

13: The amplitude of a sound wave is measured in terms of its maximum gauge pressure. By what factor does the amplitude of a sound wave increase if the sound intensity level goes up by 40.0 dB?

14: If a sound intensity level of 0 dB at 1000 Hz corresponds to a maximum gauge pressure (sound amplitude) of $10^{-9}\ \mathrm{atm}$, what is the maximum gauge pressure in a 60-dB sound? What is the maximum gauge pressure in a 120-dB sound?

15: An 8-hour exposure to a sound intensity level of 90.0 dB may cause hearing damage. What energy in joules falls on a 0.800-cm-diameter eardrum so exposed?

16: (a) Ear trumpets were never very common, but they did aid people with hearing losses by gathering sound over a large area and concentrating it on the smaller area of the eardrum. What decibel increase does an ear trumpet produce if its sound gathering area is $900\ \mathrm{cm^2}$ and the area of the eardrum is $0.500\ \mathrm{cm^2}$, but the trumpet only has an efficiency of 5.00% in transmitting the sound to the eardrum? (b) Comment on the usefulness of the decibel increase found in part (a).

17: Sound is more effectively transmitted into a stethoscope by direct contact than through the air, and it is further intensified by being concentrated on the smaller area of the eardrum. It is reasonable to assume that sound is transmitted into a stethoscope 100 times as effectively compared with transmission though the air. What, then, is the gain in decibels produced by a stethoscope that has a sound gathering area of $15.0\ \mathrm{cm^2}$, and concentrates the sound onto two eardrums with a total area of $0.900\ \mathrm{cm^2}$ with an efficiency of 40.0%?

18: Loudspeakers can produce intense sounds with surprisingly small energy input in spite of their low efficiencies. Calculate the power input needed to produce a 90.0-dB sound intensity level for a 12.0-cm-diameter speaker that has an efficiency of 1.00%. (This value is the sound intensity level right at the speaker.)

Footnotes

1. 1 Several government agencies and health-related professional associations recommend that 85 dB not be exceeded for 8-hour daily exposures in the absence of hearing protection.

Solutions

Check Your Understanding 1

Amplitude is directly proportional to the experience of loudness. As amplitude increases, loudness increases.

Check Your Understanding 2

10 dB: Running fingers through your hair.

50 dB: Inside a quiet home with no television or radio.

100 dB: Take-off of a jet plane.

Problems & Exercises

1: $3.16 \times 10^{-4}\ \mathrm{W/m^2}$

3: $3.04 \times 10^{-4}\ \mathrm{W/m^2}$

5: 106 dB

7: (a) 93 dB (b) 83 dB

9: (a) 50.1 (b)5.01×10^{-3} or $\frac{1}{200}$

11: 70.0 dB

13: 100

15: $1.45 \times 10^{-3}\ \mathrm{J}$

17: 28.2 dB

12.4 Doppler Effect and Sonic Booms

Summary

- Define Doppler effect, Doppler shift, and sonic boom.
- Calculate the frequency of a sound heard by someone observing Doppler shift.
- Describe the sounds produced by objects moving faster than the speed of sound.

The characteristic sound of a motorcycle buzzing by is an example of the **Doppler effect**. The high-pitch scream shifts dramatically to a lower-pitch roar as the motorcycle passes by a stationary observer. The closer the motorcycle brushes by, the more abrupt the shift. The faster the motorcycle moves, the greater the shift. We also hear this characteristic shift in frequency for passing race cars, airplanes, and trains. It is so familiar that it is used to imply motion and children often mimic it in play.

The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer. Although less familiar, this effect is easily noticed for a stationary source and moving observer. For example, if you ride a train past a stationary warning bell, you will hear the bell's frequency shift from high to low as you pass by. The actual change in frequency due to relative motion of source and observer is called a **Doppler shift**. The Doppler effect and Doppler shift are named for the Austrian physicist and mathematician Christian Johann Doppler (1803–1853), who did experiments with both moving sources and moving observers. Doppler, for example, had musicians play on a moving open train car and also play standing next to the train tracks as a train passed by. Their music was observed both on and off the train, and changes in frequency were measured.

What causes the Doppler shift? Figure 1, Figure 2, and Figure 3 compare sound waves emitted by stationary and moving sources in a stationary air mass. Each disturbance spreads out spherically from the point where the sound was emitted. If the source is stationary, then all of the spheres representing the air compressions in the sound wave centered on the same point, and the stationary observers on either side see the same wavelength and frequency as emitted by the source, as in Figure 1. If the source is moving, as in Figure 2, then the situation is different. Each compression of the air moves out in a sphere from the point where it was emitted, but the point of emission moves. This moving emission point causes the air compressions to be closer together on one side and farther apart on the other. Thus, the wavelength is shorter in the direction the source is moving (on the right in Figure 2), and longer in the opposite direction (on the left in Figure 2). Finally, if the observers move, as in Figure 3, the frequency at which they receive the compressions changes. The observer moving toward the source receives them at a higher frequency, and the person moving away from the source receives them at a lower frequency.

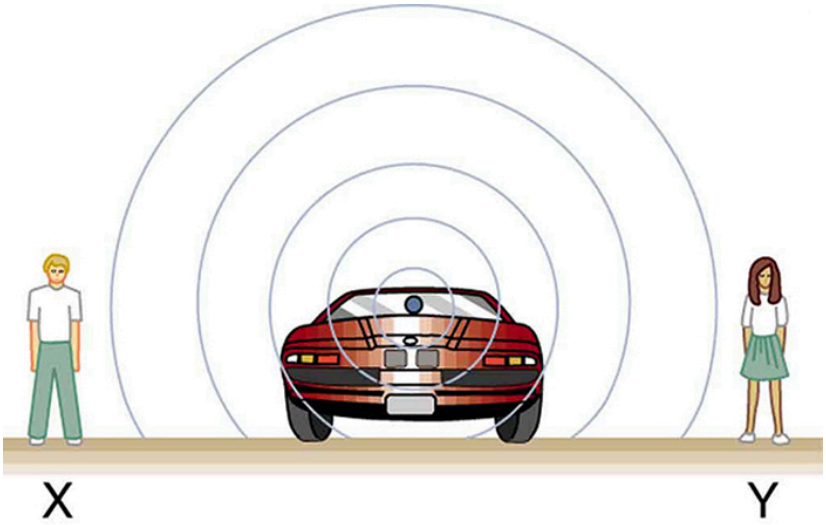


Figure 1. *Sounds emitted by a source spread out in spherical waves. Because the source, observers, and air are stationary, the wavelength and frequency are the same in all directions and to all observers.*

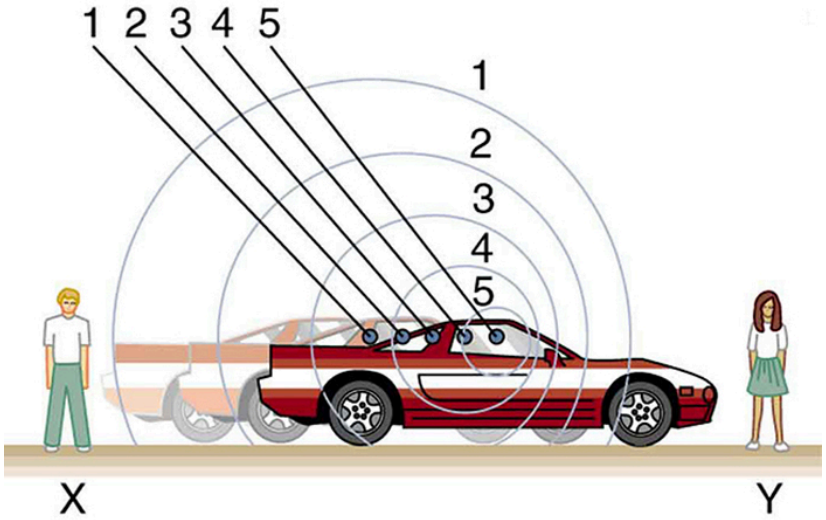


Figure 2. *Sounds emitted by a source moving to the right spread out from the points at which they were emitted. The wavelength is reduced and, consequently, the frequency is increased in the direction of motion, so that the observer on the right hears a higher-pitch sound. The opposite is true for the observer on the left, where the wavelength is increased and the frequency is reduced.*

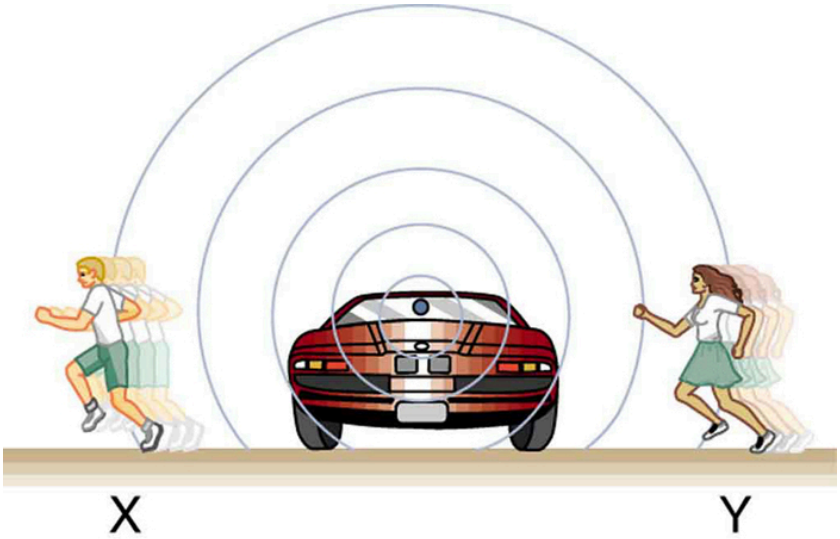


Figure 3. *The same effect is produced when the observers move relative to the source. Motion toward the source increases frequency as the observer on the right passes through more wave crests than she would if stationary. Motion away from the source decreases frequency as the observer on the left passes through fewer wave crests than he would if stationary.*

We know that wavelength and frequency are related by $v_{w=f\lambda}$, where v_w is the fixed speed of sound. The sound moves in a medium and has the same speed v_w in that medium whether the source is moving or not. Thus f multiplied by λ is a constant. Because the observer on the right in Figure 2 receives a shorter wavelength, the frequency she receives must be higher. Similarly, the observer on the left receives a longer wavelength, and hence he hears a lower frequency. The same thing happens in Figure 3. A higher frequency is received by the observer moving toward the source, and a lower frequency is received by an observer moving away from the source. In general, then, relative motion of source and observer toward one another increases the received frequency. Relative motion apart decreases frequency. The greater the relative speed is, the greater the effect.

The Doppler Effect

The Doppler effect occurs not only for sound but for any wave when there is relative motion between the observer and the source. There are Doppler shifts in the frequency of sound, light, and water waves, for example. Doppler shifts can be used to determine velocity, such as when ultrasound is reflected from blood in a medical diagnostic. The recession of galaxies is determined by the shift in the frequencies of light received from them and has implied much about the origins of the universe. Modern physics has been profoundly affected by observations of Doppler shifts.

For a stationary observer and a moving source, the frequency f_{obs} received by the observer can be shown to be

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}}}{v_{\text{w}} \pm v_{\text{s}}}\right),$$

where f_{s} is the frequency of the source, v_{s} is the speed of the source along a line joining the source and observer, and v_{w} is the speed of sound. The minus sign is used for motion toward the observer and the plus sign for motion away from the observer, producing the appropriate shifts up and down in frequency. Note that the greater the speed of the source, the greater the effect. Similarly, for a stationary source and moving observer, the frequency received by the observer f_{obs} is given by

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}} \pm v_{\text{obs}}}{v_{\text{w}}}\right),$$

where v_{obs} is the speed of the observer along a line joining the source and observer. Here the plus sign is for motion toward the source, and the minus is for motion away from the source.

Example 1: Calculate Doppler Shift: A Train Horn

Suppose a train that has a 150-Hz horn is moving at 35.0 m/s in still air on a day when the speed of sound is 340 m/s.

(a) What frequencies are observed by a stationary person at the side of the tracks as the train approaches and after it passes?

(b) What frequency is observed by the train's engineer traveling on the train?

Strategy

To find the observed frequency in (a), $f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}}}{v_{\text{w}} \pm v_{\text{s}}}\right)$, must be used because the source is moving. The minus sign is used for the approaching train, and the plus sign for the receding train. In (b), there are two Doppler shifts—one for a moving source and the other for a moving observer.

Solution for (a)

(1) Enter known values into $f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}}}{v_{\text{w}} - v_{\text{s}}}\right)$.

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}}}{v_{\text{w}} - v_{\text{s}}}\right) = (150\ \text{Hz})\left(\frac{340\ \text{m/s}}{340\ \text{m/s} - 35.0\ \text{m/s}}\right)$$

(2) Calculate the frequency observed by a stationary person as the train approaches.

$$f_{\text{obs}} = (150\ \text{Hz})(1.11) = 167\ \text{Hz}$$

(3) Use the same equation with the plus sign to find the frequency heard by a stationary person as the train recedes.

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{w}}{v_{w} + v_{s}}\right) = (150\ \text{Hz})\left(\frac{340\ \text{m/s}}{340\ \text{m/s} + 35.0\ \text{m/s}}\right)$$

(4) Calculate the second frequency.

$$f_{\text{obs}} = (150 \text{ Hz})(0.907) = 136 \text{ Hz}$$

Discussion on (a)

The numbers calculated are valid when the train is far enough away that the motion is nearly along the line joining train and observer. In both cases, the shift is significant and easily noticed. Note that the shift is 17.0 Hz for motion toward and 14.0 Hz for motion away. The shifts are not symmetric.

Solution for (b)

(1) Identify knowns:

- It seems reasonable that the engineer would receive the same frequency as emitted by the horn, because the relative velocity between them is zero.
- Relative to the medium (air), the speeds are $v_{\text{s}} = v_{\text{obs}=35.0 \text{ m/s}}$.
- The first Doppler shift is for the moving observer; the second is for the moving source.

(2) Use the following equation:

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}} \pm v_{\text{obs}}}{v_{\text{w}}}\right)\left(\frac{v_{\text{w}}}{v_{\text{w}} \pm v_{\text{s}}}\right).$$

The quantity in the square brackets is the Doppler-shifted frequency due to a moving observer. The factor on the right is the effect of the moving source.

(3) Because the train engineer is moving in the direction toward the horn, we must use the plus sign for v_{obs} however, because the horn is also moving in the direction away from the engineer, we also use the plus sign for v_{s}. But the train is carrying both the engineer and the horn at the same velocity, so $v_{\text{s}} = v_{\text{obs}}$. As a result, everything but f_{s} cancels, yielding

$$f_{\text{obs}} = f_{\text{s}}.$$

Discussion for (b)

We may expect that there is no change in frequency when source and observer move together because it fits your experience. For example, there is no Doppler shift in the frequency of conversations between driver and passenger on a motorcycle. People talking when a wind moves the air between them also observe no Doppler shift in their conversation. The crucial point is that source and observer are not moving relative to each other.

Optional: Sonic Booms to Bow Wakes

What happens to the sound produced by a moving source, such as a jet airplane, that approaches or even exceeds the speed of sound? The answer to this question applies not only to sound but to all other waves as well.

Suppose a jet airplane is coming nearly straight at you, emitting a sound of frequency f_{s}. The greater the plane's speed v_{s}, the greater the Doppler shift and the greater the value observed for f_{obs}. Now, as v_{s} approaches the speed of sound, fobsfobs approaches infinity, because the denominator in

$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}}}{v_{\text{w}} \pm v_{\text{s}}}\right)$ approaches zero. At the speed of sound, this result means that in front of the source, each successive wave is superimposed on the previous one because the source moves forward at the speed of sound. The observer gets them all at the same instant, and so the frequency is infinite. (Before airplanes exceeded the speed of sound, some people argued it would be impossible because such constructive superposition would produce pressures great enough to destroy the airplane.) If the source exceeds the speed of sound, no sound is received by the observer until the source has passed, so that the sounds from the approaching source are mixed with those from it when receding. This mixing appears messy, but something interesting happens—a sonic boom is created. (See Figure 4.)

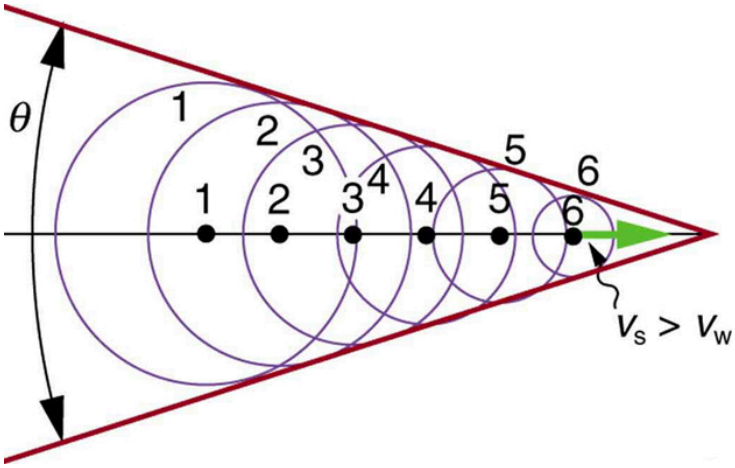


Figure 4. *Sound waves from a source that moves faster than the speed of sound spread spherically from the point where they are emitted, but the source moves ahead of each. Constructive interference along the lines shown (actually a cone in three dimensions) creates a shock wave called a sonic boom. The faster the speed of the source, the smaller the angle θ.*

There is constructive interference along the lines shown (a cone in three dimensions) from similar sound waves arriving there simultaneously. This superposition forms a disturbance called a **sonic boom**, a constructive interference of sound created by an object moving faster than sound. Inside the cone, the interference is mostly destructive, and so the sound intensity there is much less than on the shock wave. An aircraft creates two sonic booms, one from its nose and one from its tail. (See Figure 5.) During television coverage of space shuttle landings, two distinct booms could often be heard. These were separated by exactly the time it would take the shuttle to pass by a point. Observers on the ground often do not see the aircraft creating the sonic boom, because it has passed by before the shock wave reaches them, as seen in Figure 5. If the aircraft flies close by at low altitude, pressures in the sonic boom can be destructive and break windows as well as rattle nerves. Because of how destructive sonic booms can be, supersonic flights are banned over populated areas of the United States.

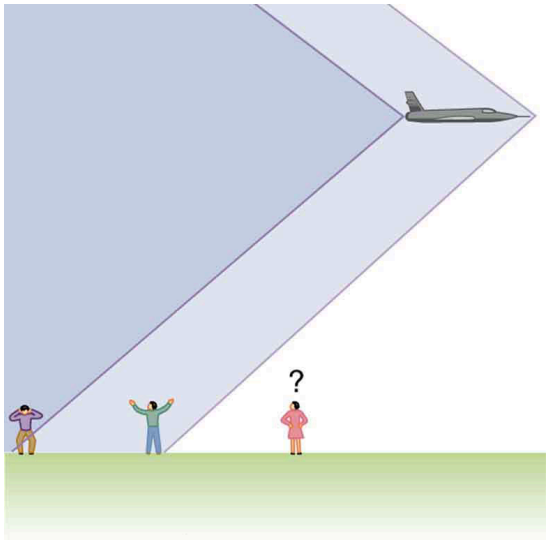


Figure 5. *Two sonic booms, created by the nose and tail of an aircraft, are observed on the ground after the plane has passed by.*

Sonic booms are one example of a broader phenomenon called bow wakes. A **bow wake**, such as the one in Figure 6, is created when the wave source moves faster than the wave propagation speed. Water waves spread out in circles from the point where created, and the bow wake is the familiar V-shaped wake trailing the source. A more exotic bow wake is created when a subatomic particle travels through a medium faster than the speed of light travels in that medium. (In a vacuum, the maximum speed of light will be $c = 3.00 \times 10^8 \text{ m/s}$in the medium of water, the speed of light is closer to $0.75c$. If the particle creates light in its passage, that light spreads on a cone with an angle indicative of the speed of the particle, as illustrated in Figure 7. Such a bow wake is called Cerenkov radiation and is commonly observed in particle physics.

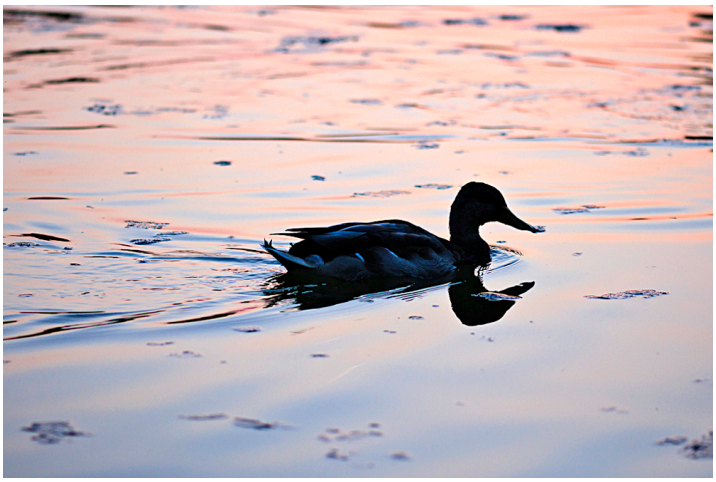


Figure 6. *Bow wake created by a duck. Constructive interference produces the rather structured wake, while there is relatively little wave action inside the wake, where interference is mostly destructive. (credit: Horia Varlan, Flickr)*

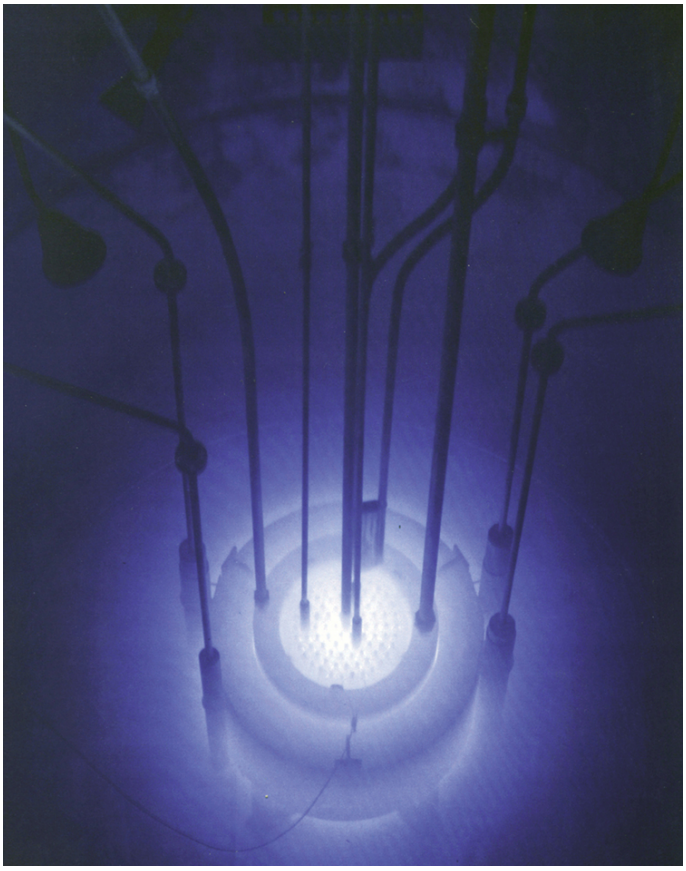


Figure 7. *The blue glow in this research reactor pool is Cerenkov radiation caused by subatomic particles traveling faster than the speed of light in water. (credit: U.S. Nuclear Regulatory Commission)*

Doppler shifts and sonic booms are interesting sound phenomena that occur in all types of waves. They can be of considerable use. For example, the Doppler shift in ultrasound can be used to measure blood velocity, while police use the Doppler shift in radar (a microwave) to measure car velocities. In meteorology, the Doppler shift is used to track the motion of storm clouds; such "Doppler Radar" can give velocity and direction and rain or snow potential of imposing weather fronts. In astronomy, we can examine the light emitted from distant galaxies and determine their speed relative to ours. As galaxies move away from us, their light is shifted to a lower frequency, and so to a longer wavelength—the so-called red shift. Such information from galaxies far, far away has allowed us to estimate the age of the universe (from the Big Bang) as about 14 billion years.

Check Your Understanding 1

Why did scientist Christian Doppler observe musicians both on a moving train and also from a stationary point not on the train?

Check Your Understanding 2

Describe a situation in your life when you might rely on the Doppler shift to help you either while driving a car or walking near traffic.

Section Summary

- The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer.
- The actual change in frequency is called the Doppler shift.
- A sonic boom is constructive interference of sound created by an object moving faster than sound.
- A sonic boom is a type of bow wake created when any wave source moves faster than the wave propagation speed.
- For a stationary observer and a moving source, the observed frequency f_{obs} is:

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}}}{v_{\text{w}} \pm v_{\text{s}}}\right),$$

  where f_{s} is the frequency of the source, v_{s} is the speed of the source, and v_{w} is the speed of sound. The minus sign is used for motion toward the observer and the plus sign for motion away.
- For a stationary source and moving observer, the observed frequency is:

$$f_{\text{obs}} = f_{\text{s}}\left(\frac{v_{\text{w}} \pm v_{\text{obs}}}{v_{\text{w}}}\right),$$

  where v_{obs} is the speed of the observer.

Glossary

Doppler effect
: an alteration in the observed frequency of a sound due to motion of either the source or the observer

Doppler shift
: the actual change in frequency due to relative motion of source and observer

sonic boom
: a constructive interference of sound created by an object moving faster than sound

bow wake
: V-shaped disturbance created when the wave source moves faster than the wave propagation speed

Conceptual Questions

1: Is the Doppler shift real or just a sensory illusion?

2: Due to efficiency considerations related to its bow wake, the supersonic transport aircraft must maintain a cruising speed that is a constant ratio to the speed of sound (a constant Mach number). If the aircraft flies from warm air into colder air, should it increase or decrease its speed? Explain your answer.

3: When you hear a sonic boom, you often cannot see the plane that made it. Why is that?

Problems & Exercises

1: (a) What frequency is received by a person watching an oncoming ambulance moving at 110 km/h and emitting a steady 800-Hz sound from its siren? The speed of sound on this day is 345 m/s. (b) What frequency does she receive after the ambulance has passed?

2: (a) At an air show a jet flies directly toward the stands at a speed of 1200 km/h, emitting a frequency of 3500 Hz, on a day when the speed of sound is 342 m/s. What frequency is received by the observers? (b) What frequency do they receive as the plane flies directly away from them?

3: What frequency is received by a mouse just before being dispatched by a hawk flying at it at 25.0 m/s and emitting a screech of frequency 3500 Hz? Take the speed of sound to be 331 m/s.

4: A spectator at a parade receives an 888-Hz tone from an oncoming trumpeter who is playing an 880-Hz note. At what speed is the musician approaching if the speed of sound is 338 m/s?

5: A commuter train blows its 200-Hz horn as it approaches a crossing. The speed of sound is 335 m/s. (a) An observer waiting at the crossing receives a frequency of 208 Hz. What is the speed of the train? (b) What frequency does the observer receive as the train moves away?

6: Can you perceive the shift in frequency produced when you pull a tuning fork toward you at 10.0 m/s on a day when the speed of sound is 344 m/s? To answer this question, calculate the factor by which the frequency shifts and see if it is greater than 0.300%.

7: Two eagles fly directly toward one another, the first at 15.0 m/s and the second at 20.0 m/s. Both screech, the first one emitting a frequency of 3200 Hz and the second one emitting a frequency of 3800 Hz. What frequencies do they receive if the speed of sound is 330 m/s?

8: What is the minimum speed at which a source must travel toward you for you to be able to hear that its frequency is Doppler shifted? That is, what speed produces a shift of 0.300% on a day when the speed of sound is 331 m/s?

Solutions

Check Your Understanding 1

Doppler needed to compare the perception of sound when the observer is stationary and the sound source moves, as well as when the sound source and the observer are both in motion.

Check Your Understanding 2

If I am driving and I hear Doppler shift in an ambulance siren, I would be able to tell when it was getting closer and also if it has passed by. This would help me to know whether I needed to pull over and let the ambulance through.

Problems & Exercises

1: (a) 878 Hz (b) 735 Hz

3: 3790 Hz

5: (a) 12.9 m/s (b) 193 Hz

7: First eagle hears 4230 Hz, and the second eagle hears 3560 Hz.

12.5 Sound Interference and Resonance: Standing Waves in Air Columns

Learning Objectives

- Define antinode, node, fundamental, overtones, and harmonics.
- Identify instances of sound interference in everyday situations.
- Describe how sound interference occurring inside open and closed tubes changes the characteristics of the sound, and how this applies to sounds produced by musical instruments.
- Calculate the length of a tube using sound wave measurements.

Figure 1. *Noise cancelling headphones. Some types of headphones use the phenomena of constructive and destructive interference to cancel out outside noises. (credit: JVC America, Flickr)*

Interference is the hallmark of waves, all of which exhibit constructive and destructive interference exactly analogous to that seen for water waves. In fact, one way to prove something “is a wave” is

to observe interference effects. So, sound being a wave, we expect it to exhibit interference; we have already mentioned a few such effects, such as the beats from two similar notes played simultaneously.

Figure 2 shows a clever use of sound interference to cancel noise. Larger-scale applications of active noise reduction by destructive interference are contemplated for entire passenger compartments in commercial aircraft. To obtain destructive interference, a fast electronic analysis is performed, and a second sound is introduced with its maxima and minima exactly reversed from the incoming noise. Sound waves in fluids are pressure waves and consistent with Pascal's principle; pressures from two different sources add and subtract like simple numbers; that is, positive and negative gauge pressures add to a much smaller pressure, producing a lower-intensity sound. Although completely destructive interference is possible only under the simplest conditions, it is possible to reduce noise levels by 30 dB or more using this technique.

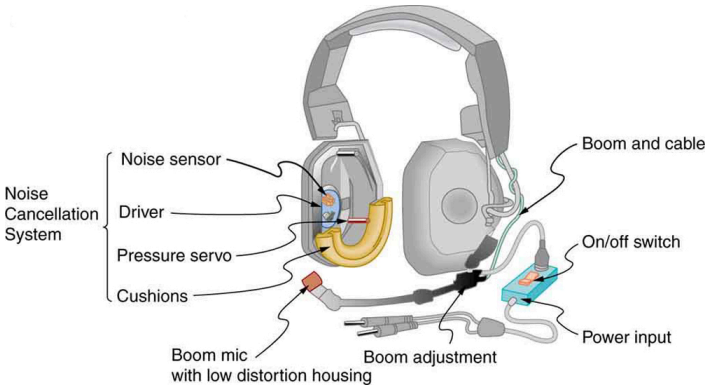


Figure 2. *Headphones designed to cancel noise with destructive interference create a sound wave exactly opposite to the incoming sound. These headphones can be more effective than the simple passive attenuation used in most ear protection. Such headphones were used on the record-setting, around the world nonstop flight of the Voyager aircraft to protect the pilots' hearing from engine noise.*

Where else can we observe sound interference? All sound resonances, such as in musical instruments, are due to constructive and destructive interference. Only the resonant frequencies interfere constructively to form standing waves, while others interfere destructively and are absent. From the toot made by blowing over a bottle, to the characteristic flavour of a violin's sounding box, to the recognizability of a great singer's voice, resonance and standing waves play a vital role.

Interference

Interference is such a fundamental aspect of waves that observing interference is proof that something is a wave. The wave nature of light was established by experiments showing interference. Similarly, when electrons scattered from crystals exhibited interference, their wave nature was confirmed to be exactly as predicted by symmetry with certain wave characteristics of light.

Suppose we hold a tuning fork near the end of a tube that is closed at the other end, as shown in Figure 3, Figure 4, Figure 5, and Figure 6. If the tuning fork has just the right frequency, the air column in the tube resonates loudly, but at most frequencies it vibrates very little. This observation just means that the air column has only certain natural frequencies. The figures show how a resonance at the lowest of these natural frequencies is formed. A disturbance travels down the tube at the speed of sound and bounces off the closed end. If the tube is just the right length, the reflected sound arrives back at the tuning fork exactly half a cycle later, and it interferes constructively with the continuing sound produced by the tuning fork. The incoming and reflected sounds form a standing wave in the tube as shown.

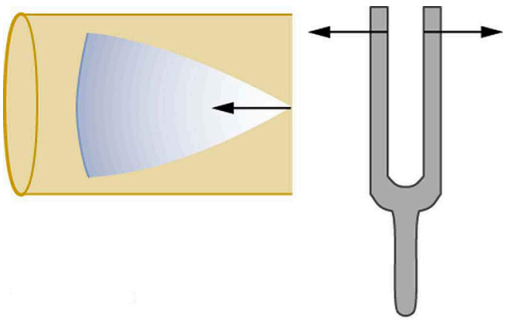


Figure 3. *Resonance of air in a tube closed at one end, caused by a tuning fork. A disturbance moves down the tube.*

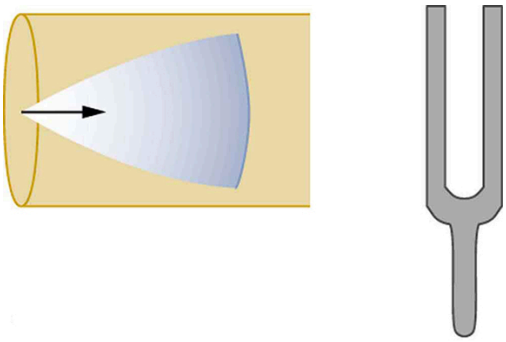


Figure 4. *Resonance of air in a tube closed at one end, caused by a tuning fork. The disturbance reflects from the closed end of the tube.*

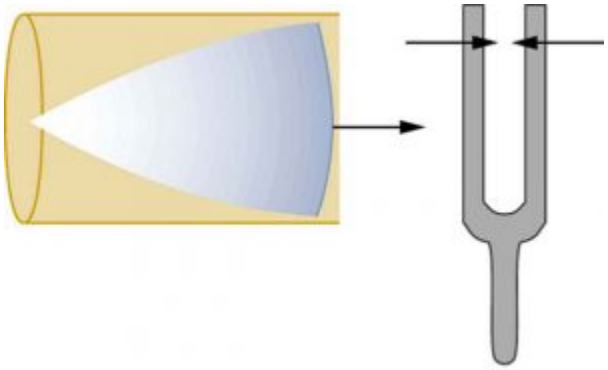


Figure 5. *Resonance of air in a tube closed at one end, caused by a tuning fork. If the length of the tube L is just right, the disturbance gets back to the tuning fork half a cycle later and interferes constructively with the continuing sound from the tuning fork. This interference forms a standing wave, and the air column resonates.*

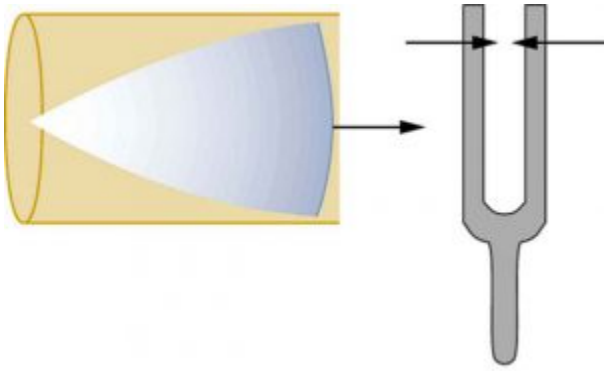


Figure 6. *Resonance of air in a tube closed at one end, caused by a tuning fork. A graph of air displacement along the length of the tube shows none at the closed end, where the motion is constrained, and a maximum at the open end. This standing wave has one-fourth of its wavelength in the tube, so that λ = 4L.*

The standing wave formed in the tube has its maximum air displacement (an **antinode**) at the open end, where motion is unconstrained, and no displacement (a **node**) at the closed end, where air movement is halted. The distance from a node to an antinode is one-fourth of a wavelength, and this equals the length of the tube; thus, $\lambda = 4L$.This same resonance can be produced by a vibration introduced at or near the closed end of the tube, as shown in Figure 7. It is best to consider this a natural vibration of the air column independently of how it is induced.

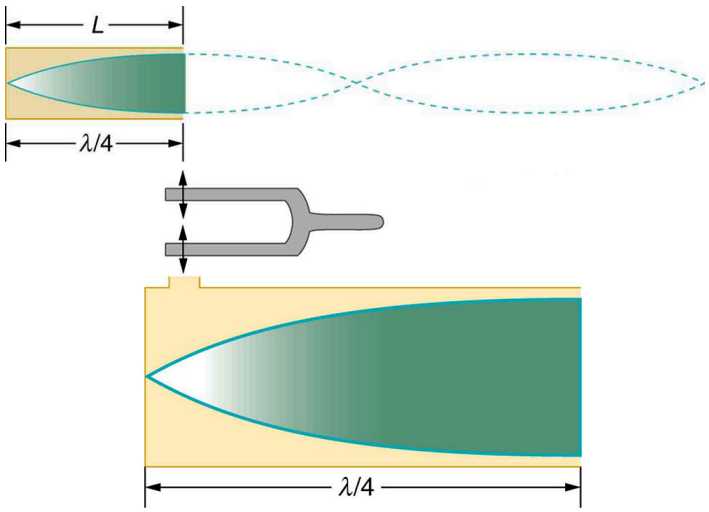


Figure 7. *The same standing wave is created in the tube by a vibration introduced near its closed end.*

Given that maximum air displacements are possible at the open end and none at the closed end, there are other, shorter wavelengths that can resonate in the tube, such as the one shown in Figure 8. Here the standing wave has three-fourths of its wavelength in the tube, or $L = (3/4)\lambda'$, so that $\lambda' = 4L/3$. Continuing this process reveals a whole series of shorter-wavelength and higher-frequency sounds that resonate in the tube. We use specific terms for the resonances in any system. The lowest resonant frequency is called the **fundamental**, while all higher resonant frequencies are called **overtones**. All resonant frequencies are integral multiples of the fundamental, and they are collectively called **harmonics**. The fundamental is the first harmonic, the first overtone is the second harmonic, and so on. Figure 9 shows the fundamental and the first three overtones (the first four harmonics) in a tube closed at one end.

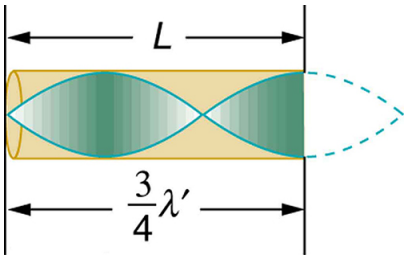


Figure 8. *Another resonance for a tube closed at one end. This has maximum air displacements at the open end, and none at the closed end. The wavelength is shorter, with three-fourths λ′ equaling the length of the tube, so that λ′=4L/3. This higher-frequency vibration is the first overtone.*

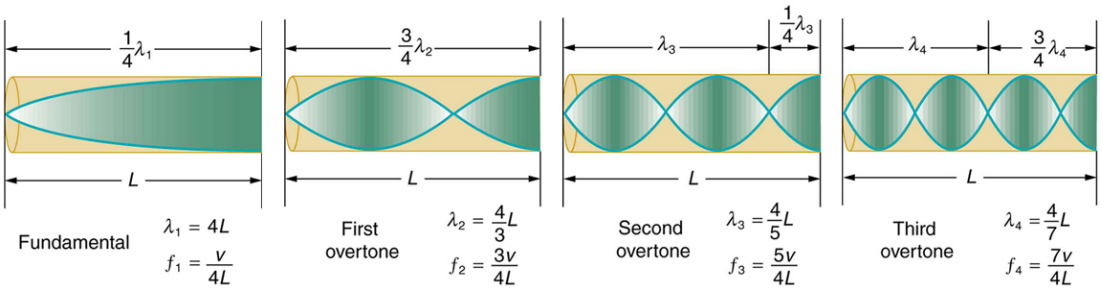


Figure 9. *The fundamental and three lowest overtones for a tube closed at one end. All have maximum air displacements at the open end and none at the closed end.*

The fundamental and overtones can be present simultaneously in a variety of combinations. For example, middle C on a trumpet has a sound distinctively different from middle C on a clarinet, both instruments being modified versions of a tube closed at one end. The fundamental frequency is the same (and usually the most intense), but the overtones and their mix of intensities are different and subject to shading by the musician. This mix is what gives various musical instruments (and human voices) their distinctive characteristics, whether they have air columns, strings, sounding boxes, or drumheads. In fact, much of our speech is determined by shaping the cavity formed by the throat and mouth and positioning the tongue to adjust the fundamental and combination of overtones. Simple resonant cavities can be made to resonate with the sound of the vowels, for example. (See Figure 10.) In boys, at puberty, the larynx grows and the shape of the resonant cavity changes giving rise to the difference in predominant frequencies in speech between men and women.

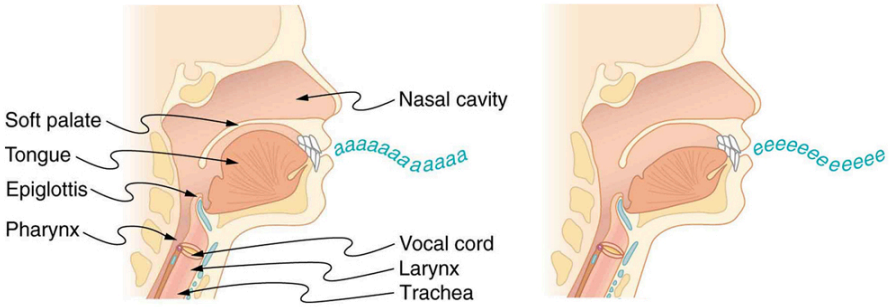


Figure 10. *The throat and mouth form an air column closed at one end that resonates in response to vibrations in the voice box. The spectrum of overtones and their intensities vary with mouth shaping and tongue position to form different sounds. The voice box can be replaced with a mechanical vibrator, and understandable speech is still possible. Variations in basic shapes make different voices recognizable.*

Now let us look for a pattern in the resonant frequencies for a simple tube that is closed at one end. The fundamental has $\lambda = 4L$, and frequency is related to wavelength and the speed of sound as given by:

$$v_{\text{w}} = f\lambda\,.$$

Solving for f in this equation gives

$$f = \frac{v_{\text{w}}}{\lambda} = \frac{v_{\text{w}}}{4L},$$

where v_{w} is the speed of sound in air. Similarly, the first overtone has $\lambda' = 4L/3$ (see Figure 9), so that

$$f' = 3\frac{v_{\text{w}}}{4L} = 3f$$

Because $f' = 3f$, we call the first overtone the third harmonic. Continuing this process, we see a pattern that can be generalized in a single expression. The resonant frequencies of a tube closed at one end are

$$f_{\text{n}} = n\frac{v_{\text{w}}}{4L}, \text{ n = 1, 3, 5, 7 ....}$$

where f_1 is the fundamental, f_3 is the first overtone, and so on. It is interesting that the resonant frequencies depend on the speed of sound and, hence, on temperature. This dependence poses a noticeable problem for organs in old unheated cathedrals, and it is also the reason why musicians commonly bring their wind instruments to room temperature before playing them.

Example 1: Find the Length of a Tube with a 128 Hz Fundamental

(a) What length should a tube closed at one end have on a day when the air temperature, is 22..0 $^{\text{o}}$C if its fundamental frequency is to be 128 Hz (C below middle C)?

(b) What is the frequency of its fourth overtone?

Strategy

The length L can be found from the relationship in $f_{\text{n}} = n\frac{v_{\text{w}}}{4L}$, but we will first need to find the speed of sound v_{w}.

Solution for (a)

(1) Identify knowns:

- the fundamental frequency is 128 Hz
- the air temperature is 22.0 $^{\text{o}}$C.

(2) Use $f_{\text{n}} = n\frac{v_{\text{w}}}{4L}$ to find the fundamental frequency $(n = 1)$

$$f_1 = \frac{v_{\text{w}}}{4L}$$

(3) Solve this equation for length.

$$L = \frac{v_{\text{w}}}{4f_1}$$

(4) Find the speed of sound using $v_{\text{w}} = (331 \text{ m/s})\sqrt{\frac{T}{273 \text{ K}}}$.

$$v_{\text{w}} = (331 \text{ m/s})\sqrt{\frac{295 \text{ K}}{273 \text{ K}}} = 344 \text{ m/s}$$

(5) Enter the values of the speed of sound and frequency into the expression for L.

$$L = \frac{v_{\text{w}}}{4f_1} = \frac{344 \text{ m/s}}{4128 \text{ Hz}} = 0.672 \text{ m}$$

Discussion on (a)

Many wind instruments are modified tubes that have finger holes, valves, and other devices for changing the length of the resonating air column and hence, the frequency of the note played. Horns producing very low frequencies, such as tubas, require tubes so long that they are coiled into loops.

Solution for (b)

(1) Identify knowns:

- the first overtone has $n = 3$
- the second overtone has $n = 5$
- the third overtone has $n = 7$
- the fourth overtone has $n = 9$

(2) Enter the value for the fourth overtone into $f_{\text{n}} = n\frac{v_{\text{w}}}{4L}$.

$$f_9 = 9\frac{v_{\text{w}}}{4L} = 9f_1 = 1.15 \text{ kHz}$$

Discussion on (b)

Whether this overtone occurs in a simple tube or a musical instrument depends on how it is stimulated to vibrate and the details of its shape. The trombone, for example, does not produce its fundamental frequency and only makes overtones.

Another type of tube is one that is *open* at both ends. Examples are some organ pipes, flutes, and oboes. The resonances of tubes open at both ends can be analyzed in a very similar fashion to those for tubes closed at one end. The air columns in tubes open at both ends have maximum air displacements at both ends, as illustrated in Figure 11. Standing waves form as shown.

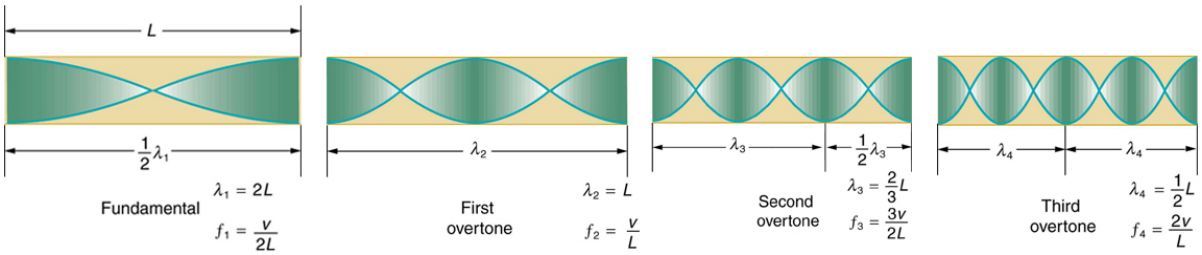


Figure 11. *The resonant frequencies of a tube open at both ends are shown, including the fundamental and the first three overtones. In all cases the maximum air displacements occur at both ends of the tube, giving it different natural frequencies than a tube closed at one end.*

Based on the fact that a tube open at both ends has maximum air displacements at both ends, and using Figure 11 as a guide, we can see that the resonant frequencies of a tube open at both ends are:

$$f_{\mathrm{n}} = n\frac{v_{\mathrm{w}}}{2L}, n = 1, 2, 3 \ldots,$$

where f_1 is the fundamental, f_2 is the first overtone, f_3 is the second overtone, and so on. Note that a tube open at both ends has a fundamental frequency twice what it would have if closed at one end. It also has a different spectrum of overtones than a tube closed at one end. So if you had two tubes with the same fundamental frequency but one was open at both ends and the other was closed at one end, they would sound different when played because they have different overtones. Middle C, for example, would sound richer played on an open tube, because it has even multiples of the fundamental as well as odd. A closed tube has only odd multiples.

Real-World Applications: Resonance in Everyday Systems

Resonance occurs in many different systems, including strings, air columns, and atoms. Resonance is the driven or forced oscillation of a system at its natural frequency. At resonance, energy is transferred rapidly to the oscillating system, and the amplitude of its oscillations grows until the system can no longer be described by Hooke's law. An example of this is the distorted sound intentionally produced in certain types of rock music.

Wind instruments use resonance in air columns to amplify tones made by lips or vibrating reeds. Other instruments also use air resonance in clever ways to amplify sound. Figure 12 shows a violin and a guitar, both of which have sounding boxes but with different shapes, resulting in different overtone structures. The vibrating string creates a sound that resonates in the sounding box, greatly amplifying the sound and creating overtones that give the instrument its characteristic flavor. The more complex the shape of the sounding box, the greater its ability to resonate over a wide range of frequencies. The marimba, like the one shown in Figure 13 uses pots or gourds below the wooden slats to amplify their tones. The resonance of the pot can be adjusted by adding water.

Figure 12. *String instruments such as violins and guitars use resonance in their sounding boxes to amplify and enrich the sound created by their vibrating strings. The bridge and supports couple the string vibrations to the sounding boxes and air within. (credits: guitar, Feliciano Guimares, Fotopedia; violin, Steve Snodgrass, Flickr)*

Figure 13. *Resonance has been used in musical instruments since prehistoric times. This marimba uses gourds as resonance chambers to amplify its sound. (credit: APC Events, Flickr)*

We have emphasized sound applications in our discussions of resonance and standing waves, but these ideas apply to any system that has wave characteristics. Vibrating strings, for example, are actually

resonating and have fundamentals and overtones similar to those for air columns. More subtle are the resonances in atoms due to the wave character of their electrons. Their orbitals can be viewed as standing waves, which have a fundamental (ground state) and overtones (excited states). It is fascinating that wave characteristics apply to such a wide range of physical systems.

Check Your Understanding 1

Describe how noise-canceling headphones differ from standard headphones used to block outside sounds.

Check Your Understanding 2

How is it possible to use a standing wave's node and antinode to determine the length of a closed-end tube?

PhET Explorations: Sound

This simulation lets you see sound waves. Adjust the frequency or volume and you can see and hear how the wave changes. Move the listener around and hear what she hears.

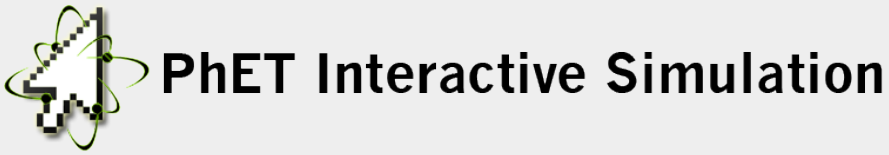


Figure 14. *Sound*

Section Summary

- Sound interference and resonance have the same properties as defined for all waves.
- In air columns, the lowest-frequency resonance is called the fundamental, whereas all higher resonant frequencies are called overtones. Collectively, they are called harmonics.
- The resonant frequencies of a tube closed at one end are:

$$f_{\text{n}} = n\frac{v_{\text{w}}}{4L}, n = 1, 3, 5 \ldots,$$

  f_1 is the fundamental and L is the length of the tube.
- The resonant frequencies of a tube open at both ends are:

$$f_{\mathrm{n}} = n\frac{v_{\mathrm{w}}}{2L}, n = 1, 2, 3 \ldots$$

Glossary

antinode
: point of maximum displacement

node
: point of zero displacement

fundamental
: the lowest-frequency resonance

overtones
: all resonant frequencies higher than the fundamental

harmonics
: the term used to refer collectively to the fundamental and its overtones

Conceptual Questions

1: How does an unamplified guitar produce sounds so much more intense than those of a plucked string held taut by a simple stick?

2: You are given two wind instruments of identical length. One is open at both ends, whereas the other is closed at one end. Which is able to produce the lowest frequency?

3: What is the difference between an overtone and a harmonic? Are all harmonics overtones? Are all overtones harmonics?

Problems & Exercises

1: A “showy” custom-built car has two brass horns that are supposed to produce the same frequency but actually emit 263.8 and 264.5 Hz. What beat frequency is produced?

2: What beat frequencies will be present: (a) If the musical notes A and C are played together (frequencies of 220 and 264 Hz)? (b) If D and F are played together (frequencies of 297 and 352 Hz)? (c) If all four are played together?

3: What beat frequencies result if a piano hammer hits three strings that emit frequencies of 127.8, 128.1, and 128.3 Hz?

4: A piano tuner hears a beat every 2.00 s when listening to a 264.0-Hz tuning fork and a single piano string. What are the two possible frequencies of the string?

5: (a) What is the fundamental frequency of a 0.672-m-long tube, open at both ends, on a day when the speed of sound is 344 m/s? (b) What is the frequency of its second harmonic?

6: If a wind instrument, such as a tuba, has a fundamental frequency of 32.0 Hz, what are its first three overtones? It is closed at one end. (The overtones of a real tuba are more complex than this example, because it is a tapered tube.)

7: What are the first three overtones of a bassoon that has a fundamental frequency of 90.0 Hz? It is open at both ends. (The overtones of a real bassoon are more complex than this example, because its double reed makes it act more like a tube closed at one end.)

8: How long must a flute be in order to have a fundamental frequency of 262 Hz (this frequency corresponds to middle C on the evenly tempered chromatic scale) on a day when air temperature is 20.0 ^{o}C. It is open at both ends.

9: What length should an oboe have to produce a fundamental frequency of 110 Hz on a day when the speed of sound is 343 m/s? It is open at both ends.

10: What is the length of a tube that has a fundamental frequency of 176 Hz and a first overtone of 352 Hz if the speed of sound is 343 m/s?

11: (a) Find the length of an organ pipe closed at one end that produces a fundamental frequency of 256 Hz when air temperature is 18.0 ^{o}C. (b) What is its fundamental frequency at 25.0 ^{o}C?

12: By what fraction will the frequencies produced by a wind instrument change when air temperature goes from 10.0 ^{o}C. to 30.0 ^{o}C? That is, find the ratio of the frequencies at those temperatures.

13: The ear canal resonates like a tube closed at one end. (See Chapter 17.6 The Hearing Mechanism.) If ear canals range in length from 1.80 to 2.60 cm in an average population, what is the range of fundamental resonant frequencies? Take air temperature to be 37.0°C, which is the same as body temperature. How does this result correlate with the intensity versus frequency graph (Figure 3 of the human ear?

14: Calculate the first overtone in an ear canal, which resonates like a 2.40-cm-long tube closed at one end, by taking air temperature to be 37.0°C. Is the ear particularly sensitive to such a frequency? (The resonances of the ear canal are complicated by its nonuniform shape, which we shall ignore.)

15: A crude approximation of voice production is to consider the breathing passages and mouth to be a resonating tube closed at one end. (See Figure 10.) (a) What is the fundamental frequency if the tube is 0.240-m long, by taking air temperature to be 37.0 ^{o}C?

(b) What would this frequency become if the person replaced the air with helium? Assume the same temperature dependence for helium as for air.

16: (a) Students in a physics lab are asked to find the length of an air column in a tube closed at one end that has a fundamental frequency of 256 Hz. They hold the tube vertically and fill it with water to the top, then lower the water while a 256-Hz tuning fork is rung and listen for the first resonance. What is the air temperature if the resonance occurs for a length of 0.336 m? (b) At what length will they observe the second resonance (first overtone)?

17: What frequencies will a 1.80-m-long tube produce in the audible range at 20.0 ^{o}C if: (a) The tube is closed at one end? (b) It is open at both ends?

Solutions

Check Your Understanding 1

Regular headphones only block sound waves with a physical barrier. Noise-canceling headphones use destructive interference to reduce the loudness of outside sounds.

Check Your Understanding 2

When the tube resonates at its natural frequency, the wave's node is located at the closed end of the tube, and the antinode is located at the open end. The length of the tube is equal to one-fourth of the wavelength of this wave. Thus, if we know the wavelength of the wave, we can determine the length of the tube.

Problems & Exercises

1: 0.7 Hz

3: 0.3 Hz, 0.2 Hz, 0.5 Hz

5: (a) 256 Hz (b) 512 Hz

7: 180 Hz, 270 Hz, 360 Hz

9: 1.56 m

11: (a) 0.334 m (b) 259 Hz

13: 3.39 to 4.90 kHz

15: (a) 367 Hz (b) 1.07 kHz

17: (a) $f_{\text{n}} = n(47.6\text{ Hz}), n = 1, 3, 5, \ldots, 419$

(b) $f_{\text{n}} = n(95.3\text{ Hz}), n = 1, 2, 3, \ldots, 210$

12.6 Optional: Hearing

Summary

- Define hearing, pitch, loudness, timbre, note, tone, phon, ultrasound, and infrasound.
- Compare loudness to frequency and intensity of a sound.
- Identify structures of the inner ear and explain how they relate to sound perception.

Figure 1. *Hearing allows this vocalist, his band, and his fans to enjoy music. (credit: West Point Public Affairs, Flickr)*

The human ear has a tremendous range and sensitivity. It can give us a wealth of simple information—such as pitch, loudness, and direction. And from its input we can detect musical quality and nuances of voiced emotion. How is our hearing related to the physical qualities of sound, and how does the hearing mechanism work?

Hearing is the perception of sound. (Perception is commonly defined to be awareness through the senses, a typically circular definition of higher-level processes in living organisms.) Normal human hearing encompasses frequencies from 20 to 20,000 Hz, an impressive range. Sounds below 20 Hz are called **infrasound**, whereas those above 20,000 Hz are **ultrasound**. Neither is perceived by the ear, although infrasound can sometimes be felt as vibrations. When we do hear low-frequency vibrations, such as the sounds of a diving board, we hear the individual vibrations only because there are higher-frequency sounds in each. Other animals have hearing ranges different from that of humans. Dogs can hear sounds as high as 30,000 Hz, whereas bats and dolphins can hear up to 100,000-Hz sounds. You may have noticed that dogs respond to the sound of a dog whistle which produces sound out of the range of human hearing. Elephants are known to respond to frequencies below 20 Hz.

The perception of frequency is called **pitch**. Most of us have excellent relative pitch, which means that we can tell whether one sound has a different frequency from another. Typically, we can discriminate between two sounds if their frequencies differ by 0.3% or more. For example, 500.0 and 501.5 Hz are noticeably different. Pitch perception is directly related to frequency and is not greatly affected by

other physical quantities such as intensity. Musical **notes** are particular sounds that can be produced by most instruments and in Western music have particular names. Combinations of notes constitute music. Some people can identify musical notes, such as A-sharp, C, or E-flat, just by listening to them. This uncommon ability is called perfect pitch.

The ear is remarkably sensitive to low-intensity sounds. The lowest audible intensity or threshold is about 10^{-12} W/m^2 or 0 dB. Sounds as much as 10^{12} more intense can be briefly tolerated. Very few measuring devices are capable of observations over a range of a trillion. The perception of intensity is called **loudness**. At a given frequency, it is possible to discern differences of about 1 dB, and a change of 3 dB is easily noticed. But loudness is not related to intensity alone. Frequency has a major effect on how loud a sound seems. The ear has its maximum sensitivity to frequencies in the range of 2000 to 5000 Hz, so that sounds in this range are perceived as being louder than, say, those at 500 or 10,000 Hz, even when they all have the same intensity. Sounds near the high- and low-frequency extremes of the hearing range seem even less loud, because the ear is even less sensitive at those frequencies. Table 4 gives the dependence of certain human hearing perceptions on physical quantities.

Perception	Physical quantity
Pitch	Frequency
Loudness	Intensity and Frequency
Timbre	Number and relative intensity of multiple frequencies. Subtle craftsmanship leads to non-linear effects and more detail.
Note	Basic unit of music with specific names, combined to generate tunes
Tone	Number and relative intensity of multiple frequencies.

Table 4. Sound Perceptions.

When a violin plays middle C, there is no mistaking it for a piano playing the same note. The reason is that each instrument produces a distinctive set of frequencies and intensities. We call our perception of these combinations of frequencies and intensities **tone** quality, or more commonly the **timbre** of the sound. It is more difficult to correlate timbre perception to physical quantities than it is for loudness or pitch perception. Timbre is more subjective. Terms such as dull, brilliant, warm, cold, pure, and rich are employed to describe the timbre of a sound. So the consideration of timbre takes us into the realm of perceptual psychology, where higher-level processes in the brain are dominant. This is true for other perceptions of sound, such as music and noise. We shall not delve further into them; rather, we will concentrate on the question of loudness perception.

A unit called a **phon** is used to express loudness numerically. Phons differ from decibels because the phon is a unit of loudness perception, whereas the decibel is a unit of physical intensity. Figure 2 shows the relationship of loudness to intensity (or intensity level) and frequency for persons with normal hearing. The curved lines are equal-loudness curves. Each curve is labeled with its loudness in phons. Any sound along a given curve will be perceived as equally loud by the average person. The curves were determined by having large numbers of people compare the loudness of sounds at different frequencies and sound intensity levels. At a frequency of 1000 Hz, phons are taken to be numerically equal to decibels. The following example helps illustrate how to use the graph:

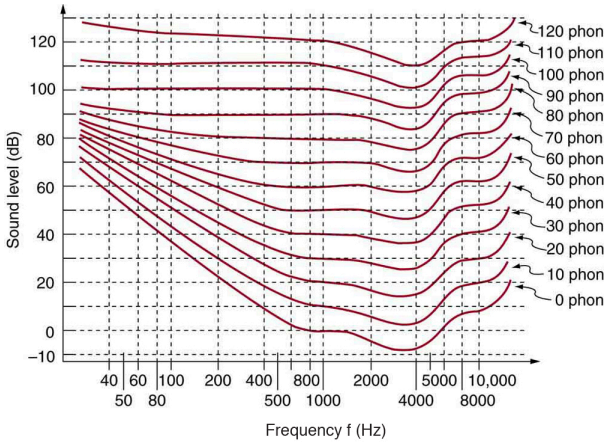


Figure 2. *The relationship of loudness in phons to intensity level (in decibels) and intensity (in watts per meter squared) for persons with normal hearing. The curved lines are equal-loudness curves—all sounds on a given curve are perceived as equally loud. Phons and decibels are defined to be the same at 1000 Hz.*

Example 1: Measuring Loudness: Loudness Versus Intensity Level and Frequency

(a) What is the loudness in phons of a 100-Hz sound that has an intensity level of 80 dB? (b) What is the intensity level in decibels of a 4000-Hz sound having a loudness of 70 phons? (c) At what intensity level will an 8000-Hz sound have the same loudness as a 200-Hz sound at 60 dB?

Strategy for (a)

The graph in Figure 2 should be referenced in order to solve this example. To find the loudness of a given sound, you must know its frequency and intensity level and locate that point on the square grid, then interpolate between loudness curves to get the loudness in phons.

Solution for (a)

(1) Identify knowns:

- The square grid of the graph relating phons and decibels is a plot of intensity level versus frequency—both physical quantities.
- 100 Hz at 80 dB lies halfway between the curves marked 70 and 80 phons.

(2) Find the loudness: 75 phons.

Strategy for (b)

The graph in Figure 2 should be referenced in order to solve this example. To find the intensity level of a sound, you must have its frequency and loudness. Once that point is located, the intensity level can be determined from the vertical axis.

Solution for (b)

(1) Identify knowns:

- Values are given to be 4000 Hz at 70 phons.

(2) Follow the 70-phon curve until it reaches 4000 Hz. At that point, it is below the 70 dB line at about 67 dB.

(3) Find the intensity level:

67 dB

Strategy for (c)

The graph in Figure 2 should be referenced in order to solve this example.

Solution for (c)

(1) Locate the point for a 200 Hz and 60 dB sound.

(2) Find the loudness: This point lies just slightly above the 50-phon curve, and so its loudness is 51 phons.

(3) Look for the 51-phon level is at 8000 Hz: 63 dB.

Discussion

These answers, like all information extracted from Figure 2, have uncertainties of several phons or several decibels, partly due to difficulties in interpolation, but mostly related to uncertainties in the equal-loudness curves.

Further examination of the graph in Figure 2 reveals some interesting facts about human hearing. First, sounds below the 0-phon curve are not perceived by most people. So, for example, a 60 Hz sound at 40 dB is inaudible. The 0-phon curve represents the threshold of normal hearing. We can hear some sounds at intensity levels below 0 dB. For example, a 3-dB, 5000-Hz sound is audible, because it lies above the 0-phon curve. The loudness curves all have dips in them between about 2000 and 5000 Hz. These dips mean the ear is most sensitive to frequencies in that range. For example, a 15-dB sound at 4000 Hz has a loudness of 20 phons, the same as a 20-dB sound at 1000 Hz. The curves rise at both extremes of the frequency range, indicating that a greater-intensity level sound is needed at those frequencies to be perceived to be as loud as at middle frequencies. For example, a sound at 10,000 Hz must have an intensity level of 30 dB to seem as loud as a 20 dB sound at 1000 Hz. Sounds above 120 phons are painful as well as damaging.

We do not often utilize our full range of hearing. This is particularly true for frequencies above 8000 Hz, which are rare in the environment and are unnecessary for understanding conversation or appreciating music. In fact, people who have lost the ability to hear such high frequencies are usually unaware of their loss until tested. The shaded region in Figure 3 is the frequency and intensity region where most conversational sounds fall. The curved lines indicate what effect hearing losses of 40 and 60 phons will have. A 40-phon hearing loss at all frequencies still allows a person to understand conversation, although it will seem very quiet. A person with a 60-phon loss at all frequencies will hear only the lowest frequencies and will not be able to understand speech unless it is much louder than normal. Even so, speech may seem indistinct, because higher frequencies are not as well perceived. The conversational

speech region also has a gender component, in that female voices are usually characterized by higher frequencies. So the person with a 60-phon hearing impediment might have difficulty understanding the normal conversation of a woman.

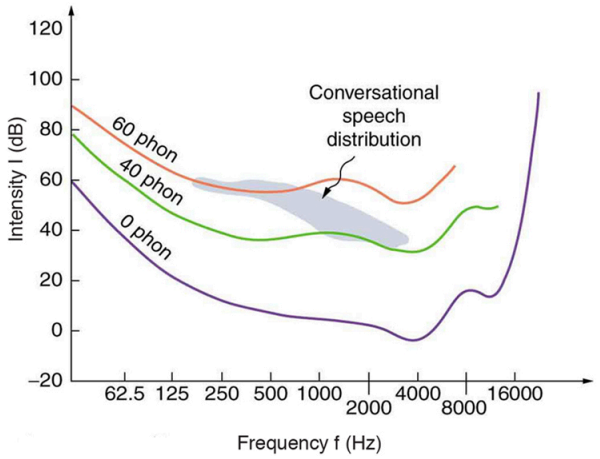


Figure 3. *The shaded region represents frequencies and intensity levels found in normal conversational speech. The 0-phon line represents the normal hearing threshold, while those at 40 and 60 represent thresholds for people with 40- and 60-phon hearing losses, respectively.*

Hearing tests are performed over a range of frequencies, usually from 250 to 8000 Hz, and can be displayed graphically in an audiogram like that in Figure 4. The hearing threshold is measured in dB *relative to the normal threshold*, so that normal hearing registers as 0 dB at all frequencies. Hearing loss caused by noise typically shows a dip near the 4000 Hz frequency, irrespective of the frequency that caused the loss and often affects both ears. The most common form of hearing loss comes with age and is called *presbycusis*—literally elder ear. Such loss is increasingly severe at higher frequencies, and interferes with music appreciation and speech recognition.

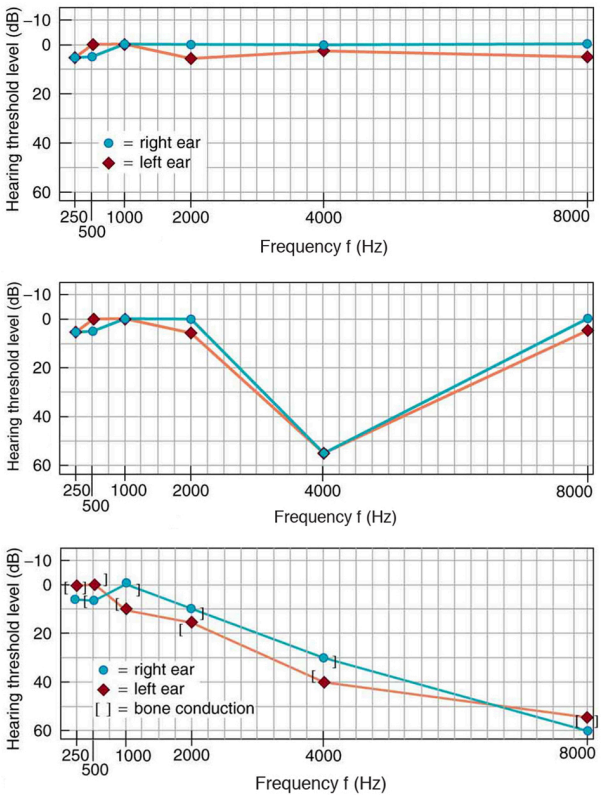


Figure 4. *Audiograms showing the threshold in intensity level versus frequency for three different individuals. Intensity level is measured relative to the normal threshold. The top left graph is that of a person with normal hearing. The graph to its right has a dip at 4000 Hz and is that of a child who suffered hearing loss due to a cap gun. The third graph is typical of presbycusis, the progressive loss of higher frequency hearing with age. Tests performed by bone conduction (brackets) can distinguish nerve damage from middle ear damage.*

The Hearing Mechanism

The hearing mechanism involves some interesting physics. The sound wave that impinges upon our ear is a pressure wave. The ear is a transducer that converts sound waves into electrical nerve impulses in a manner much more sophisticated than, but analogous to, a microphone. Figure 5. shows the gross anatomy of the ear with its division into three parts: the outer ear or ear canal; the middle ear, which runs from the eardrum to the cochlea; and the inner ear, which is the cochlea itself. The body part normally referred to as the ear is technically called the pinna.

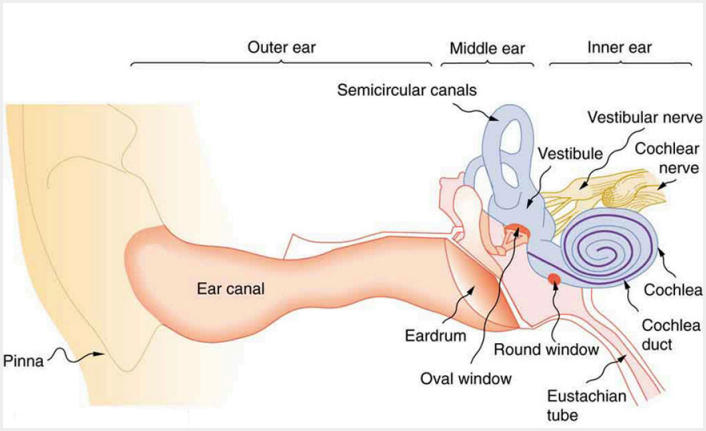


Figure 5. *The illustration shows the gross anatomy of the human ear.*

The outer ear, or ear canal, carries sound to the recessed protected eardrum. The air column in the ear canal resonates and is partially responsible for the sensitivity of the ear to sounds in the 2000 to 5000 Hz range. The middle ear converts sound into mechanical vibrations and applies these vibrations to the cochlea. The lever system of the middle ear takes the force exerted on the eardrum by sound pressure variations, amplifies it and transmits it to the inner ear via the oval window, creating pressure waves in the cochlea approximately 40 times greater than those impinging on the eardrum. (See Figure 6.) Two muscles in the middle ear (not shown) protect the inner ear from very intense sounds. They react to intense sound in a few milliseconds and reduce the force transmitted to the cochlea. This protective reaction can also be triggered by your own voice, so that humming while shooting a gun, for example, can reduce noise damage.

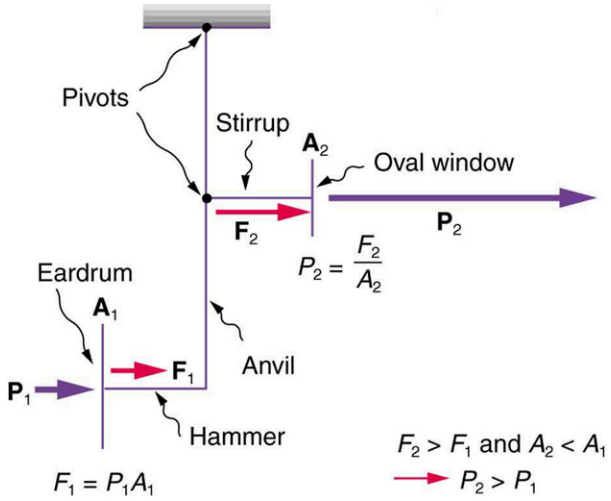


Figure 6. *This schematic shows the middle ear's system for converting sound pressure into force, increasing that force through a lever system, and applying the increased force to a small area of the cochlea, thereby creating a pressure about 40 times that in the original sound wave. A protective muscle reaction to intense sounds greatly reduces the mechanical advantage of the lever system.*

Figure 7 shows the middle and inner ear in greater detail. Pressure waves moving through the cochlea cause the tectorial membrane to vibrate, rubbing cilia (called hair cells), which stimulate nerves that send electrical signals to the brain. The membrane resonates at different positions for different frequencies, with high frequencies stimulating nerves at the near end and low frequencies at the far end. The complete operation of the cochlea is still not understood, but several mechanisms for sending information to the brain are known to be involved. For sounds below about 1000 Hz, the nerves send signals at the same frequency as the sound. For frequencies greater than about 1000 Hz, the nerves signal frequency by position. There is a structure to the cilia, and there are connections between nerve cells that perform signal processing before information is sent to the brain. Intensity information is partly indicated by the number of nerve signals and by volleys of signals. The brain processes the cochlear nerve signals to provide additional information such as source direction (based on time and intensity comparisons of sounds from both ears). Higher-level processing produces many nuances, such as music appreciation.

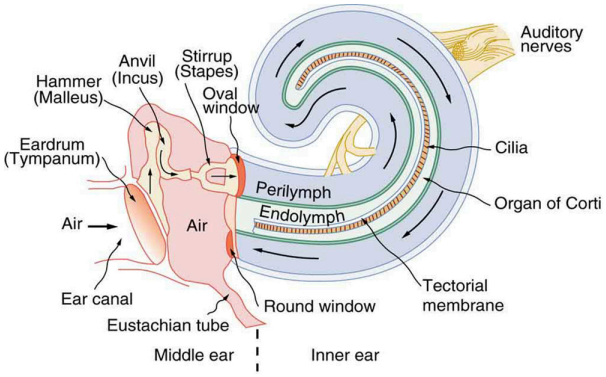


Figure 7. *The inner ear, or cochlea, is a coiled tube about 3 mm in diameter and 3 cm in length if uncoiled. When the oval window is forced inward, as shown, a pressure wave travels through the perilymph in the direction of the arrows, stimulating nerves at the base of cilia in the organ of Corti.*

Hearing losses can occur because of problems in the middle or inner ear. Conductive losses in the middle ear can be partially overcome by sending sound vibrations to the cochlea through the skull. Hearing aids for this purpose usually press against the bone behind the ear, rather than simply amplifying the sound sent into the ear canal as many hearing aids do. Damage to the nerves in the cochlea is not repairable, but amplification can partially compensate. There is a risk that amplification will produce further damage. Another common failure in the cochlea is damage or loss of the cilia but with nerves remaining functional. Cochlear implants that stimulate the nerves directly are now available and widely accepted. Over 100,000 implants are in use, in about equal numbers of adults and children.

The cochlear implant was pioneered in Melbourne, Australia, by Graeme Clark in the 1970s for his deaf father. The implant consists of three external components and two internal components. The external components are a microphone for picking up sound and converting it into an electrical signal, a speech processor to select certain frequencies and a transmitter to transfer the signal to the internal components through electromagnetic induction. The internal components consist of a receiver/transmitter secured in the bone beneath the skin, which converts the signals into electric impulses and sends them through an internal cable to the cochlea and an array of about 24 electrodes wound through the cochlea. These electrodes in turn send the impulses directly into the brain. The electrodes basically emulate the cilia.

Check Your Understanding

1: Are ultrasound and infrasound imperceptible to all hearing organisms? Explain your answer.

Section Summary

- The range of audible frequencies is 20 to 20,000 Hz.
- Those sounds above 20,000 Hz are ultrasound, whereas those below 20 Hz are infrasound.
- The perception of frequency is pitch.

- The perception of intensity is loudness.
- Loudness has units of phons.

Glossary

loudness
the perception of sound intensity
timbre
number and relative intensity of multiple sound frequencies
note
basic unit of music with specific names, combined to generate tunes
tone
number and relative intensity of multiple sound frequencies
phon
the numerical unit of loudness
ultrasound
sounds above 20,000 Hz
infrasound
sounds below 20 Hz

Conceptual Questions

1: Why can a hearing test show that your threshold of hearing is 0 dB at 250 Hz, when Figure 3 implies that no one can hear such a frequency at less than 20 dB?

Problems & Exercises

1: The factor of 10^{12} in the range of intensities to which the ear can respond, from threshold to that causing damage after brief exposure, is truly remarkable. If you could measure distances over the same range with a single instrument and the smallest distance you could measure was 1 mm, what would the largest be?

2: The frequencies to which the ear responds vary by a factor of 10^{3}. Suppose the speedometer on your car measured speeds differing by the same factor of 10^{3} and the greatest speed it reads is 90.0 mi/h. What would be the slowest nonzero speed it could read?

3: What are the closest frequencies to 500 Hz that an average person can clearly distinguish as being different in frequency from 500 Hz? The sounds are not present simultaneously.

4: Can the average person tell that a 2002-Hz sound has a different frequency than a 1999-Hz sound without playing them simultaneously?

5: If your radio is producing an average sound intensity level of 85 dB, what is the next lowest sound intensity level that is clearly less intense?

6: Can you tell that your roommate turned up the sound on the TV if its average sound intensity level goes from 70 to 73 dB?

7: Based on the graph in Figure 2, what is the threshold of hearing in decibels for frequencies of 60, 400, 1000, 4000, and 15,000 Hz? Note that many AC electrical appliances produce 60 Hz, music is commonly 400 Hz, a reference frequency is 1000 Hz, your maximum sensitivity is near 4000 Hz, and many older TVs produce a 15,750 Hz whine.

8: What sound intensity levels must sounds of frequencies 60, 3000, and 8000 Hz have in order to have the same loudness as a 40-dB sound of frequency 1000 Hz (that is, to have a loudness of 40 phons)?

9: What is the approximate sound intensity level in decibels of a 600-Hz tone if it has a loudness of 20 phons? If it has a loudness of 70 phons?

1): (a) What are the loudnesses in phons of sounds having frequencies of 200, 1000, 5000, and 10,000 Hz, if they are all at the same 60.0-dB sound intensity level? (b) If they are all at 110 dB? (c) If they are all at 20.0 dB?

11: Suppose a person has a 50-dB hearing loss at all frequencies. By how many factors of 10 will low-intensity sounds need to be amplified to seem normal to this person? Note that smaller amplification is appropriate for more intense sounds to avoid further hearing damage.

12: If a woman needs an amplification of 5.0×10^{12} times the threshold intensity to enable her to hear at all frequencies, what is her overall hearing loss in dB? Note that smaller amplification is appropriate for more intense sounds to avoid further damage to her hearing from levels above 90 dB.

13: (a) What is the intensity in watts per meter squared of a just barely audible 200-Hz sound? (b) What is the intensity in watts per meter squared of a barely audible 4000-Hz sound?

14: (a) Find the intensity in watts per meter squared of a 60.0-Hz sound having a loudness of 60 phons. (b) Find the intensity in watts per meter squared of a 10,000-Hz sound having a loudness of 60 phons.

15: A person has a hearing threshold 10 dB above normal at 100 Hz and 50 dB above normal at 4000 Hz. How much more intense must a 100-Hz tone be than a 4000-Hz tone if they are both barely audible to this person?

16: A child has a hearing loss of 60 dB near 5000 Hz, due to noise exposure, and normal hearing elsewhere. How much more intense is a 5000-Hz tone than a 400-Hz tone if they are both barely audible to the child?

17: What is the ratio of intensities of two sounds of identical frequency if the first is just barely discernible as louder to a person than the second?

Solutions

Check Your understanding

1: No, the range of perceptible sound is based in the range of human hearing. Many other organisms perceive either infrasound or ultrasound.

Problems & Exercises

1: 1×10^{6} km

3: 498.5 or 501.5 Hz

5: 82 dB

7: approximately 48, 9, 0, –7, and 20 dB, respectively

9: (a) 23 dB (b) 70 dB

11: Five factors of 10

13: (a) 2×10^{-10} W/m^2 (b) 2×10^{-13} W/m^2

15: 2.5

17: 1.26

Appendix A Useful Information - Constants, Units, Formulae

This appendix is broken into several tables.

- Table 3, Important Constants
- Table 4, Submicroscopic Masses
- Table 5, Solar System Data
- Table 6, Metric Prefixes for Powers of Ten and Their Symbols
- Table 7, The Greek Alphabet
- Table 8, SI units
- Table 9, Selected British Units
- Table 10, Other Units
- Table 11, Useful Formulae

Table 3. Important Constants[1]

Symbol	Meaning	Best Value	Approximate Value
c	Speed of light in vacuum	$2.99792458 \times 10^8 \mathrm{m/s}$	$3.00 \times 10^8 \mathrm{m/s}$
G	Gravitational constant	$6.67408(31) \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$	$6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
N_A	Avogadro's number	$6.02214129(27) \times 10^{23}$	6.02×10^{23}
k	Boltzmann's constant	$1.3806488(13) \times 10^{-23}\ \mathrm{J/K}$	$1.38 \times 10^{-23}\ \mathrm{J/K}$
R	Gas constant	$8.3144621(75)\ \mathrm{J/mol \cdot K}$	$8.31\ \mathrm{J/mol \cdot K} = 1.99\mathrm{cal/mol \cdot K}$
σ	Stefan-Boltzmann constant	$5.670373(21) \times 10^{-8}\ \mathrm{W/m^2 \cdot K}$	$5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K}$
k	Coulomb force constant	$8.987551788\cdots \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$
q_e	Charge on electron	$-1.602176565(35) \times 10^{-19}\ \mathrm{C}$	$-1.60 \times 10^{-19}\ \mathrm{C}$
ε_0	Permittivity of free space	$8.854187817\cdots \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$	$8.85\cdots \times 10^{-12}\ \mathrm{C^2/Nm^2}$
μ_0	Permeability of free space	$4\pi \times 10^{-7}\ \mathrm{T \cdot\ m/A}$	$1.26 \times 10^{-6}\ \mathrm{T \cdot m/A}$
h	Planck's constant	$6.62606957(29) \times 10^{-34}\ \mathrm{J \cdot s}$	$6.63 \times 10^{-34}\ \mathrm{J \cdot s}$

Symbol	Meaning	Best Value	Approximate Value
m_e	Electron mass	$9.10938291(40) \times 10^{-31}\mathrm{kg}$	$9.11 \times 10^{-31}\mathrm{kg}$
m_p	Proton mass	$1.672621777(74) \times 10^{-27}\mathrm{kg}$	$1.6726 \times 10^{-27}\mathrm{kg}$
m_n	Neutron mass	$1.674927351(74) \times 10^{-27}\mathrm{kg}$	$1.6749 \times 10^{-27}\mathrm{kg}$
u	Atomic mass unit	$1.660538921(73) \times 10^{-27}\mathrm{kg}$	$1.6605 \times 10^{-27}\mathrm{kg}$

Table 4. Submicroscopic Masses[2]

Sun	mass	$1.99 \times 10^{30}\mathrm{kg}$
	average radius	$6.96 \times 10^{8}\mathrm{m}$
	Earth-sun distance (average)	$1.496 \times 10^{11}\mathrm{m}$
Earth	mass	$5.9736 \times 10^{24}\mathrm{kg}$
	average radius	$6.376 \times 10^{6}\mathrm{m}$
	orbital period	$3.16 \times 10^{7}\mathrm{s}$
Moon	mass	$7.35 \times 10^{22}\mathrm{kg}$
	average radius	$1.74 \times 10^{6}\mathrm{m}$
	orbital period (average)	$2.36 \times 10^{6}\mathrm{s}$
	Earth-moon distance (average)	$3.84 \times 10^{8}\mathrm{m}$

Table 5. Solar System Data

Prefix	**Symbol**	**Value**	**Prefix**	**Symbol**	**Value**
tera	T	10^{12}	deci	d	10^{-1}
giga	G	10^{9}	centi	c	10^{-2}
mega	M	10^{6}	milli	m	10^{-3}
kilo	k	10^{3}	micro	μ	10^{-6}
hecto	h	10^{2}	nano	n	10^{-9}
deka	da	10^{1}	pico	p	10^{-12}
—	—	$10^{0}(=1)$	femto	f	10^{-15}

Table 6. Metric Prefixes for Powers of Ten and Their Symbols

Alpha	A	α	Eta	H	η	Nu	N	ν		Tau	T	τ
Beta	B	β	Theta	Θ	θ	Xi	Ξ	ξ		Upsilon	Y	υ
Gamma	Γ	γ	Iota	I	ι	Omicron	O	Formula does not parse		Phi	Φ	ϕ
Delta	Δ	δ	Kappa	K	κ	Pi	Π	π		Chi	X	χ
Epsilon	E	ϵ	Lambda	Λ	λ	Rho	P	ρ		Psi	Ψ	ψ
Zeta	Z	ζ	Mu	M	μ	Sigma	Σ	σ		Omega	Ω	ω

Table 7. The Greek Alphabet

	Entity	**Abbreviation**	**Name**
Fundamental units	Length	m	meter
	Mass	kg	kilogram
	Time	s	second
	Current	A	ampere
Supplementary unit	Angle	rad	radian
Derived units	Force	$\mathrm{N} = \mathrm{kg} \cdot \mathrm{m}/\mathrm{s}^2$	newton
	Energy	$\mathrm{J} = \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$	joule
	Power	$\mathrm{W} = \mathrm{J}/\mathrm{s}$	watt
	Pressure	$\mathrm{Pa} = \mathrm{N}/\mathrm{m}^2$	pascal
	Frequency	$\mathrm{Hz} = 1/\mathrm{s}$	hertz
	Electronic potential	$\mathrm{V} = \mathrm{J}/\mathrm{C}$	volt
	Capacitance	$\mathrm{F} = \mathrm{C}/\mathrm{V}$	farad
	Charge	$\mathrm{C} = \mathrm{s} \cdot \mathrm{A}$	coulomb
	Resistance	$\Omega = \mathrm{V}/\mathrm{A}$	ohm
	Magnetic field	$\mathrm{T} = \mathrm{N}/(\mathrm{A} \cdot \mathrm{m})$	tesla
	Nuclear decay rate	$\mathrm{Bq} = 1/\mathrm{s}$	becquerel

Table 8. SI Units

Length	$1 \text{ inch (in.)} = 2.54 \text{ cm (exactly)}$
	$1 \text{ foot (ft)} = 0.3048 \text{ m}$
	$1 \text{ mile (mi)} = 1.609 \text{ km}$
Force	$1 \text{ pound (lb)} = 4.448 \text{ N}$
Energy	$1 \text{ British thermal unit (Btu)} = 1.055 \times 10^3 \text{J}$
Power	$1 \text{ horsepower (hp)} = 746 \text{ W}$
Pressure	$1 \text{ lb/in}^2 = 6.895 \times 10^3 \text{ Pa}$

Table 9. Selected British Units

Length	$1 \text{ light year (ly)} = 9.46 \times 10^{15} \text{ m}$
	$1 \text{ astronomical unit (au)} = 1.50 \times 10^{11} \text{ m}$
	$1 \text{ nautical mile} = 1.852 \text{ km}$
	Formula does not parse
Area	$1 \text{ acre (ac)} = 4.05 \times 10^{3} \text{m}^2$
	$1 \text{ square foot (ft}^2\text{)} = 9.29 \times 10^{-2} \text{ m}^2$
	$1 \text{ barn } (b) = 10^{-28} \text{ m}^2$
Volume	$1 \text{ liter } (L) = 10^{-3} \text{ m}^3$
	$1 \text{ U.S. gallon (gal)} = 3.785 \times 10^{-3} \text{ m}^3$
Mass	$1 \text{ solar mass} = 1.99 \times 10^{30} \text{ kg}$
	$1 \text{ metric ton} = 10^{3} \text{ kg}$
	$1 \text{ atomic mass unit } (u) = 1.6605 \times 10^{-27} \text{ kg}$
Time	$1 \text{ year (y)} = 3.16 \times 10^{7} \text{ s}$
	$1 \text{ day (d)} = 86,400 \text{ s}$
Speed	$1 \text{ mile per hour (mph)} = 1.609 \text{ km/h}$
	$1 \text{ nautical mile per hour (naut)} = 1.852 \text{ km/h}$
Angle	$1 \text{ degree } (^\circ) = 1.745 \times 10^{-2} \text{ rad}$
	$1 \text{ minute of arc } (‘) = 1/60 \text{ degree}$
	$1 \text{ second of arc } (”) = 1/60 \text{ minute of arc}$
	$1 \text{ grad} = 1.571 \times 10^{-2} \text{ rad}$
Energy	$1 \text{ kiloton TNT (kT)} = 4.2 \times 10^{12} \text{ J}$
	$1 \text{ kilowatt hour (kW} \cdot h) = 3.60 \times 10^{6} \text{ J}$
	$1 \text{ food calorie (kcal)} = 4186 \text{ J}$
	$1 \text{ calorie (cal)} = 4.186 \text{ J}$

	$1 \text{ electron volt (eV)} = 1.60 \times 10^{-19} \text{ J}$
Pressure	$1 \text{ atmosphere (atm)} = 1.013 \times 10^{5} \text{ Pa}$
	$1 \text{ millimeter of mercury (mm Hg)} = 133.3 \text{ Pa}$
	$1 \text{ torricelli (torr)} = 1 \text{ mm Hg} = 133.3 \text{ Pa}$
Nuclear decay rate	$1 \text{ curie (Ci)} = 3.70 \times 10^{10} \text{ Bq}$

Table 10. Other Units

Circumference of a circle with radius r or diameter d	$C = 2\pi r = \pi d$
Area of a circle with radius r or diameter$d$	$A = \pi r^2 = \pi d^2/4$
Area of a sphere with radius r	$A = 4\pi r^2$
Volume of a sphere with radius r	$V = (4/3)(\pi r^3)$

Table 11. Useful Formulae

Units of Length

meter (m)	= 39.37 inches (in.) = 1.094 yards (yd)	angstrom (Å)	= 10^{-8} cm (exact, definition) = 10^{-10} m (exact, definition)
centimeter (cm)	= 0.01 m (exact, definition)	yard (yd)	= 0.9144 m
millimeter (mm)	= 0.001 m (exact, definition)	inch (in.)	= 2.54 cm (exact, definition)
kilometer (km)	= 1000 m (exact, definition)	mile (US)	= 1.60934 km

Units of Volume

liter (L)	= 0.001 m^3 (exact, definition) = 1000 cm^3 (exact, definition) = 1.057 (US) quarts
milliliter (mL)	= 0.001 L (exact, definition) = 1 cm^3 (exact, definition)
microliter (μL)(μL)	= 10^{-6} L (exact, definition) = 10^{-3} cm^3 (exact, definition)
liquid quart (US)	= 32 (US) liquid ounces (exact, definition) = 0.25 (US) gallon (exact, definition) = 0.9463 L
dry quart	= 1.1012 L
cubic foot (US)	= 28.316 L

Units of Mass

gram (g)	= 0.001 kg (exact, definition)	ounce (oz) (avoirdupois)	= 28.35 g
milligram (mg)	= 0.001 g (exact, definition)	pound (lb) (avoirdupois)	= 0.4535924 kg
kilogram (kg)	= 1000 g (exact, definition) = 2.205 lb	ton (short)	=2000 lb (exact, definition) = 907.185 kg
ton (metric)	=1000 kg (exact, definition) = 2204.62 lb	ton (long)	= 2240 lb (exact, definition) = 1.016 metric ton

Units of Energy

4.184 joule (J)	= 1 thermochemical calorie (cal)
1 thermochemical calorie (cal)	= 4.184×10^7 erg
erg	= 10^{-7} J (exact, definition)
electron-volt (eV)	= 1.60218×10^{-19} J = 23.061 kcal mol^{-1}
liter·atmosphere	= 24.217 cal = 101.325 J (exact, definition)
nutritional calorie (Cal)	= 1000 cal (exact, definition) = 4184 J
British thermal unit (BTU)	= 1054.804 J BTU is the amount of energy needed to heat one pound of water by one degree Fahrenheit. Therefore, the exact relationship of BTU to joules and other energy units depends on the temperature at which BTU is measured. 59 °F (15 °C) is the most widely used reference temperature for BTU definition in the United States. At this temperature, the conversion factor is the one provided in this table.

Footnotes

1. 1 Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, www.physics.nist.gov/cuu (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.
2. 2 Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, www.physics.nist.gov/cuu (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.

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Exponential Arithmetic

Exponential notation is used to express very large and very small numbers as a product of two numbers. The first number of the product, the *digit term*, is usually a number not less than 1 and not greater than 10. The second number of the product, the *exponential term*, is written as 10 with an exponent. Some examples of exponential notation are:

$$1000 \quad 1 \times 10^{3}$$

$$100 \quad 1 \times 10^{2}$$

$$10 \quad 1 \times 10^{1}$$

$$1 \quad 1 \times 10^{0}$$

$$0.1 \quad 1 \times 10^{-1}$$

$$0.001 \quad 1 \times 10^{-3}$$

$$2386 \quad 2.386 \times 1000 = 2.386 \times 10^{3}$$

$$0.123 \quad 1.23 \times 0.1 = 1.23 \times 10^{-1}$$

The power (exponent) of 10 is equal to the number of places the decimal is shifted to give the digit number. The exponential method is particularly useful notation for every large and very small numbers. For example, 1,230,000,000 = 1.23×10^{9}, and 0.00000000036 = 3.6×10^{-10}.

Addition of Exponentials

Convert all numbers to the same power of 10, add the digit terms of the numbers, and if appropriate, convert the digit term back to a number between 1 and 10 by adjusting the exponential term.

Example 1 **Adding Exponentials**
Add 5.00×10^{-5} and 3.00×10^{-3}.

Solution

$$3.00 \times 10^{-3} = 300 \times 10^{-5}$$
$$(5.00 \times 10^{-5}) + (300 \times 10^{-5}) = 305 \times 10^{-5} = 3.05 \times 10^{-3}$$

Subtraction of Exponentials

Convert all numbers to the same power of 10, take the difference of the digit terms, and if appropriate, convert the digit term back to a number between 1 and 10 by adjusting the exponential term.

Example 2 **Subtracting Exponentials**
Subtract 4.0×10^{-7} from 5.0×10^{-6}.

Solution

$$4.0 \times 10^{-7} = 0.40 \times 10^{-6}$$
$$(5.0 \times 10^{-6}) - (0.40 \times 10^{-6}) = 4.6 \times 10^{-6}$$

Multiplication of Exponentials

Multiply the digit terms in the usual way and add the exponents of the exponential terms.

Example 3 **Multiplying Exponentials**
Multiply 4.2×10^{-8} by 2.0×10^{3}.

Solution

$$(4.2 \times 10^{-8}) \times (2.0 \times 10^{3}) = (4.2 \times 2.0) \times 10^{(-8)+(+3)} = 8.4 \times 10^{-5}$$

Division of Exponentials

Divide the digit term of the numerator by the digit term of the denominator and subtract the exponents of the exponential terms.

Example 4 **Dividing Exponentials**
Divide 3.6×10^{5} by 6.0×10^{-4}.

Solution

$$\frac{3.6 \times 10^{-5}}{6.0 \times 10^{-4}} = \left(\frac{3.6}{6.0}\right) \times 10^{(-5)-(-4)} = 0.60 \times 10^{-1} = 6.0 \times 10^{-2}$$

Squaring of Exponentials

Square the digit term in the usual way and multiply the exponent of the exponential term by 2.

Example 5 **Squaring Exponentials**
Square the number 4.0×10^{-6}.

Solution

$$(4.0 \times 10^{-6})^{2} = 4 \times 4 \times 10^{2 \times (-6)} = 16 \times 10^{-12} = 1.6 \times 10^{-11}$$

Cubing of Exponentials

Cube the digit term in the usual way and multiply the exponent of the exponential term by 3.

Example 6 **Cubing Exponentials**
Cube the number 2×10^{4}.

Solution

$$(2 \times 10^{4})^{3} = 2 \times 2 \times 2 \times 10^{3 \times 4} = 8 \times 10^{12}$$

Taking Square Roots of Exponentials

If necessary, decrease or increase the exponential term so that the power of 10 is evenly divisible by 2. Extract the square root of the digit term and divide the exponential term by 2.

Example 7 **Finding the Square Root of Exponentials**
Find the square root of 1.6×10^{-7}.

Solution

$$1.6 \times 10^{-7} = 16 \times 10^{-8}$$

$$\sqrt{16 \times 10^{-8}} = \sqrt{16} \times \sqrt{10^{-8}} = \sqrt{16} \times \sqrt{10}^{-\frac{8}{2}} = 4.0 \times 10^{-4}$$

Significant Figures

A beekeeper reports that he has 525,341 bees. The last three figures of the number are obviously inaccurate, for during the time the keeper was counting the bees, some of them died and others hatched; this makes it quite difficult to determine the exact number of bees. It would have been more accurate if the beekeeper had reported the number 525,000. In other words, the last three figures are not significant, except to set the position of the decimal point. Their exact values have no meaning useful in this situation. In reporting any information as numbers, use only as many significant figures as the accuracy of the measurement warrants.

The importance of significant figures lies in their application to fundamental computation. In addition and subtraction, the sum or difference should contain as many digits to the right of the decimal as that in the least certain of the numbers used in the computation (indicated by underscoring in the following example).

Example 8 **Addition and Subtraction with Significant Figures**
Add 4.383 g and 0.0023 g.

Solution

$$\begin{array}{r} 4.38\underline{3}\ \text{g} \\ 0.002\underline{3}\ \text{g} \\ \hline 4.38\underline{5}\ \text{g} \end{array}$$

In multiplication and division, the product or quotient should contain no more digits than that in the factor containing the least number of significant figures.

Example 9 **Multiplication and Division with Significant Figures**
Multiply 0.6238 by 6.6.

Solution

$$0.6238 \times 6.6 = 4.1$$

When rounding numbers, increase the retained digit by 1 if it is followed by a number larger than 5 (“round up”). Do not change the retained digit if the digits that follow are less than 5 (“round down”). If the retained digit is followed by 5, round up if the retained digit is odd, or round down if it is even (after rounding, the retained digit will thus always be even).

The Use of Logarithms and Exponential Numbers

The common logarithm of a number (log) is the power to which 10 must be raised to equal that number. For example, the common logarithm of 100 is 2, because 10 must be raised to the second power to equal 100. Additional examples follow.

Number	Number Expressed Exponentially	Common Logarithm
1000	10^3	3
10	10^1	1
1	10^0	0
0.1	10^{-1}	−1
0.001	10^{-3}	−3

Table 1. Logarithms and Exponential Numbers

What is the common logarithm of 60? Because 60 lies between 10 and 100, which have logarithms of 1 and 2, respectively, the logarithm of 60 is 1.7782; that is,

$$60 = 10^{1.7782}$$

The common logarithm of a number less than 1 has a negative value. The logarithm of 0.03918 is −1.4069, or

$$0.03918 = 10^{-1.4069} = \frac{1}{10^{1.4069}}$$

To obtain the common logarithm of a number, use the *log* button on your calculator. To calculate a number from its logarithm, take the inverse log of the logarithm, or calculate 10^x (where x is the logarithm of the number).

The natural logarithm of a number (ln) is the power to which e must be raised to equal the number; e is the constant 2.7182818. For example, the natural logarithm of 10 is 2.303; that is,

$$10 = e^{2.303} = 2.7182818^{2.303}$$

To obtain the natural logarithm of a number, use the *ln* button on your calculator. To calculate a number from its natural logarithm, enter the natural logarithm and take the inverse ln of the natural logarithm, or calculate e^x (where x is the natural logarithm of the number).

Logarithms are exponents; thus, operations involving logarithms follow the same rules as operations involving exponents.

1. The logarithm of a product of two numbers is the sum of the logarithms of the two numbers.
$$\log xy = \log x \ + \ \log y, \;\text{and}\; \ln xy = \ln x \ + \ \ln y$$
2. The logarithm of the number resulting from the division of two numbers is the difference between the logarithms of the two numbers.
$$\log \frac{x}{y} = \log x \ - \ \log y, \;\text{and}\; \ln \frac{x}{y} = \ln x \ - \ \ln y$$
3. The logarithm of a number raised to an exponent is the product of the exponent and the logarithm of the number.
$$\log x^n = n\log x \text{ and } \ln x^n = n\ln x$$

The Solution of Quadratic Equations

Mathematical functions of this form are known as second-order polynomials or, more commonly, quadratic functions.

$$ax^2 \ + \ bx \ + \ c = 0$$

The solution or roots for any quadratic equation can be calculated using the following formula:

$$x = \frac{-b \pm \sqrt{b^2 \ - \ 4ac}}{2a}$$

Example 10 **Solving Quadratic Equations**
Solve the quadratic equation $3x^2 + 13x - 10 = 0$.

Solution Substituting the values $a = 3$, $b = 13$, $c = -10$ in the formula, we obtain

$$x = \frac{-13 \pm \sqrt{(13)^2 - 4 \times 3 \times (-10)}}{2 \times 3}$$

$$x = \frac{-13 \pm \sqrt{169 + 120}}{6} = \frac{-13 \pm \sqrt{289}}{6} = \frac{-13 \pm 17}{6}$$

The two roots are therefore

$$x = \frac{-13 + 17}{6} = \frac{2}{3} \text{ and } x = \frac{-13 - 17}{6} = -5$$

Quadratic equations constructed on physical data always have real roots, and of these real roots, often only those having positive values are of any significance.

Two-Dimensional (*x*–*y*) Graphing

The relationship between any two properties of a system can be represented graphically by a two-dimensional data plot. Such a graph has two axes: a horizontal one corresponding to the independent variable, or the variable whose value is being controlled (*x*), and a vertical axis corresponding to the dependent variable, or the variable whose value is being observed or measured (*y*).

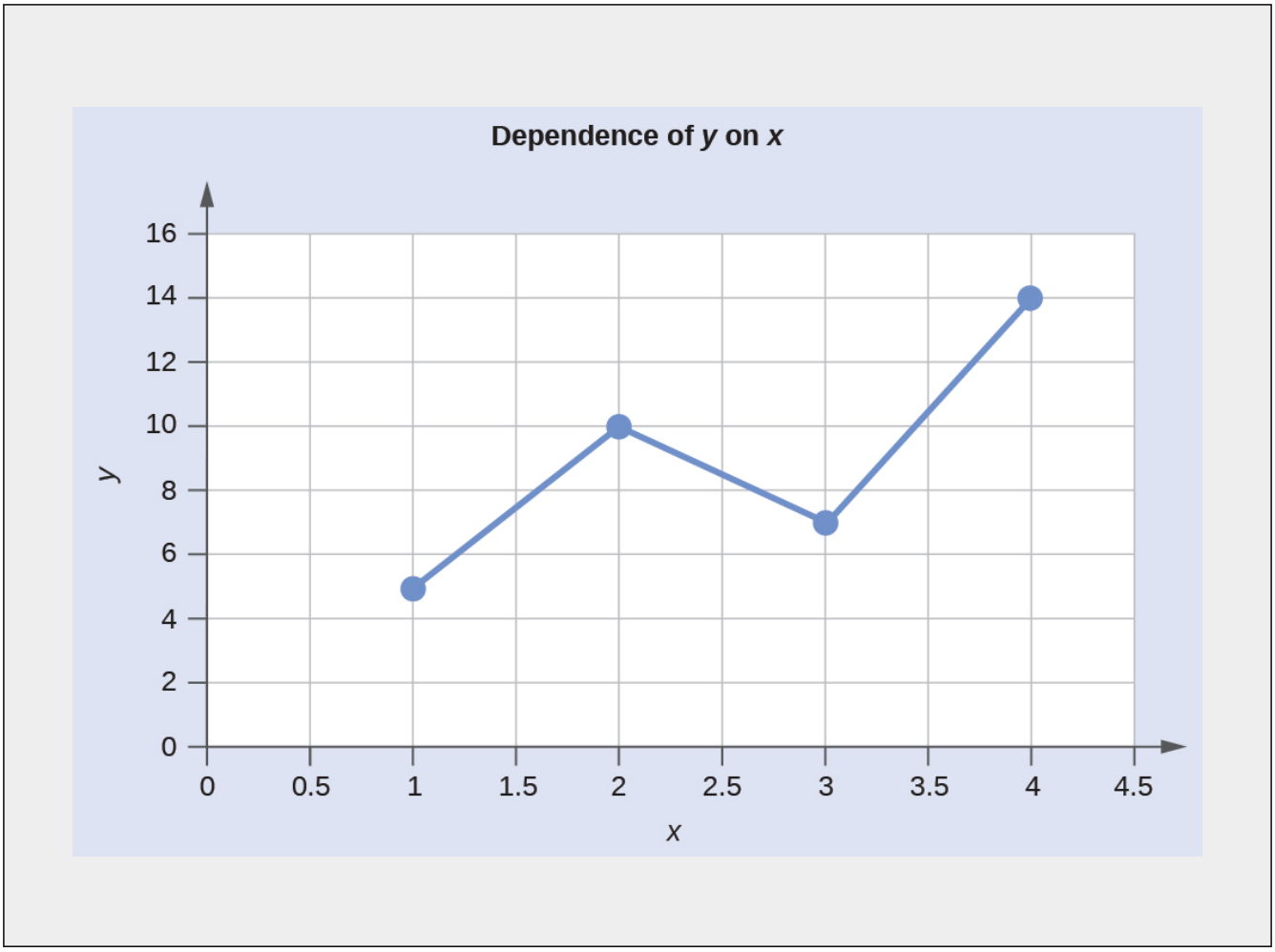
When the value of *y* is changing as a function of *x* (that is, different values of *x* correspond to different values of *y*), a graph of this change can be plotted or sketched. The graph can be produced by using specific values for (*x*,*y*) data pairs.

Example 11 **Graphing the Dependence of *y* on *x***

x	***y***
1	5
2	10
3	7
4	14

Table 2.

This table contains the following points: (1,5), (2,10), (3,7), and (4,14). Each of these points can be plotted on a graph and connected to produce a graphical representation of the dependence of *y* on *x*.


Dependence of y on x
16
14
12
10
8
6
4
2
0
y
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
x

Appendix C Units, Numbers, and Significant Figures

This content is originally from OpenStax College Chemistry 1st Canadian Edition.

This appendix is broken into several sections:

- Introduction to Measurement
- Expressing Numbers
- Significant Figures
- Converting Units
- Other Units: Temperature and Density
- Expressing Units
- Additional Exercises

Introduction to Measurement

Data suggest that a male child will weigh 50% of his adult weight at about 11 years of age. However, he will reach 50% of his adult height at only 2 years of age. It is obvious, then, that people eventually stop growing up but continue to grow out. Data also suggest that the average human height has been increasing over time. In industrialized countries, the average height of people increased 5.5 inches from 1810 to 1984. Most scientists attribute this simple, basic measurement of the human body to better health and nutrition.

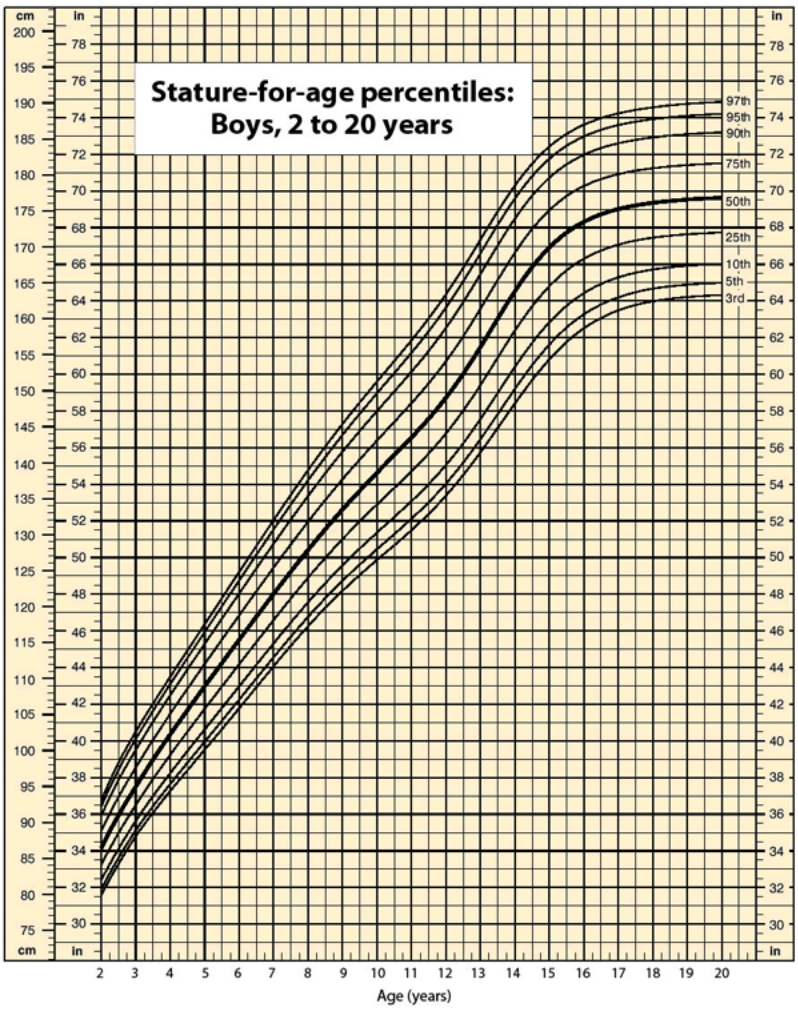


Figure 1. *Stature-for-age percentiles: Boys, 2 to 20 years. Source: Chart courtesy of Centers for Disease Control and Prevention, http://www.cdc.gov/nchs/nhanes.htm#Set%201.*

In 1983, an Air Canada airplane had to make an emergency landing because it unexpectedly ran out of fuel; ground personnel had filled the fuel tanks with a certain number of pounds of fuel, not kilograms of fuel. In 1999, the Mars Climate Orbiter spacecraft was lost attempting to orbit Mars because the thrusters were programmed in terms of English units, even though the engineers built the spacecraft using metric units. In 1993, a nurse mistakenly administered 23 units of morphine to a patient rather than the "2–3" units prescribed. (The patient ultimately survived.) These incidents occurred because people weren't paying attention to quantities.

Physics and chemistry, like all sciences, are quantitative. they deals with *quantities*, things that have amounts and units. Dealing with quantities is very important in chemistry and physics, as is relating quantities to each other. In this chapter, we will discuss how we deal with numbers and units, including how they are combined and manipulated.

Expressing Numbers

Quantities have two parts: the number and the unit. The number tells "how many." It is important to be able to express numbers properly so that the quantities can be communicated properly.

Standard notation is the straightforward expression of a number. Numbers such as 17, 101.5, and 0.00446 are expressed in standard notation. For relatively small numbers, standard notation is fine. However, for very large numbers, such as 306,000,000, or for very small numbers, such as 0.000000419, standard notation can be cumbersome because of the number of zeros needed to place nonzero numbers in the proper position.

Scientific notation is an expression of a number using powers of 10. Powers of 10 are used to express numbers that have many zeros:

$10^0 = 1$

$10^1 = 10$

$10^2 = 100 = 10 \times 10$

$10^3 = 1{,}000 = 10 \times 10 \times 10$

$10^4 = 10{,}000 = 10 \times 10 \times 10 \times 10$

and so forth. The raised number to the right of the 10 indicating the number of factors of 10 in the original number is the exponent. (Scientific notation is sometimes called *exponential notation*.) The exponent's value is equal to the number of zeros in the number expressed in standard notation.

Small numbers can also be expressed in scientific notation but with negative exponents:

$10^{-1} = 0.1 = 1/10$

$10^{-2} = 0.01 = 1/100$

$10^{-3} = 0.001 = 1/1{,}000$

$10^{-4} = 0.0001 = 1/10{,}000$

and so forth. Again, the value of the exponent is equal to the number of zeros in the denominator of the associated fraction. A negative exponent implies a decimal number less than one.

A number is expressed in scientific notation by writing the first nonzero digit, then a decimal point, and then the rest of the digits. The part of a number in scientific notation that is multiplied by a power of 10 is called the coefficient. Then determine the power of 10 needed to make that number into the original number and multiply the written number by the proper power of 10. For example, to write 79,345 in scientific notation,

$79{,}345 = 7.9345 \times 10{,}000 = 7.9345 \times 10^4$

Thus, the number in scientific notation is 7.9345×10^4. For small numbers, the same process is used, but the exponent for the power of 10 is negative:

$0.000411 = 4.11 \times 1/10{,}000 = 4.11 \times 10^{-4}$

Typically, the extra zero digits at the end or the beginning of a number are not included.

Example 1

Express these numbers in scientific notation.

1. 306,000
2. 0.00884
3. 2,760,000
4. 0.000000559

Solution

1. The number 306,000 is 3.06 times 100,000, or 3.06 times 10^5. In scientific notation, the number is 3.06×10^5.
2. The number 0.00884 is 8.84 times 1/1,000, which is 8.84 times 10^{-3}. In scientific notation, the number is 8.84×10^{-3}.
3. The number 2,760,000 is 2.76 times 1,000,000, which is the same as 2.76 times 10^6. In scientific notation, the number is written as 2.76×10^6. Note that we omit the zeros at the end of the original number.
4. The number 0.000000559 is 5.59 times 1/10,000,000, which is 5.59 times 10^{-7}. In scientific notation, the number is written as 5.59×10^{-7}.

Test Yourself

Express these numbers in scientific notation.

1. 23,070
2. 0.0009706

Answers

1. 2.307×10^4
2. 9.706×10^{-4}

Another way to determine the power of 10 in scientific notation is to count the number of places you need to move the decimal point to get a numerical value between 1 and 10. The number of places equals the power of 10. This number is positive if you move the decimal point to the right and negative if you move the decimal point to the left.

Many quantities in chemistry are expressed in scientific notation. When performing calculations, you may have to enter a number in scientific notation into a calculator. Be sure you know how to correctly enter a number in scientific notation into your calculator. Different models of calculators require different actions for properly entering scientific notation. If in doubt, consult your instructor immediately.

Key Takeaways

- Standard notation expresses a number normally.
- Scientific notation expresses a number as a coefficient times a power of 10.
- The power of 10 is positive for numbers greater than 1 and negative for numbers between 0 and 1.

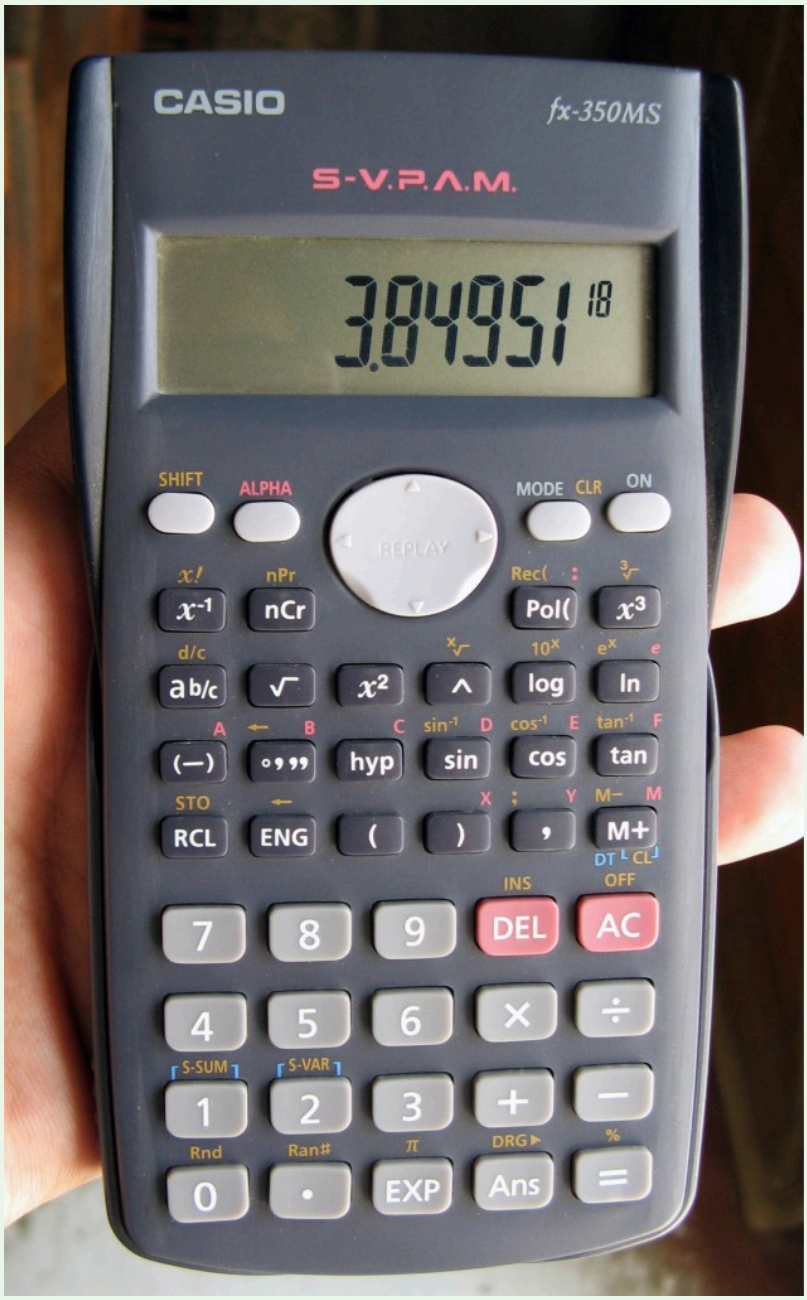


Figure 1. *This calculator shows only the coefficient and the power of 10 to represent the number in scientific notation. Thus, the number being displayed is 3.84951 × 10^{18}, or 3,849,510,000,000,000,000.*
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Problems & Exercises

1. Express these numbers in scientific notation.

(a) 56.9 (b) 563,100 (c) 0.0804 (d) 0.00000667

2. Express these numbers in scientific notation.

(a) −890,000 (b) 602,000,000,000 (c) 0.0000004099 (d) 0.000000000000011

3. Express these numbers in scientific notation.

(a) 0.00656 (b) 65,600 (c) 4,567,000 (d) 0.000005507

4. Express these numbers in scientific notation.

(a) 65 (b) −321.09 (c) 0.000077099 (d) 0.000000000218

5. Express these numbers in standard notation.

(a) 1.381×10^{5} (b) 5.22×10^{-7} (c) 9.998×10^{4}

6. Express these numbers in standard notation.

(a) 7.11×10^{-2} (b) 9.18×10^{2} (c) 3.09×10^{-10}

7. Express these numbers in standard notation.

(a) 8.09×10^{0} (b) 3.088×10^{-5} (c) -4.239×10^{2}

8. Express these numbers in standard notation.

(a) 2.87×10^{-8} (b) 1.78×10^{11} (c) 1.381×10^{-23}

9. These numbers are not written in proper scientific notation. Rewrite them so that they are in proper scientific notation.

(a) 72.44×10^{3} (b) $9{,}943 \times 10^{-5}$ (c) $588{,}399 \times 10^{2}$

10. These numbers are not written in proper scientific notation. Rewrite them so that they are in proper scientific notation.

(a) 0.000077×10^{-7} (b) 0.000111×10^{8} (c) $602{,}000 \times 10^{18}$

11. These numbers are not written in proper scientific notation. Rewrite them so that they are in proper scientific notation.

(a) 345.1×10^{2} (b) 0.234×10^{-3} (c) $1{,}800 \times 10^{-2}$

12. These numbers are not written in proper scientific notation. Rewrite them so that they are in proper scientific notation.

(a) $8{,}099 \times 10^{-8}$ (b) 34.5×10^{0} (c) 0.000332×10^{4}

13. Write these numbers in scientific notation by counting the number of places the decimal point is moved.

(a) 123,456.78 (b) 98,490 (c) 0.000000445

14. Write these numbers in scientific notation by counting the number of places the decimal point is moved.

(a) 0.000552 (b) 1,987 (c) 0.00000000887

15. Use your calculator to evaluate these expressions. Express the final answer in proper scientific notation.
(a) $456 \times (7.4 \times 10^{8}) = ?$ (b) $(3.02 \times 10^{5}) \div (9.04 \times 10^{15}) = ?$ (c) $0.0044 \times 0.000833 = ?$

16. Use your calculator to evaluate these expressions. Express the final answer in proper scientific notation.
(a) $98{,}000 \times 23{,}000 = ?$ (b) $98{,}000 \div 23{,}000 = ?$ (c) $(4.6 \times 10^{-5}) \times (2.09 \times 10^{3}) = ?$

17. Use your calculator to evaluate these expressions. Express the final answer in proper scientific notation.
(a) $45 \times 132 \div 882 = ?$ (b) $[(6.37 \times 10^{4}) \times (8.44 \times 10^{-4})] \div (3.2209 \times 10^{15}) = ?$

18. Use your calculator to evaluate these expressions. Express the final answer in proper scientific notation.
(a) $(9.09 \times 10^{8}) \div [(6.33 \times 10^{9}) \times (4.066 \times 10^{-7})] = ?$ (b) $9{,}345 \times 34.866 \div 0.00665 = ?$

Solutions

Problems & Exercises

1. a) 5.69×10^{1} b) 5.631×10^{5} c) 8.04×10^{-2} d) 6.67×10^{-6}

3. a) 6.56×10^{-3} b) 6.56×10^{4} c) 4.567×10^{6} d) 5.507×10^{-6}

5. a) 138,100 b) 0.000000522 c) 99,980

7. a) 8.09 b) 0.00003088 c) −423.9

9. a) 7.244×10^{4} b) 9.943×10^{-2} c) 5.88399×10^{7}

11. a) 3.451×10^{4} b) 2.34×10^{-4} c) 1.8×10^{1}

13. a) 1.2345678×10^{5} b) 9.849×10^{4} c) 4.45×10^{-7}

15. a) 3.3744×10^{11} b) 3.3407×10^{-11} c) 3.665×10^{-6}

17. a) 6.7346×10^{0} b) 1.6691×10^{-14}

Significant Figures

Learning Objectives

1. Apply the concept of significant figures to limit a measurement to the proper number of digits.
2. Recognize the number of significant figures in a given quantity.
3. Limit mathematical results to the proper number of significant figures.

If you use a calculator to evaluate the expression 337/217, you will get the following:

337217=1.55299539171…

and so on for many more digits. Although this answer is correct, it is somewhat presumptuous. You start with two values that each have three digits, and the answer has *twelve* digits? That does not make much sense from a strict numerical point of view.

Consider using a ruler to measure the width of an object, as shown in Figure 2.6 “Expressing Width”. The object is definitely more than 1 cm long, so we know that the first digit in our measurement is 1. We see by counting the tick marks on the ruler that the object is at least three ticks after the 1. If each tick represents 0.1 cm, then we know the object is at least 1.3 cm wide. But our ruler does not have any more ticks between the 0.3 and the 0.4 marks, so we can’t know exactly how much the next decimal place is. But with a practiced eye we can estimate it. Let us estimate it as about six-tenths of the way between the third and fourth tick marks, which estimates our hundredths place as 6, so we identify a measurement of 1.36 cm for the width of the object.

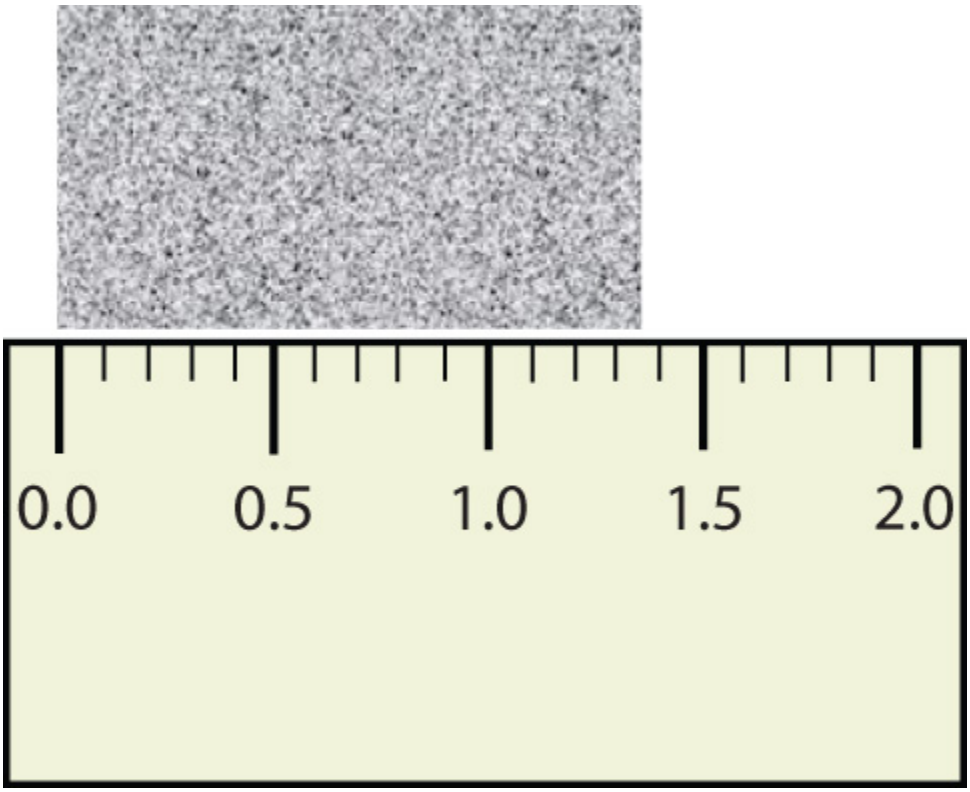


Figure 1. *Expressing Width*

What is the proper way to express the width of this object?

Does it make any sense to try to report a thousandths place for the measurement? No, it doesn’t; we are not exactly sure of the hundredths place (after all, it was an estimate only), so it would be fruitless to estimate a thousandths place. Our best measurement, then, stops at the hundredths place, and we report 1.36 cm as proper measurement.

This concept of reporting the proper number of digits in a measurement or a calculation is called significant figures. Significant figures (sometimes called significant digits) represent the limits of what values of a measurement or a calculation we are sure of. The convention for a measurement is that

the quantity reported should be all known values and the first estimated value. The conventions for calculations are discussed as follows.

Example 3

Use each diagram to report a measurement to the proper number of significant figures.

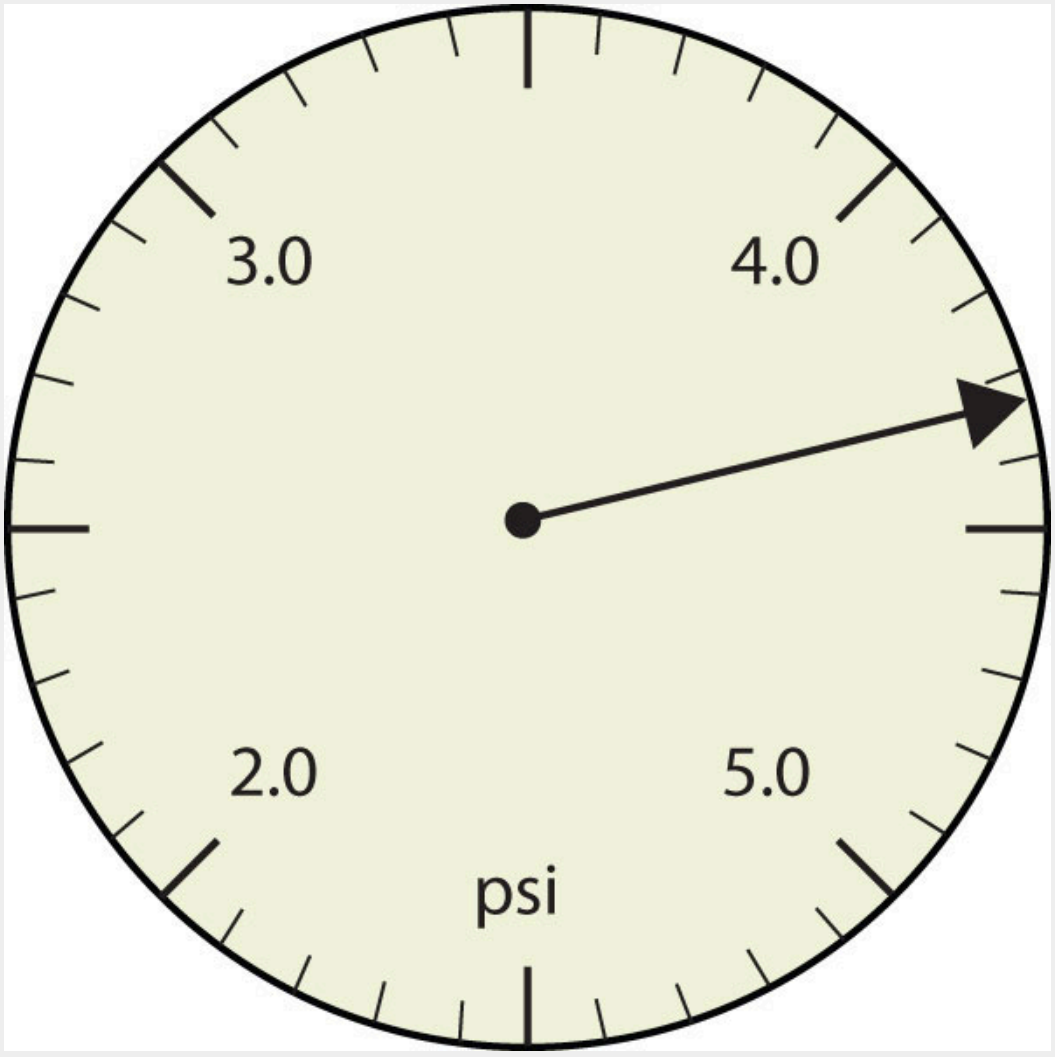
1. 

Figure 2. *Pressure gauge in units of pounds per square inch*

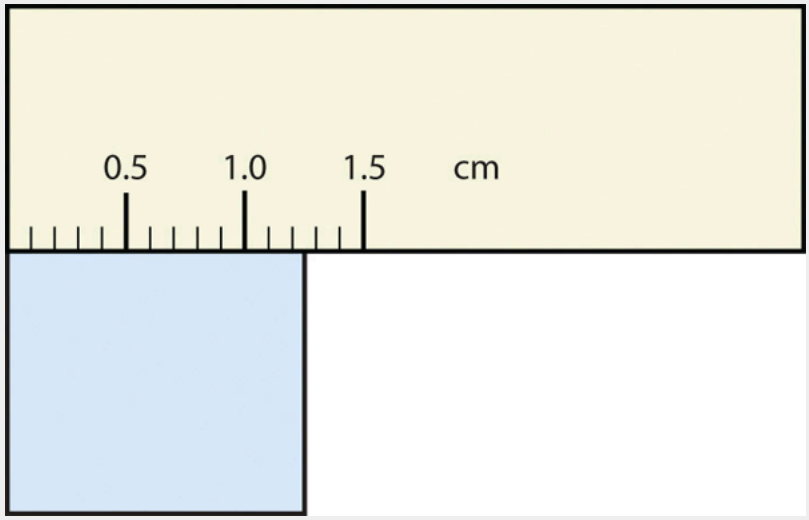
2. 

Figure 3. *A measuring ruler*

Solution

1. The arrow is between 4.0 and 5.0, so the measurement is at least 4.0. The arrow is between the third and fourth small tick marks, so it's at least 0.3. We will have to estimate the last place. It looks like about one-third of the way across the space, so let us estimate the hundredths place as 3. Combining the digits, we have a measurement of 4.33 psi (psi stands for "pounds per square inch" and is a unit of pressure, like air in a tire). We say that the measurement is reported to three significant figures.
2. The rectangle is at least 1.0 cm wide but certainly not 2.0 cm wide, so the first significant digit is 1. The rectangle's width is past the second tick mark but not the third; if each tick mark represents 0.1, then the rectangle is at least 0.2 in the next significant digit. We have to estimate the next place because there are no markings to guide us. It appears to be about halfway between 0.2 and 0.3, so we will estimate the next place to be a 5. Thus, the measured width of the rectangle is 1.25 cm. Again, the measurement is reported to three significant figures.

Check Your Understanding 1

What would be the reported width of this rectangle?

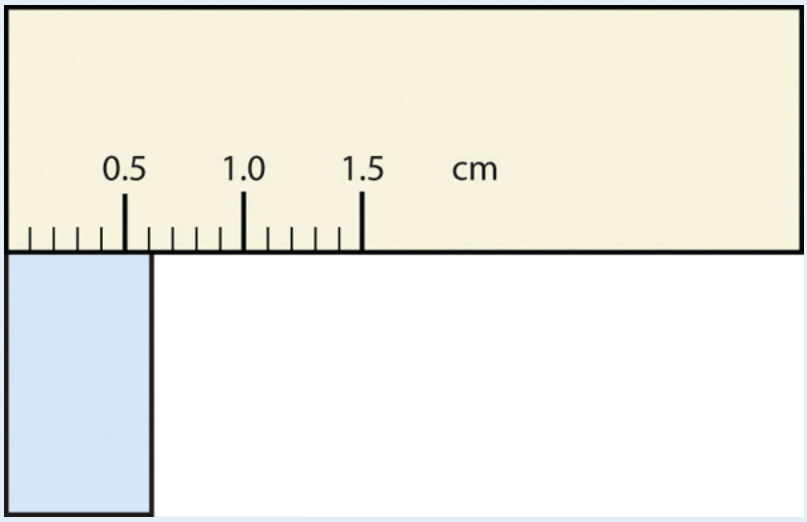


Figure 4. *A measuring ruler*

In many cases, you will be given a measurement. How can you tell by looking what digits are significant? For example, the reported population of the United States is 306,000,000. Does that mean that it is *exactly* three hundred six million or is some estimation occurring?

The following conventions dictate which numbers in a reported measurement are significant and which are not significant:

1. Any nonzero digit is significant.
2. Any zeros between nonzero digits (i.e., embedded zeros) are significant.
3. Zeros at the end of a number without a decimal point (i.e., trailing zeros) are not significant; they serve only to put the significant digits in the correct positions. However, zeros at the end of any number with a decimal point are significant.
4. Zeros at the beginning of a decimal number (i.e., leading zeros) are not significant; again, they serve only to put the significant digits in the correct positions.

So, by these rules, the population figure of the United States has only three significant figures: the 3, the 6, and the zero between them. The remaining six zeros simply put the 306 in the millions position.

Example 4

Give the number of significant figures in each measurement.

1. 36.7 m
2. 0.006606 s
3. 2,002 kg
4. 306,490,000 people

Solution

1. By rule 1, all nonzero digits are significant, so this measurement has three significant figures.
2. By rule 4, the first three zeros are not significant, but by rule 2 the zero between the sixes is; therefore, this number has four significant figures.
3. By rule 2, the two zeros between the twos are significant, so this measurement has four significant figures.
4. The four trailing zeros in the number are not significant, but the other five numbers are, so this number has five significant figures.

Check Your Understanding 2

Give the number of significant figures in each measurement.

1. 0.000601 m
2. 65.080 kg

How are significant figures handled in calculations? It depends on what type of calculation is being performed. If the calculation is an addition or a subtraction, the rule is as follows: limit the reported answer to the rightmost column that all numbers have significant figures in common. For example, if you were to add 1.2 and 4.71, we note that the first number stops its significant figures in the tenths column, while the second number stops its significant figures in the hundredths column. We therefore limit our answer to the tenths column.

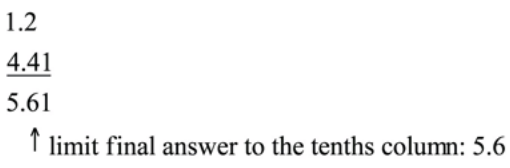


Figure 5. *Math*

We drop the last digit—the 1—because it is not significant to the final answer.

The dropping of positions in sums and differences brings up the topic of rounding. Although there are several conventions, in this text we will adopt the following rule: the final answer should be rounded up if the first dropped digit is 5 or greater and rounded down if the first dropped digit is less than 5.

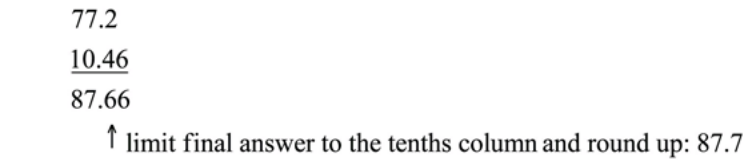


Figure 6. *More Math*

Example 5

Express the final answer to the proper number of significant figures.

1. 101.2 + 18.702 = ?
2. 202.88 − 1.013 = ?

Solution

1. If we use a calculator to add these two numbers, we would get 119.902. However, most calculators do not understand significant figures, and we need to limit the final answer to the tenths place. Thus, we drop the 02 and report a final answer of 119.9 (rounding down).
2. A calculator would answer 201.867. However, we have to limit our final answer to the hundredths place. Because the first number being dropped is 7, which is greater than 7, we round up and report a final answer of 201.87.

Check Your Understanding 3

Express the answer for 3.445 + 90.83 − 72.4 to the proper number of significant figures.

If the operations being performed are multiplication or division, the rule is as follows: limit the answer to the number of significant figures that the data value with the *least* number of significant figures has. So if we are dividing 23 by 448, which have two and three significant figures each, we should limit the final reported answer to two significant figures (the lesser of two and three significant figures):

$$\frac{23}{448} = 0.051339286\ldots = 0.051$$

The same rounding rules apply in multiplication and division as they do in addition and subtraction.

Example 6

Express the final answer to the proper number of significant figures.

1. $76.4 \times 180.4 = ?$
2. $934.9 \div 0.00455 = ?$

Solution

1. The first number has three significant figures, while the second number has four significant figures. Therefore, we limit our final answer to three significant figures: $76.4 \times 180.4 = 13{,}782.56 = 13{,}800$.
2. The first number has four significant figures, while the second number has three significant figures. Therefore we limit our final answer to three significant figures: $934.9 \div 0.00455 = 205{,}472.5275\ldots = 205{,}000$.

Check Your Understanding 4

Express the final answer to the proper number of significant figures.

1. $22.4 \times 8.314 = ?$
2. $1.381 \div 6.02 = ?$

As you have probably realized by now, the biggest issue in determining the number of significant figures in a value is the zero. Is the zero significant or not? One way to unambiguously determine whether a zero is significant or not is to write a number in scientific notation. Scientific notation will include zeros in the coefficient of the number *only if they are significant*. Thus, the number 8.666×10^6 has four significant figures. However, the number 8.6660×10^6 has five significant figures. That last zero is significant; if it were not, it would not be written in the coefficient. So when in doubt about expressing the number of significant figures in a quantity, use scientific notation and include the number of zeros that are truly significant.

Figure 7. *Video source: Significant figures by keyj (https://viutube.viu.ca/public/media/Significant+Figures/0_0j38j93r)*

Key Takeaways

- Significant figures in a quantity indicate the number of known values plus one place that is estimated.
- There are rules for which numbers in a quantity are significant and which are not significant.
- In calculations involving addition and subtraction, limit significant figures based on the rightmost place that all values have in common.
- In calculations involving multiplication and division, limit significant figures to the least number of significant figures in all the data values.

Problems & Exercises

1. Express each measurement to the correct number of significant figures.

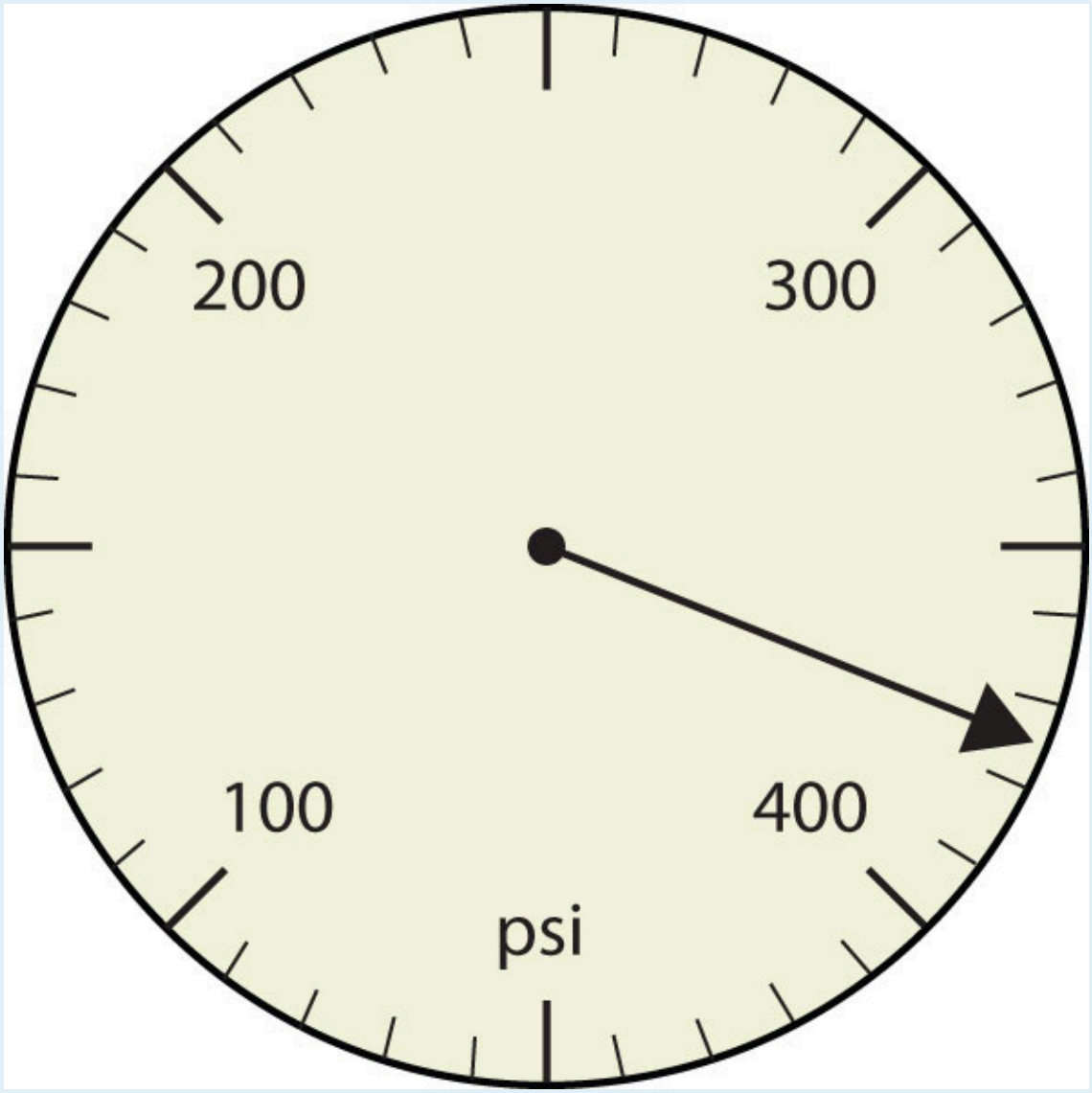


Figure 8. *Pressure gauge in units of pounds per square inch*

a)

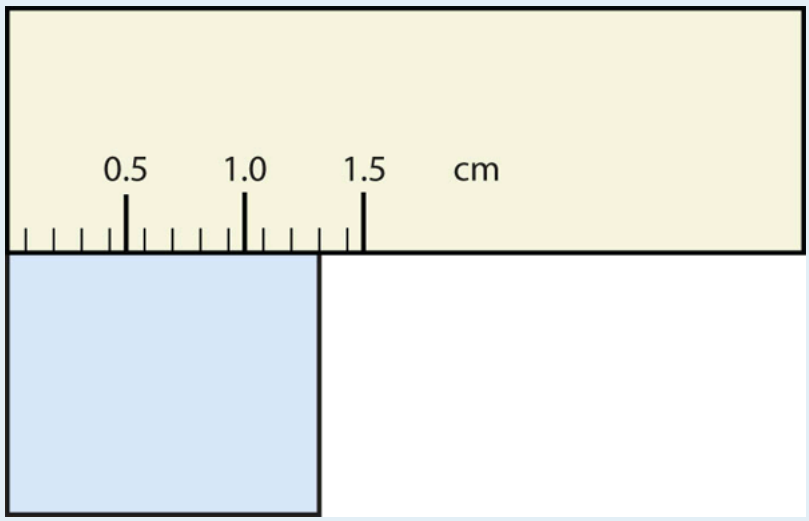


Figure 9. *A measuring ruler*

b)

2. Express each measurement to the correct number of significant figures.

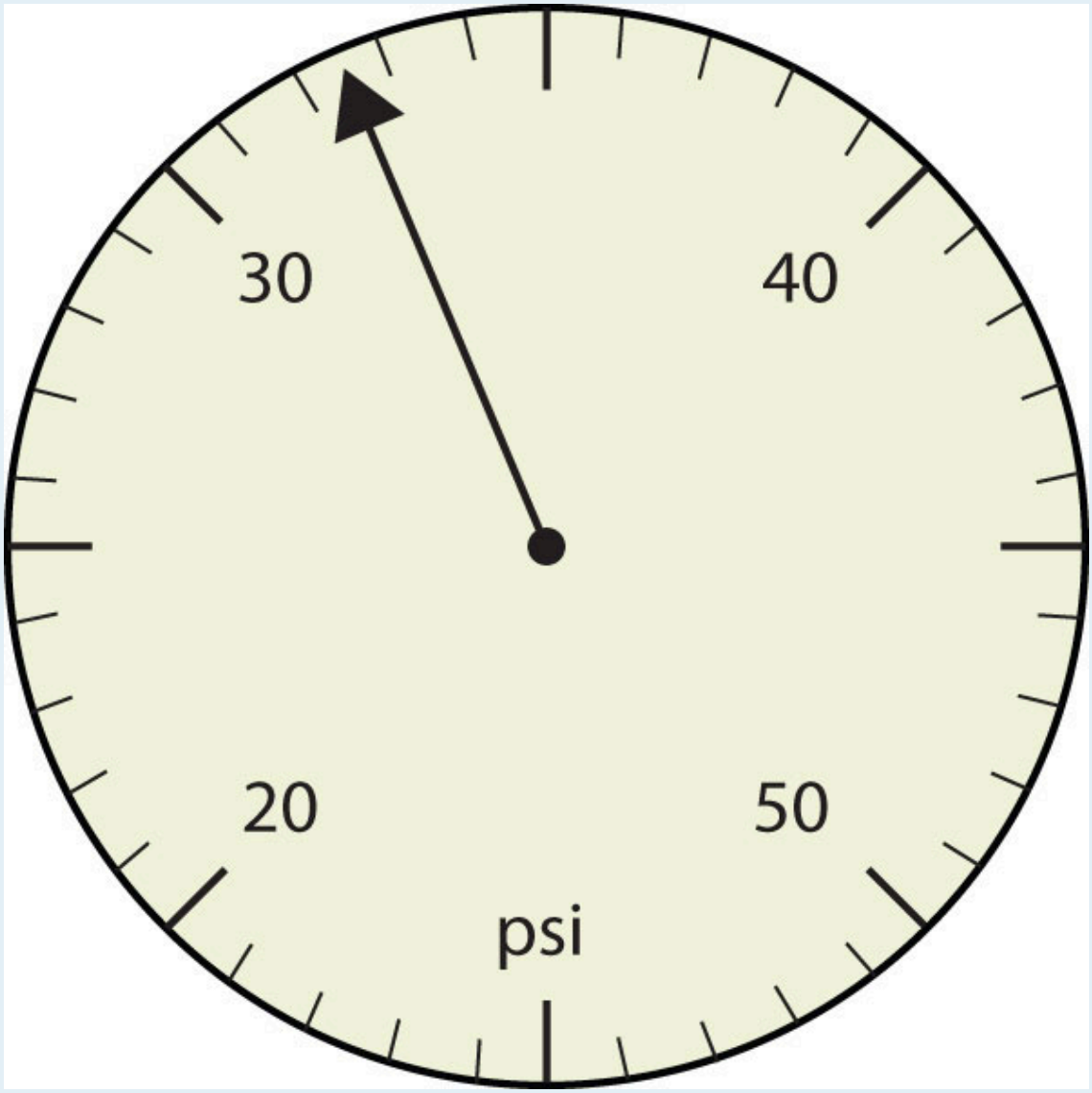


Figure 10. *Pressure gauge in units of pounds per square inch*

a)

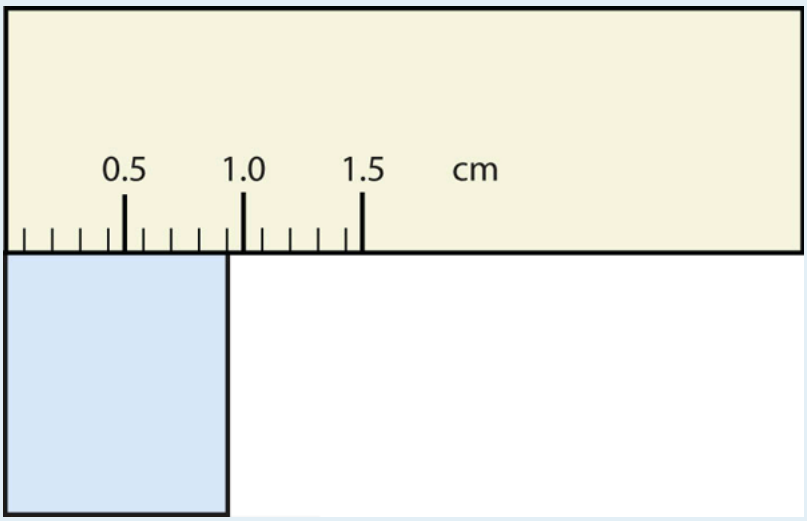


Figure 11. *A measuring ruler*

b)

3. How many significant figures do these numbers have?

(a) 23 (b) 23.0 (c) 0.00023 (d) 0.0002302

4. How many significant figures do these numbers have?

(a) 5.44×10^{8} (b) 1.008×10^{-5} (c) 43.09 (d) 0.0000001381

5. How many significant figures do these numbers have?

(a) 765,890 (b) 765,890.0 (c) 1.2000×10^{5} (d) 0.0005060

6) How many significant figures do these numbers have?

(a) 0.009 (b) 0.0000009 (c) 65,444 (d) 65,040

7. Compute and express each answer with the proper number of significant figures, rounding as necessary.

(a) 56.0 + 3.44 = ? (b) 0.00665 + 1.004 = ? (c) 45.99 − 32.8 = ? (d) 45.99 − 32.8 + 75.02 = ?

8. Compute and express each answer with the proper number of significant figures, rounding as necessary.

(a) 1.005 + 17.88 = ? (b) 56,700 − 324 = ? (c) 405,007 − 123.3 = ? (d) 55.5 + 66.66 − 77.777 = ?

9. Compute and express each answer with the proper number of significant figures, rounding as necessary.

(a) 56.7 × 66.99 = ? (b) 1.000 ÷ 77 = ? (c) 1.000 ÷ 77.0 = ? (d) 6.022 × 1.89 = ?

10. Compute and express each answer with the proper number of significant figures, rounding as necessary.

(a) 0.000440 × 17.22 = ? (b) 203,000 ÷ 0.044 = ? (c) 67 × 85.0 × 0.0028 = ? (d) 999,999 ÷ 3,310 = ?

11. Write the number 87,449 in scientific notation with four significant figures.

12. Write the number 0.000066600 in scientific notation with five significant figures.

13. Write the number 306,000,000 in scientific notation to the proper number of significant figures.

14. Write the number 0.0000558 in scientific notation with two significant figures.

15. Perform each calculation and limit each answer to three significant figures.

(a) 67,883 × 0.004321 = ? (b) $(9.67 \times 10^3) \times 0.0055087 = ?$

16. Perform each calculation and limit each answer to four significant figures.

(a) 18,900 × 76.33 ÷ 0.00336 = ? (b) 0.77604 ÷ 76,003 × 8.888 = ?

Solutions

Check Your Understanding 1

0.63 cm

Check Your Understanding 2

1. three significant figures
2. five significant figures

Check Your Understanding 3

21.9

Check Your Understanding 4

1. 186
2. 0.229

Problems & Exercises

1. (a) 375 psi (b) 1.30 cm

3. (a) two (b) three (c) two (d) four

5. (a) five (b) seven (c) five (d) four

7. (a) 59.4 (b) 1.011 (c) 13.2 (d) 88.2

9. (a) 3.80×10^3 (b) 0.013 (c) 0.0130 (d) 11.4

11. (a) 8.745×10^4 (b) 6.6600×10^{-5}

13. (a) 293 (b) 53.3

Converting Units

In “Expressing Units”, we showed some examples of how to replace initial units with other units of the same type to get a numerical value that is easier to comprehend. In this section, we will formalize the process.

Consider a simple example: how many feet are there in 4 yards? Most people will almost automatically

answer that there are 12 feet in 4 yards. How did you make this determination? Well, if there are 3 feet in 1 yard and there are 4 yards, then there are 4 × 3 = 12 feet in 4 yards.

This is correct, of course, but it is informal. Let us formalize it in a way that can be applied more generally. We know that 1 yard (yd) equals 3 feet (ft):

1 yd = 3 ft

In math, this expression is called an *equality*. The rules of algebra say that you can change (i.e., multiply or divide or add or subtract) the equality (as long as you don't divide by zero) and the new expression will still be an equality. For example, if we divide both sides by 2, we get

$$\frac{1}{2}\,yd = \frac{3}{2}\,ft$$

We see that one-half of a yard equals 3/2, or one and a half, feet—something we also know to be true, so the above equation is still an equality. Going back to the original equality, suppose we divide both sides of the equation by 1 yard (number *and* unit):

$$\frac{1\,yd}{1\,yd} = \frac{3\,ft}{1\,yd}$$

The expression is still an equality, by the rules of algebra. The left fraction equals 1. It has the same quantity in the numerator and the denominator, so it must equal 1. The quantities in the numerator and denominator cancel, both the number *and* the unit:

$$\frac{\cancel{1\,yd}}{\cancel{1\,yd}} = \frac{3\,ft}{1\,yd}$$

When everything cancels in a fraction, the fraction reduces to 1:

$$1 = \frac{3\ ft}{1\ yd}$$

We have an expression, 3 ft1 yd, that equals 1. This is a strange way to write 1, but it makes sense: 3 ft equal 1 yd, so the quantities in the numerator and denominator are the same quantity, just expressed with different units. The expression 3 ft1 yd is called a conversion factor, and it is used to formally change the unit of a quantity into another unit. (The process of converting units in such a formal fashion is sometimes called *dimensional analysis* or the *factor label method*.)

To see how this happens, let us start with the original quantity:

4 yd

Now let us multiply this quantity by 1. When you multiply anything by 1, you don't change the value of the quantity. Rather than multiplying by just 1, let us write 1 as 3 ft1 yd:

$$4\ yd\ \times \frac{3\ ft}{1\ yd}$$

The 4 yd term can be thought of as 4 yd/1; that is, it can be thought of as a fraction with 1 in the denominator. We are essentially multiplying fractions. If the same thing appears in the numerator and denominator of a fraction, they cancel. In this case, what cancels is the unit *yard*:

$$4\ \cancel{yd}\ \times \frac{3\ ft}{1\ \cancel{yd}}$$

$$\frac{4\ \times 3\ ft}{1} = \frac{12\ ft}{1} = 12\ ft$$

That is all that we can cancel. Now, multiply and divide all the numbers to get the final answer:

Again, we get an answer of 12 ft, just as we did originally. But in this case, we used a more formal procedure that is applicable to a variety of problems.

How many millimeters are in 14.66 m? To answer this, we need to construct a conversion factor between millimeters and meters and apply it correctly to the original quantity. We start with the definition of a millimeter, which is

1 mm = 1/1,000 m

The 1/1,000 is what the prefix *milli-* means. Most people are more comfortable working without fractions, so we will rewrite this equation by bringing the 1,000 into the numerator of the other side of the equation:

1,000 mm = 1 m

Now we construct a conversion factor by dividing one quantity into both sides. But now a question arises: which quantity do we divide by? It turns out that we have two choices, and the two choices will give us different conversion factors, both of which equal 1:

$$\frac{1000\ mm}{1000\ mm} = \frac{1\ m}{1000\ mm} \quad or \quad \frac{1000\ mm}{1\ m} = \frac{1\ m}{1\ m}$$

$$1 = \frac{1\ m}{1000\ mm} \quad or \quad \frac{1000\ mm}{1\ m} = 1$$

Which conversion factor do we use? The answer is based on *what unit you want to get rid of in your initial quantity*. The original unit of our quantity is meters, which we want to convert to millimeters. Because the original unit is assumed to be in the numerator, to get rid of it, we want the meter unit in the *denominator*; then they will cancel. Therefore, we will use the second conversion factor. Canceling units and performing the mathematics, we get

$$14.66\ \cancel{m} \times \frac{1000\ mm}{1\ \cancel{m}} = 14660\ mm$$

Note how m cancels, leaving mm, which is the unit of interest.

The ability to construct and apply proper conversion factors is a very powerful mathematical technique

in chemistry. You need to master this technique if you are going to be successful in this and future courses.

Example 7

1. Convert 35.9 kL to liters.
2. Convert 555 nm to meters.

Solution

1. We will use the fact that 1 kL = 1,000 L. Of the two conversion factors that can be defined, the one that will work is 1,000 L/1 kL. Applying this conversion factor, we get

$$35.9\ \cancel{kL} \times \frac{1000\ L}{1\ \cancel{kL}} = 35900\ L$$

2. We will use the fact that 1 nm = 1/1,000,000,000 m, which we will rewrite as 1,000,000,000 nm = 1 m, or 10^9 nm = 1 m. Of the two possible conversion factors, the appropriate one has the nm unit in the denominator: 1 m/10^9 nm. Applying this conversion factor, we get

$$555\ \cancel{nm} \times \frac{1\ m}{10^9\ \cancel{nm}} = 0.000000555\ m = 5.55 \times 10^{-7}\ m$$

In the final step, we expressed the answer in scientific notation.

Check Your Understanding 1

1. Convert 67.08 µL to liters.
2. Convert 56.8 m to kilometers.

What if we have a derived unit that is the product of more than one unit, such as m^2? Suppose we want to convert square meters to square centimeters? The key is to remember that m^2 means m × m, which means we have *two* meter units in our derived unit. That means we have to include *two* conversion factors, one for each unit. For example, to convert 17.6 m^2 to square centimeters, we perform the conversion as follows:

$$17.6\ m^2 = 17.6\ (\cancel{m} \times \cancel{m}) \times \frac{100\ cm}{1\ \cancel{m}} \times \frac{100\ cm}{1\ \cancel{m}} = 176000\ cm \times cm = 1.76 \times 10^5\ cm^2$$

Example 8

How many cubic centimeters are in 0.883 m^3?

Solution

With an exponent of 3, we have three length units, so by extension we need to use three conversion factors between meters and centimeters. Thus, we have

$$0.833\ \cancel{m}^3 \times \frac{100\ cm}{1\ \cancel{m}} \times \frac{100\ cm}{1\ \cancel{m}} \times \frac{100\ cm}{1\ \cancel{m}} = 883000\ cm^3 = 8.83 \times 10^5\ cm^3$$

You should demonstrate to yourself that the three meter units do indeed cancel.

Check Your Understanding 2

How many cubic millimeters are present in 0.0923 m^3?

Suppose the unit you want to convert is in the denominator of a derived unit; what then? Then, in the conversion factor, the unit you want to remove must be in the *numerator*. This will cancel with the original unit in the denominator and introduce a new unit in the denominator. The following example illustrates this situation.

Example 9

Convert 88.4 m/min to meters/second.

Solution

We want to change the unit in the denominator from minutes to seconds. Because there are 60 seconds in 1 minute (60 s = 1 min), we construct a conversion factor so that the unit we want to remove, minutes, is in the numerator: 1 min/60 s. Apply and perform the math:

$$\frac{88.4\ m}{\cancel{min}} \times \frac{1\ \cancel{min}}{60\ s} = 1.47\ m/s$$

Notice how the 88.4 automatically goes in the numerator. That's because any number can be thought of as being in the numerator of a fraction divided by 1.

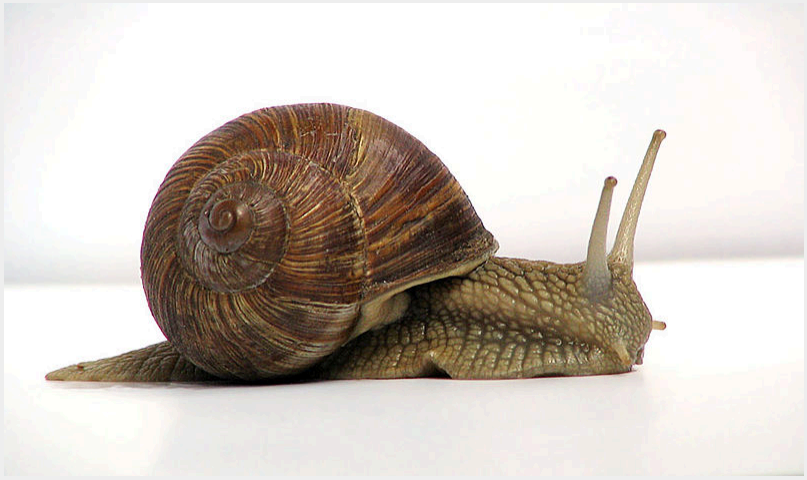


Figure 1. *How Fast Is Fast? A common garden snail moves at a rate of about 0.2 m/min, which is about 0.003 m/s, which is 3 mm/s!*
Source: "Grapevine snail" by Jürgen Schoneris licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license.

Check Your Understanding 3

Convert 0.203 m/min to meters/second.

Sometimes there will be a need to convert from one unit with one numerical prefix to another unit with a different numerical prefix. How do we handle those conversions? Well, you could memorize the conversion factors that interrelate all numerical prefixes. Or you can go the easier route: first convert the quantity to the base unit, the unit with no numerical prefix, using the definition of the original prefix. Then convert the quantity in the base unit to the desired unit using the definition of the second prefix. You can do the conversion in two separate steps or as one long algebraic step. For example, to convert 2.77 kg to milligrams:

$$2.77\ \cancel{kg} \times \frac{1000\ g}{1\ \cancel{kg}} = 2770\ g\ (convert\ to\ the\ base\ unit\ of\ grams)$$

$$2770\ \cancel{g} \times \frac{1000\ mg}{1\ \cancel{g}} = 2770000\ mg = 2.77 \times 10^6\ mg\ (convert\ to\ the\ desired\ unit)$$

Alternatively, it can be done in a single multistep process:

$$2.77\ \cancel{kg} \times \frac{1000\ \cancel{g}}{1\ \cancel{kg}} \times \frac{1000\ mg}{1\ \cancel{g}} = 2770000\ mg = 2.77 \times 10^6\ mg$$

You get the same answer either way.

Example 10

How many nanoseconds are in 368.09 μs?

Solution

You can either do this as a one-step conversion from microseconds to nanoseconds or convert to the base unit first and then to the final desired unit. We will use the second method here, showing the two steps in a single line. Using the definitions of the prefixes *micro-* and *nano-*,

$$368.09\ \cancel{\mu s} \times \frac{1\ \cancel{s}}{10^6\ \cancel{\mu s}} \times \frac{10^9\ ns}{1\ \cancel{s}} = 368090\ ns = 3.6809\ x\ 10^5\ ns$$

Check Your Understanding 4

How many milliliters are in 607.8 kL?

When considering the significant figures of a final numerical answer in a conversion, there is one important case where a number does not impact the number of significant figures in a final answer—the so-called exact number. An exact number is a number from a defined relationship, not a measured one. For example, the prefix *kilo-* means 1,000—*exactly* 1,000, no more or no less. Thus, in constructing the conversion factor

$$\frac{1000\ g}{1\ kg}$$

neither the 1,000 nor the 1 enter into our consideration of significant figures. The numbers in the numerator and denominator are defined exactly by what the prefix *kilo-* means. Another way of thinking about it is that these numbers can be thought of as having an infinite number of significant figures, such as

$$\frac{1000.0000000000\ \ldots\ g}{1.0000000000\ \ldots\ kg}$$

The other numbers in the calculation will determine the number of significant figures in the final answer.

Example 11

A rectangular plot in a garden has the dimensions 36.7 cm by 128.8 cm. What is the area of the garden plot in square meters? Express your answer in the proper number of significant figures.

Solution

Area is defined as the product of the two dimensions, which we then have to convert to square meters and express our final answer to the correct number of significant figures, which in this case will be three.

$$36.7\ \cancel{cm} \times 128.8\ \cancel{cm} \times \frac{1\ m}{100\ \cancel{cm}} \times \frac{1\ m}{100\ \cancel{cm}} = 0.472696\ m^2 = 0.473\ m^2$$

The 1 and 100 in the conversion factors do not affect the determination of significant figures because they are exact numbers, defined by the centi- prefix.

Check Your Understanding 5

What is the volume of a block in cubic meters whose dimensions are 2.1 cm × 34.0 cm × 118 cm?

Chemistry (and physics and math…) is Everywhere: The Gimli Glider

On July 23, 1983, an Air Canada Boeing 767 jet had to glide to an emergency landing at Gimli Industrial Park Airport in Gimli, Manitoba, because it unexpectedly ran out of fuel during flight. There was no loss of life in the course of the emergency landing, only some minor injuries associated in part with the evacuation of the craft after landing. For the remainder of its operational life (the plane was retired in 2008), the aircraft was nicknamed "the Gimli Glider."

Figure 2. *The Gimli Glider is the Boeing 767 that ran out of fuel and glided to safety at Gimli Airport. The aircraft ran out of fuel because of confusion over the units used to express the amount of fuel. "Aircanada.b767" is in the the public domain.*

The 767 took off from Montreal on its way to Ottawa, ultimately heading for Edmonton, Canada. About halfway through the flight, all the engines on the plane began to shut down because of a lack of fuel. When the final engine cut off, all electricity (which was generated by the engines) was lost; the plane became, essentially, a powerless glider. Captain Robert Pearson was an experienced glider pilot, although he had never flown a glider the size of a 767. First Officer Maurice Quintal quickly determined that the aircraft would not be able make it to Winnipeg, the next large airport. He suggested his old Royal Air Force base at Gimli Station, one of whose runways was still being used as a community airport. Between the efforts of the pilots and the flight crew, they managed to get the airplane safely on the ground (although with buckled landing gear) and all passengers off safely.

What happened? At the time, Canada was transitioning from the older English system to the metric system. The Boeing 767s were the first aircraft whose gauges were calibrated in the metric system of units (liters and kilograms) rather than the English system of units (gallons and pounds). Thus, when the fuel gauge read 22,300, the gauge meant kilograms, but the ground crew mistakenly fueled the plane with 22,300 *pounds* of fuel. This ended up being just less than half of the fuel needed to make the trip, causing the engines to quit about halfway to Ottawa. Quick thinking and extraordinary skill saved the lives of 61 passengers and 8 crew members—an incident that would not have occurred if people were watching their units.

Figure 3. *Video source: Unit conversion by keyj (https://viutube.viu.ca/public/media/Unit+Conversion/0_h2w068q1)*

Key Takeaways

- Units can be converted to other units using the proper conversion factors.
- Conversion factors are constructed from equalities that relate two different units.
- Conversions can be a single step or multistep.
- Unit conversion is a powerful mathematical technique in chemistry that must be mastered.
- Exact numbers do not affect the determination of significant figures.

Exercises

1. Write the two conversion factors that exist between the two given units.

(a) milliliters and liters (b) microseconds and seconds (c) kilometers and meters

2. Write the two conversion factors that exist between the two given units.

(a) kilograms and grams (b) milliseconds and seconds (c) centimeters and meters

3. Perform the following conversions.

(a) 5.4 km to meters (b) 0.665 m to millimeters (c) 0.665 m to kilometers

4. Perform the following conversions.

(a) 90.6 mL to liters (b) 0.00066 ML to liters (c) 750 L to kiloliters

5. Perform the following conversions.

(a) 17.8 µg to grams (b) 7.22×10^2 kg to grams (c) 0.00118 g to nanograms

6. Perform the following conversions.

(a) 833 ns to seconds (b) 5.809 s to milliseconds (c) 2.77×10^6 s to megaseconds

7. Perform the following conversions.

(a) 9.44 m^2 to square centimeters (b) 3.44×10^8 mm^3 to cubic meters

8. Perform the following conversions.

(a) 0.00444 cm^3 to cubic meters (b) 8.11×10^2 m^2 to square nanometers

9. Why would it be inappropriate to convert square centimeters to cubic meters?

10. Why would it be inappropriate to convert from cubic meters to cubic seconds?

11. Perform the following conversions.

(a) 45.0 m/min to meters/second (b) 0.000444 m/s to micrometers/second (c) 60.0 km/h to kilometers/second

12. Perform the following conversions.

(a) 3.4×10^2 cm/s to centimeters/minute (b) 26.6 mm/s to millimeters/hour (c) 13.7 kg/L to kilograms/milliliters

13. Perform the following conversions.

(a) 0.674 kL to milliliters (b) 2.81×10^{12} mm to kilometers (c) 94.5 kg to milligrams

14. Perform the following conversions.

(a) 6.79×10^{-6} kg to micrograms (b) 1.22 mL to kiloliters (c) 9.508×10^{-9} ks to milliseconds

15. Perform the following conversions.

(a) 6.77×10^{14} ms to kiloseconds (b) 34,550,000 cm to kilometers

16. Perform the following conversions.

(a) 4.701×10^{15} mL to kiloliters (b) 8.022×10^{-11} ks to microseconds

17. Perform the following conversions. Note that you will have to convert units in both the numerator and the denominator.

(a) 88 ft/s to miles/hour (Hint: use 5,280 ft = 1 mi.) (b) 0.00667 km/h to meters/second

18. Perform the following conversions. Note that you will have to convert units in both the numerator and the denominator.

(a) 3.88×10^2 mm/s to kilometers/hour (b) 1.004 kg/L to grams/milliliter

19. What is the area in square millimeters of a rectangle whose sides are 2.44 cm × 6.077 cm? Express the answer to the proper number of significant figures.

20. What is the volume in cubic centimeters of a cube with sides of 0.774 m? Express the answer to the proper number of significant figures.

21. The formula for the area of a triangle is 1/2 × base × height. What is the area of a triangle in square centimeters if its base is 1.007 m and its height is 0.665 m? Express the answer to the proper number of significant figures.

22. The formula for the area of a triangle is 1/2 × base × height. What is the area of a triangle in square meters if its base is 166 mm and its height is 930.0 mm? Express the answer to the proper number of significant figures.

Solutions

Check Your Understanding 1

1. 6.708×10^{-5} L
2. 5.68×10^{-2} km

Check Your Understanding 2

9.23×10^{7} mm^3

Check Your Understanding 3

0.00338 m/s or 3.38×10^{-3} m/s

Check Your Understanding 4

6.078×10^{8} mL

Check Your Understanding 5

0.0084 m^3

Problems & Exercises

1. (a) 1,000 mL/1 L and 1 L/1,000 mL (b) 1,000,000 μs/1 s and 1 s/1,000,000 μs (c) 1,000 m/1 km and 1 km1,000 m

3. (a) 5,400 m (b) 665 mm (c) 6.65×10^{-4} km

5. (a) 1.78×10^{-5} g (b) 7.22×10^{5} g (c) 1.18×10^{6} ng

7. (a) 94,400 cm^2 (b) 0.344 m^3

9. One is a unit of area, and the other is a unit of volume.

11. (a) 0.75 m/s (b) 444 μm/s (c) 1.666×10^{-2} km/s

13. (a) 674,000 mL (b) 2.81×10^{6} km (c) 9.45×10^{7} mg

15. (a) 6.77×10^{8} ks (b) 345.5 km

17. (a) 6.0×10^{1} mi/h (b) 0.00185 m/s

19. 1.48×10^{3} mm^2

21. 3.35×10^{3} cm^2

Other Units: Temperature and Density

There are other units in chemistry that are important, and we will cover others in the course of the entire book. One of the fundamental quantities in science is temperature. Temperature is a measure of the average amount of energy of motion, or *kinetic energy*, a system contains. Temperatures are expressed using scales that use units called degrees, and there are several temperature scales in use. In the United States, the commonly used temperature scale is the *Fahrenheit scale* (symbolized by °F and spoken as "degrees Fahrenheit"). On this scale, the freezing point of liquid water (the temperature at which liquid water turns to solid ice) is 32 °F, and the boiling point of water (the temperature at which liquid water turns to steam) is 212 °F.

Science also uses other scales to express temperature. The Celsius scale (symbolized by °C and spoken as "degrees Celsius") is a temperature scale where 0 °C is the freezing point of water and 100 °C is the boiling point of water; the scale is divided into 100 divisions between these two landmarks and extended higher and lower. By comparing the Fahrenheit and Celsius scales, a conversion between the two scales can be determined:

$$°\text{C} = (°\text{F} - 32) \times \frac{5}{9}$$

$$°\text{F} = \left(°\text{C} \times \frac{9}{5}\right) + 32$$

Using these formulas, we can convert from one temperature scale to another. The number 32 in the formulas is exact and does not count in significant figure determination.

Example 12

1. What is 98.6 °F in degrees Celsius?
2. What is 25.0 °C in degrees Fahrenheit?

Solution

1. Using the first formula from above, we have
   °C = (98.6 – 32) × 5/9 = 66.6 × 5/9 = 37.0 °C
2. Using the second formula from above, we have

°F = (25.0 × 9/5) + 3/2 = 45.0 + 32 = 77.0 °F

Check Your Understanding 1

1. Convert 0 °F to degrees Celsius.
2. Convert 212 °C to degrees Fahrenheit.

The fundamental unit of temperature (another fundamental unit of science, bringing us to four) in SI is the kelvin (K). The Kelvin temperature scale (note that the name of the scale capitalizes the word *Kelvin*, but the unit itself is lowercase) uses degrees that are the same size as the Celsius degree, but the numerical scale is shifted up by 273.15 units. That is, the conversion between the Kelvin and Celsius scales is as follows:

K = °C + 273.15
°C = K − 273.15

For most purposes, it is acceptable to use 273 instead of 273.15. Note that the Kelvin scale does not use the word *degrees*; a temperature of 295 K is spoken of as "two hundred ninety-five kelvins" and not "two hundred ninety-five degrees Kelvin."

The reason that the Kelvin scale is defined this way is because there exists a minimum possible temperature called absolute zero. The Kelvin temperature scale is set so that 0 K is absolute zero, and temperature is counted upward from there. Normal room temperature is about 295 K, as seen in the following example.

Example 13

If normal room temperature is 72.0 °F, what is room temperature in degrees Celsius and kelvins?

Solution

First, we use the formula to determine the temperature in degrees Celsius:

°C = (72.0 – 32) × 5/9 = 40.0 × 5/9 = 22.2 °C

Then we use the appropriate formula above to determine the temperature in the Kelvin scale:

K = 22.2 °C + 273.15 = 295.4 K

So, room temperature is about 295 K.

Check Your Understanding 2

What is 98.6 °F on the Kelvin scale?

Figure 2.9 “Fahrenheit, Celsius, and Kelvin Temperatures” compares the three temperature scales. Note that science uses the Celsius and Kelvin scales almost exclusively; virtually no practicing chemist expresses laboratory-measured temperatures with the Fahrenheit scale. (In fact, the United States is one of the few countries in the world that still uses the Fahrenheit scale on a daily basis. The other two countries are Liberia and Myanmar [formerly Burma].

Figure 2.9 Fahrenheit, Celsius, and Kelvin Temperatures

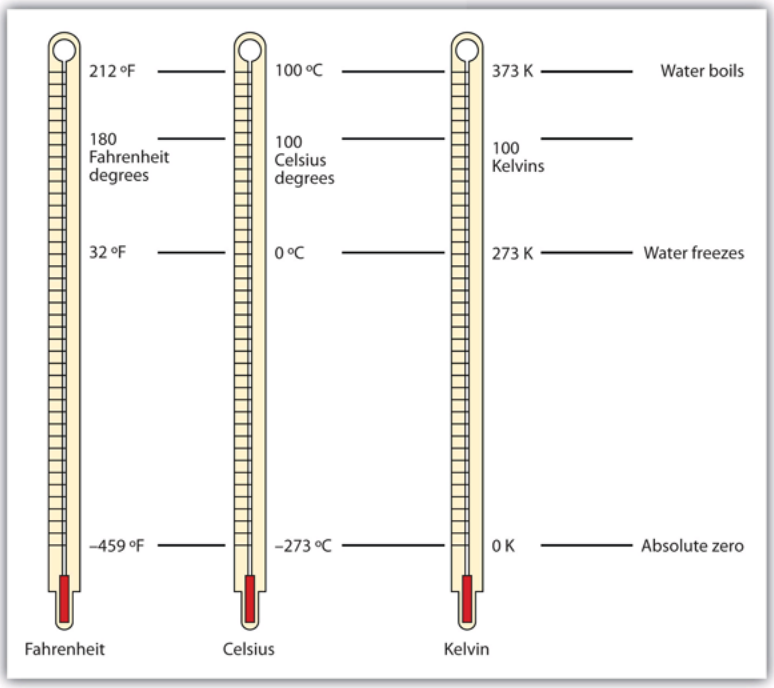


Figure 1. *Fahrenheit, Celsius, and Kelvin Temperatures*

A comparison of the three temperature scales.

Density is a physical property that is defined as a substance’s mass divided by its volume:

density = mass/volume or d = m/V

Density is usually a measured property of a substance, so its numerical value affects the significant figures in a calculation. Notice that density is defined in terms of two dissimilar units, mass and volume. That means that density overall has derived units, just like velocity. Common units for density include g/mL, g/cm^3, g/L, kg/L, and even kg/m^3. Densities for some common substances are listed in Table 2.2 “Densities of Some Common Substances”.

Table 2.2 Densities of Some Common Substances

Substance	**Density (g/mL or g/cm^3)**
water	1.0
gold	19.3
mercury	13.6
air	0.0012
cork	0.22–0.26
aluminum	2.7
iron	7.87

Because of how it is defined, density can act as a conversion factor for switching between units of mass and volume. For example, suppose you have a sample of aluminum that has a volume of 7.88 cm^3. How can you determine what mass of aluminum you have without measuring it? You can use the volume to calculate it. If you multiply the given volume by the known density (from Table 2.2 “Densities of Some Common Substances”), the volume units will cancel and leave you with mass units, telling you the mass of the sample:

7.88 cm^3 × 2.7 g/cm^3 = 21 g of aluminum

where we have limited our answer to two significant figures.

Example 14

What is the mass of 44.6 mL of mercury?

Solution

Use the density from Table 2.2 “Densities of Some Common Substances” as a conversion factor to go from volume to mass:

44.6 mL × 13.6 g/mL = 607 g

The mass of the mercury is 607 g.

Check Your Understanding 3

What is the mass of 25.0 cm^3 of iron?

Density can also be used as a conversion factor to convert mass to volume—but care must be taken. We have already demonstrated that the number that goes with density normally goes in the numerator when density is written as a fraction. Take the density of gold, for example:

$$d = 19.3\,\frac{g}{mL} = \frac{19.3\ g}{1\ mL}$$

Although this was not previously pointed out, it can be assumed that there is a 1 in the denominator:

That is, the density value tells us that we have 19.3 grams for every 1 milliliter of volume, and the 1 is an exact number. When we want to use density to convert from mass to volume, the numerator and denominator of density need to be switched—that is, we must take the *reciprocal* of the density. In so doing, we move not only the units but also the numbers:

$$\frac{1}{d} = \frac{1\ mL}{19.3\ g}$$

This reciprocal density is still a useful conversion factor, but now the mass unit will cancel and the volume unit will be introduced. Thus, if we want to know the volume of 45.9 g of gold, we would set up the conversion as follows:

$$45.9\ \cancel{g} \times \frac{1\ mL}{19.3\ \cancel{g}} = 2.38\ mL$$

Note how the mass units cancel, leaving the volume unit, which is what we're looking for.

Example 15

A cork stopper from a bottle of wine has a mass of 3.78 g. If the density of cork is 0.22 g/cm^3, what is the volume of the cork?

Solution

To use density as a conversion factor, we need to take the reciprocal so that the mass unit of density is in the denominator. Taking the reciprocal, we find

$$\frac{1}{d} = \frac{1\ cm^3}{0.22\ g}$$

We can use this expression as the conversion factor. So

$$3.78\ \cancel{g} \times \frac{1\ cm^3}{0.22\ \cancel{g}} = 17.2\ cm^3$$

Check Your Understanding 4

What is the volume of 3.78 g of gold?

Care must be used with density as a conversion factor. Make sure the mass units are the same, or the volume units are the same, before using density to convert to a different unit. Often, the unit of the given quantity must be first converted to the appropriate unit before applying density as a conversion factor.

Food and Drink App: Cooking Temperatures

Because degrees Fahrenheit is the common temperature scale in the United States, kitchen appliances, such as ovens, are calibrated in that scale. A cool oven may be only 150°F, while a cake may be baked at 350°F and a chicken roasted at 400°F. The broil setting on many ovens is 500°F, which is typically the highest temperature setting on a household oven.

People who live at high altitudes, typically 2,000 ft above sea level or higher, are sometimes urged to use slightly different cooking instructions on some products, such as cakes and bread, because water boils

at a lower temperature the higher in altitude you go, meaning that foods cook slower. For example, in Cleveland water typically boils at 212°F (100°C), but in Denver, the Mile-High City, water boils at about 200°F (93.3°C), which can significantly lengthen cooking times. Good cooks need to be aware of this.

At the other end is pressure cooking. A pressure cooker is a closed vessel that allows steam to build up additional pressure, which increases the temperature at which water boils. A good pressure cooker can get to temperatures as high as 252°F (122°C); at these temperatures, food cooks much faster than it normally would. Great care must be used with pressure cookers because of the high pressure and high temperature. (When a pressure cooker is used to sterilize medical instruments, it is called an *autoclave*.)

Other countries use the Celsius scale for everyday purposes. Therefore, oven dials in their kitchens are marked in degrees Celsius. It can be confusing for US cooks to use ovens abroad—a 425°F oven in the United States is equivalent to a 220°C oven in other countries. These days, many oven thermometers are marked with both temperature scales.

Key Takeaways

- Chemistry uses the Celsius and Kelvin scales to express temperatures.
- A temperature on the Kelvin scale is the Celsius temperature plus 273.15.
- The minimum possible temperature is absolute zero and is assigned 0 K on the Kelvin scale.
- Density relates a substance's mass and volume.
- Density can be used to calculate volume from a given mass or mass from a given volume.

Problems & Exercises

1. Perform the following conversions.

(a) 255°F to degrees Celsius (b) −255°F to degrees Celsius (c) 50.0°C to degrees Fahrenheit (d) −50.0°C to degrees Fahrenheit

2. Perform the following conversions.

(a) 1,065°C to degrees Fahrenheit (b) −222°C to degrees Fahrenheit (c) 400.0°F to degrees Celsius (d) 200.0°F to degrees Celsius

3. Perform the following conversions.

(a) 100.0°C to kelvins (b) −100.0°C to kelvins (c) 100 K to degrees Celsius (d) 300 K to degrees Celsius

4. Perform the following conversions.

(a) 1,000.0 K to degrees Celsius (b) 50.0 K to degrees Celsius (c) 37.0°C to kelvins (d) −37.0°C to kelvins

5. Convert 0 K to degrees Celsius. What is the significance of the temperature in degrees Celsius?

6. Convert 0 K to degrees Fahrenheit. What is the significance of the temperature in degrees Fahrenheit?

7. The hottest temperature ever recorded on the surface of the earth was 136°F in Libya in 1922. What is the temperature in degrees Celsius and in kelvins?

8. The coldest temperature ever recorded on the surface of the earth was −128.6°F in Vostok, Antarctica, in 1983. What is the temperature in degrees Celsius and in kelvins?

9. Give at least three possible units for density.

10. What are the units when density is inverted? Give three examples.

11. A sample of iron has a volume of 48.2 cm^3. What is its mass?

12. A sample of air has a volume of 1,015 mL. What is its mass?

13. The volume of hydrogen used by the *Hindenburg*, the German airship that exploded in New Jersey in 1937, was 2.000×10^8 L. If hydrogen gas has a density of 0.0899 g/L, what mass of hydrogen was used by the airship?

14. The volume of an Olympic-sized swimming pool is 2.50×10^9 cm^3. If the pool is filled with alcohol (d = 0.789 g/cm^3), what mass of alcohol is in the pool?

15. A typical engagement ring has 0.77 cm^3 of gold. What mass of gold is present?

16. A typical mercury thermometer has 0.039 mL of mercury in it. What mass of mercury is in the thermometer?

17. What is the volume of 100.0 g of lead if lead has a density of 11.34 g/cm^3?

18. What is the volume of 255.0 g of uranium if uranium has a density of 19.05 g/cm^3?

19. What is the volume in liters of 222 g of neon if neon has a density of 0.900 g/L?

20. What is the volume in liters of 20.5 g of sulfur hexafluoride if sulfur hexafluoride has a density of 6.164 g/L?

21. Which has the greater volume, 100.0 g of iron (d = 7.87 g/cm^3) or 75.0 g of gold (d = 19.3 g/cm^3)?

22. Which has the greater volume, 100.0 g of hydrogen gas (d = 0.0000899 g/cm^3) or 25.0 g of argon gas (d = 0.00178 g/cm^3)?

Solutions

Check Your Understanding 1

1. −17.8 °C
2. 414 °F

Check Your Understanding 2

310.2 K

Check Your Understanding 3

197 g

Check Your Understanding 4

0.196 cm^3

Problems & Exercises

1. (a) 124°C (b) −159°C (c) 122°F (d) −58°F

3. (a) 373 K (b) 173 K (c) −173°C (d) 27°C

5. −273°C. This is the lowest possible temperature in degrees Celsius.

7. 57.8°C; 331 K

9. g/mL, g/L, and kg/L (answers will vary)

11. 379 g

13. 1.80×10^7 g

15. 15 g

17. 8.818 cm^3

19. 247 L

21. The 100.0 g of iron has the greater volume.

Expressing Units

A number indicates "how much," but the unit indicates "of what." The "of what" is important when communicating a quantity. For example, if you were to ask a friend how close you are to Lake Erie and your friend says "six," then your friend isn't giving you complete information. Six *what*? Six miles? Six inches? Six city blocks? The actual distance to the lake depends on what units you use.

Chemistry, like most sciences, uses the International System of Units, or SI for short. (The letters *SI* stand for the French "le Système International d'unités.") SI specifies certain units for various types of quantities, based on seven fundamental units for various quantities. We will use most of the fundamental units in chemistry. Initially, we will deal with three fundamental units. The meter (m) is the SI unit of length. It is a little longer than a yard (see Figure 2.3 "The Meter"). The SI unit of mass is the kilogram (kg), which is about 2.2 pounds (lb). The SI unit of time is the second (s).

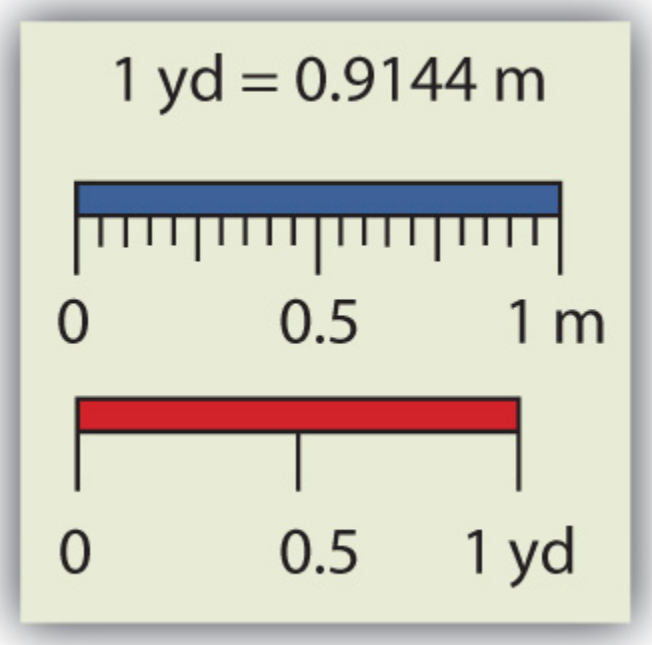


Figure 1. *The Meter*

The SI standard unit of length, the meter, is a little longer than a yard.

To express a quantity, you need to combine a number with a unit. If you have a length that is 2.4 m, then you express that length as simply 2.4 m. A time of 15,000 s can be expressed as 1.5×10^4 s in scientific notation.

Sometimes, a given unit is not an appropriate size to easily express a quantity. For example, the width of a human hair is very small, and it doesn't make much sense to express it in meters. SI also defines a series of numerical prefixes that refer to multiples or fractions of a fundamental unit to make a unit more conveniently sized for a specific quantity. Table 2.1 "Multiplicative Prefixes for SI Units" lists the prefixes, their abbreviations, and their multiplicative factors. Some of the prefixes, such as kilo-, mega-, and giga-, represent more than one of the fundamental unit, while other prefixes, such as centi-, milli-, and micro-, represent fractions of the original unit. Note, too, that once again we are using powers of 10. Each prefix is a multiple of or fraction of a power of 10.

Table 2.1 Multiplicative Prefixes for SI Units

Prefix	Abbreviation	Multiplicative Amount
giga-	G	1,000,000,000 ×
mega-	M	1,000,000 ×
kilo-	k	1,000 ×
deci-	d	1/10 ×
centi-	c	1/100 ×
milli-	m	1/1,000 ×
micro-	μ*	1/1,000,000 ×
nano-	n	1/1,000,000,000 ×
pico-	p	1/1,000,000,000,000 ×

*** The letter *μ* is the Greek letter lowercase equivalent to an m and is called "mu" (pronounced "myoo").**

To use the fractions to generate new units, simply combine the prefix with the unit itself; the abbreviation for the new unit is the combination of the abbreviation for the prefix and the abbreviation of the unit. For example, the kilometer (km) is 1,000 × meter, or 1,000 m. Thus, 5 kilometers (5 km) is equal to 5,000 m. Similarly, a millisecond (ms) is 1/1,000 × second, or one-thousandth of a second. Thus, 25 ms is 25 thousandths of a second. You will need to become proficient in combining prefixes and units. (You may recognize that one of our fundamental units, the kilogram, automatically has a prefix-unit combination, the kilogram. The word *kilogram* means 1,000 g.)

In addition to the fundamental units, SI also allows for derived units based on a fundamental unit or units. There are many derived units used in science. For example, the derived unit for area comes from the idea that area is defined as width times height. Because both width and height are lengths, they both have the fundamental unit of meter, so the unit of area is meter × meter, or meter^2 (m^2). This is sometimes spoken as "square meters." A unit with a prefix can also be used to derive a unit for area, so we can also have cm^2, mm^2, or km^2 as acceptable units for area.

Volume is defined as length times width times height, so it has units of meter × meter × meter or meter^3 (m^3), sometimes spoken as "cubic meters." The cubic meter is a rather large unit, however, so another unit is defined that is somewhat more manageable: the liter (L). A liter is 1/1,000th of a cubic meter and is a little more than 1 quart in volume (see Figure 2.4 "The Liter"). Prefixes can also be used with the liter unit, so we can speak of milliliters (1/1,000th of a liter; mL) and kiloliters (1,000 L; kL).

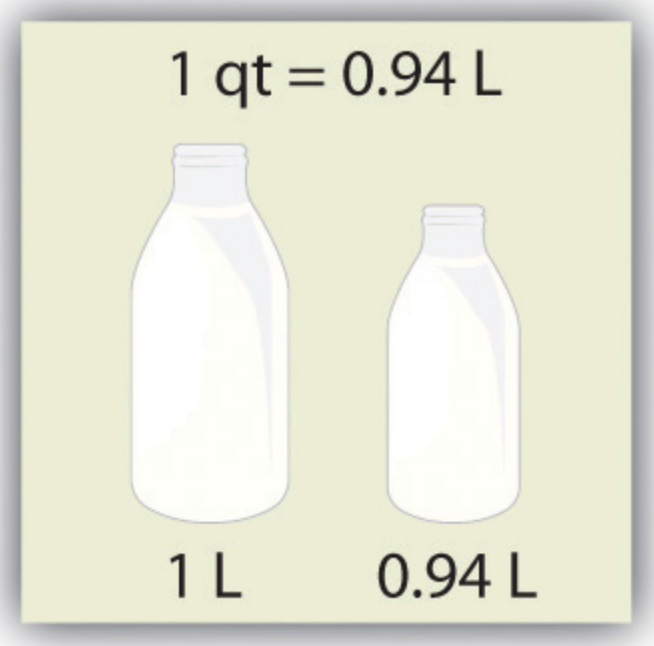


Figure 2. *The Liter*

The SI unit of volume, the liter, is slightly larger than 1 quart.

Another definition of a liter is one-tenth of a meter cubed. Because one-tenth of a meter is 10 cm, then a liter is equal to 1,000 cm^3 (Figure 2.5 “The Size of 1 Liter”). Because 1 L equals 1,000 mL, we conclude that 1 mL equals 1 cm^3; thus, these units are interchangeable.

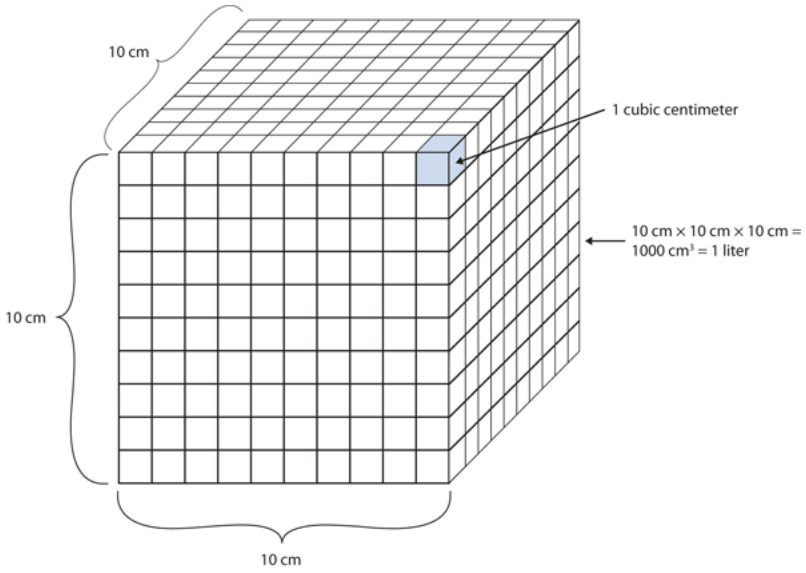


Figure 3. *The Size of 1 Liter*

One liter equals 1,000 cm^3, so 1 cm^3 is the same as 1 mL.

Units not only are multiplied together but also can be divided. For example, if you are traveling at one meter for every second of time elapsed, your velocity is 1 meter per second, or 1 m/s. The word *per* implies division, so velocity is determined by dividing a distance quantity by a time quantity. Other units for velocity include kilometers per hour (km/h) or even micrometers per nanosecond (μm/ns). Later, we will see other derived units that can be expressed as fractions.

Example 2

1. A human hair has a diameter of about 6.0×10^{-5} m. Suggest an appropriate unit for this measurement and write the diameter of a human hair in terms of that unit.
2. What is the velocity of a car if it goes 25 m in 5.0 s?

Solution

1. The scientific notation 10^{-5} is close to 10^{-6}, which defines the micro- prefix. Let us use micrometers as the unit for hair diameter. The number 6.0×10^{-5} can be written as 60×10^{-6}, and a micrometer is 10^{-6} m, so the diameter of a human hair is about 60 μm.
2. If velocity is defined as a distance quantity divided by a time quantity, then velocity is 25 meters/5.0 seconds. Dividing the numbers gives us 25/5.0 = 5.0, and dividing the units gives us meters/second, or m/s. The velocity is 5.0 m/s.

Check Your Understanding 1

1. Express the volume of an Olympic-sized swimming pool, 2,500,000 L, in more appropriate units.
2. A common garden snail moves about 6.1 m in 30 min. What is its velocity in meters per minute (m/min)?

Key Takeaways

- Numbers tell “how much,” and units tell “of what.”
- Chemistry uses a set of fundamental units and derived units from SI units.
- Chemistry uses a set of prefixes that represent multiples or fractions of units.
- Units can be multiplied and divided to generate new units for quantities.

Problems & Exercises

1. Identify the unit in each quantity.

(a) 2 boxes of crayons (b) 3.5 grams of gold

2. Identify the unit in each quantity.

(a) 32 oz of cheddar cheese (b) 0.045 cm^3 of water

3. Identify the unit in each quantity.

(a) 9.58 s (the current world record in the 100 m dash) (b) 6.14 m (the current world record in the pole vault)

4. Identify the unit in each quantity.

(a) 2 dozen eggs (b) 2.4 km/s (the escape velocity of the moon, which is the velocity you need at the surface to escape the moon's gravity)

5. Indicate what multiplier each prefix represents.

(a) k (b) m (c) M

6. Indicate what multiplier each prefix represents.

(a) c (b) G (c) μ

7. Give the prefix that represents each multiplier.

(a) 1/1,000th × (b) 1,000 × (c) 1,000,000,000 ×

8. Give the prefix that represents each multiplier.

(a) 1/1,000,000,000th × (b) 1/100th × (c) 1,000,000 ×

9. Complete the following table with the missing information.

Unit	**Abbreviation**
kilosecond	
mL	
Mg	
centimeter	

10. Complete the following table with the missing information.

Unit	Abbreviation
kilometer per second	
second	
cm^3	
μL	
nanosecond	

11. Express each quantity in a more appropriate unit. There may be more than one acceptable answer.
(a) 3.44×10^{-6} s (b) 3,500 L(c) 0.045 m

12. Express each quantity in a more appropriate unit. There may be more than one acceptable answer.
(a) 0.000066 m/s (Hint: you need consider only the unit in the numerator.) (b) 4.66×10^{6} s (c) 7,654 L

13. Express each quantity in a more appropriate unit. There may be more than one acceptable answer.
(a) 43,600 mL (b) 0.0000044 m (c) 1,438 ms

14. Express each quantity in a more appropriate unit. There may be more than one acceptable answer.
(a) 0.000000345 m^3 (b) 47,000,000 mm^3 (c) 0.00665 L

15. Multiplicative prefixes are used for other units as well, such as computer memory. The basic unit of computer memory is the byte (b). What is the unit for one million bytes?

16. You may have heard the terms *microscale* or *nanoscale* to represent the sizes of small objects. What units of length do you think are useful at these scales? What fractions of the fundamental unit of length are these units?

17. Acceleration is defined as a change in velocity per time. Propose a unit for acceleration in terms of the fundamental SI units.

18. Density is defined as the mass of an object divided by its volume. Propose a unit of density in terms of the fundamental SI units.

Solutions

Check Your Understanding 1

1. 2.5 ML
2. 0.203 m/min

Problems & Exercises

1. (a) boxes of crayons (b) grams of gold

3. (a) seconds (b) meters

5. (a) 1,000 × (b) 1/1,000 × (c) 1,000,000 ×

7. (a) milli- (b) kilo- (c) giga-

9.

Unit	Abbreviation
kilosecond	ks
milliliter	mL
megagram	Mg
centimeter	cm

11. (a) 3.44 µs (b) 3.5 kL (c) 4.5 cm

13. (a) 43.6 L (b) 4.4 µm (c) 1.438 s

15. megabytes (Mb)

17. meters/second^2

Additional Exercises

Additional Exercises

1. Evaluate 0.00000000552 × 0.0000000006188 and express the answer in scientific notation. You may have to rewrite the original numbers in scientific notation first.
2. Evaluate 333,999,500,000 ÷ 0.00000000003396 and express the answer in scientific notation. You may need to rewrite the original numbers in scientific notation first.
3. Express the number 6.022×10^{23} in standard notation.
4. Express the number 6.626×10^{-34} in standard notation.
5. When powers of 10 are multiplied together, the powers are added together. For example, $10^2 \times 10^3 = 10^{2+3} = 10^5$. With this in mind, can you evaluate $(4.506 \times 10^4) \times (1.003 \times 10^2)$ without entering scientific notation into your calculator?
6. When powers of 10 are divided into each other, the bottom exponent is subtracted from the top exponent. For example, $10^5/10^3 = 10^{5-3} = 10^2$. With this in mind, can you evaluate $(8.552 \times 10^6) \div (3.129 \times 10^3)$ without entering scientific notation into your calculator?

7. Consider the quantity two dozen eggs. Is the number in this quantity “two” or “two dozen”? Justify your choice.
8. Consider the quantity two dozen eggs. Is the unit in this quantity “eggs” or “dozen eggs”? Justify your choice.
9. Fill in the blank: 1 km = _____ μm.
10. Fill in the blank: 1 Ms = _____ ns.
11. Fill in the blank: 1 cL = _____ ML.
12. Fill in the blank: 1 mg = _____ kg.
13. Express 67.3 km/h in meters/second.
14. Express 0.00444 m/s in kilometers/hour.
15. Using the idea that 1.602 km = 1.000 mi, convert a speed of 60.0 mi/h into kilometers/hour.
16. Using the idea that 1.602 km = 1.000 mi, convert a speed of 60.0 km/h into miles/hour.
17. Convert 52.09 km/h into meters/second.
18. Convert 2.155 m/s into kilometers/hour.
19. Use the formulas for converting degrees Fahrenheit into degrees Celsius to determine the relative size of the Fahrenheit degree over the Celsius degree.
20. Use the formulas for converting degrees Celsius into kelvins to determine the relative size of the Celsius degree over kelvins.
21. What is the mass of 12.67 L of mercury?
22. What is the mass of 0.663 m^3 of air?
23. What is the volume of 2.884 kg of gold?
24. What is the volume of 40.99 kg of cork? Assume a density of 0.22 g/cm^3.

Answers

1. 3.42×10^{-18}

3. 602,200,000,000,000,000,000,000

5. 4.520×10^{6}

7 . The quantity is two; dozen is the unit.

9. 1,000,000,000

11. 1/100,000,000

13. 18.7 m/s

15. 96.1 km/h

17. 14.47 m/s

19. One Fahrenheit degree is nine-fifths the size of a Celsius degree.

21. 1.72×10^5 g

23. 149 mL